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SOME REMARKS ON ABSOLUTE SUMMABILITY METHODS

Abstract

This note deals with certain recent results concerning summability $|\overline{N}, p_n|_k$.

1. Let $\sum a_n$ be a given infinite series with $\{s_n\}$ as the sequence of its n-th partial sums. Let σ_n^α and t_n^α denote the n-th (C, α) means of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. We denote by $\{p_n\}$ a sequence of positive constants such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$. The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $\alpha > -1$, $k \geq 1$ [8] if

$$\sum_1^\infty n^{k-1} |\Delta \sigma_{n-1}^\alpha|^k = \sum_1^\infty \frac{|t_n^\alpha|^k}{n} < \infty \,. \tag{1.1}$$

It is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$ [1] if

$$\sum_1^\infty \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta T_{n-1}|^k < \infty \,, \tag{1.2}$$

where

$$T_n = \frac{1}{P_n} \sum_{k=0}^n p_k s_k \,.$$

The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$ [11] if

$$\sum_1^\infty n^{k-1} |\Delta T_{n-1}|^k < \infty \,. \tag{1.3}$$

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Obviously $|\overline{N}, p_n|_1 = |R, p_n|_1 = |\overline{N}, p_n|$. Also if $P_n = O(np_n)$ then

$$|R, p_n|_k \Rightarrow |\overline{N}, p_n|_k \tag{1.4}$$

$$\text{and if } np_n = O(P_n), \text{then } |\overline{N}, p_n|_k \Rightarrow |R, p_n|_k\,. \tag{1.5}$$

Thus if

$$np_n \asymp P_n\,, \tag{1.6}$$

then $|\overline{N}, p_n|_k \Longleftrightarrow |R, p_n|_k$.
For $p_n = 1$ we have $|\overline{N}, 1|_k = |R, 1|_k = |C, 1|_k$ while for $p_n = 1/(n+1)$ we find from (1.5) that $|\overline{N}, 1/(n+1)|_k \Rightarrow |R, 1/(n+1)|_k$ but its converse is not true as can be seen by taking $\Delta T_{n-1} = n^{-1}[\log(n+1)]^{-1}$.

It is well known [7] that $|C, 1| \Rightarrow |R, \log n, 1|$ but the converse need not be true. Also it is known that

$$|R, \log n, 1| \Longleftrightarrow \left|\overline{N}, \frac{1}{n+1}\right| = \left|R, \frac{1}{n+1}\right|_1\,. \tag{1.7}$$

Concerning summability $|C, \alpha|$, Kogbetliantz [9] had proved that $|C, \alpha| \Rightarrow |C, \beta|$, $\beta \geq \alpha$, $\alpha > -1$. The corresponding extension to summability $|C, \alpha|_k$ was proved by Flett [8] who has shown that $|C, \alpha|_k \Rightarrow |C, \beta|_k$, $k \geq 1$, $\beta \geq \alpha$, $\alpha > -1$. Consequently in view of (1.7) we should expect either

$$|C, 1|_k \Rightarrow \left|\overline{N}, \frac{1}{n+1}\right|_k \tag{1.8}$$

or

$$|C, 1|_k \Rightarrow \left|R, \frac{1}{n+1}\right|_k \tag{1.9}$$

Concerning inclusion relations between summability $|C, 1|_k$ and summability $|\overline{N}, p_n|_k$ the following results are known.

Under the condition (1.6)

$$|C, 1|_k \Rightarrow |\overline{N}, p_n|_k \quad [2] \tag{1.10}$$

and

$$|\overline{N}, p_n|_k \Rightarrow |C, 1|_k \quad [3] \tag{1.11}$$

Consequently under the condition (1.6)

$$|\overline{N}, p_n|_k \Leftrightarrow |C, 1|_k \tag{1.12}$$

For the necessity part we have the following [11].

$$\text{If } \ |C, 1|_k \Rightarrow |\overline{N}, p_n|_k, \ \text{ then } \ np_n = O(P_n)\,, \ \text{ and} \tag{1.13}$$

$$\text{if} \quad |\overline{N}, p_n|_k \Rightarrow |C,1|_k, \quad \text{then} \quad P_n = O(np_n)\,. \tag{1.14}$$

It can be easily shown that $\{(-1)^{n-1}\}$ is summable $|R, 1/(n+1)|_k$ but not summable $|\overline{N}, 1/(n+1)|_k$ or $|C,1|_k$. In view of the relation $|\overline{N}, 1/(n+1)|_k \Rightarrow |R, 1/(n+1)|_k$ we should expect (1.9) to hold true. Writing

$$\sigma_n = \frac{1}{n+1}\sum_{\nu=0}^{n} s_\nu\,, \quad T_n = \frac{1}{P_n}\sum_{\nu=0}^{n}\frac{s_\nu}{\nu+1} \quad \text{and}$$

$$P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}$$

we have

$$\begin{aligned}\Delta T_n &= \left(\frac{1}{P_n} - \frac{1}{P_{n+1}}\right)\sum_{0}^{n}\frac{s_\nu}{\nu+1} - \frac{S_{n+1}}{(n+2)P_{n+1}} \\ &= \frac{p_{n+1}}{P_nP_{n+1}}\sum_{\nu=0}^{n}(P_\nu - 1)\Delta\sigma_{\nu-1} + \frac{\Delta\sigma_n}{P_{n+1}}\end{aligned}$$

so that

$$\sum_{1}^{\infty} n^{k-1}|\Delta T_n|^k \leq$$

$$C\sum_{n=1}^{\infty}\frac{n^{k-1}}{\big(n\log^2(n+1)\big)^k}\left(\sum_{\nu=0}^{n}P_\nu|\Delta\sigma_{\nu-1}|\right)^k + C\sum_{n=1}^{\infty}\frac{n^{k-1}|\Delta\sigma_n|^k}{(\log(n+1))^k}$$

$$= O(1)\sum_{n=1}^{\infty}\frac{1}{n\log^{2k}(n+1)}\sum_{\nu=0}^{n}P_\nu(\nu+1)^{k-1}|\Delta\sigma_{\nu-1}|^k\left(\sum_{\nu=0}^{n}\frac{P_\nu}{\nu+1}\right)^{k-1}$$

$$+O\left(\sum_{1}^{\infty}n^{k-1}|\Delta\sigma_n|^k\right) = O(1)\sum_{n=1}^{\infty}\frac{1}{n\log^2(n+1)}\sum_{\nu=1}^{n}\nu^{k-1}\log(\nu+1)|\Delta\sigma_{\nu-1}|^k$$

$$+O(1) = O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}\log(\nu+1)|\Delta\sigma_{\nu-1}|^k\sum_{n=\nu}^{\infty}\frac{1}{n\log^2(n+1)} + O(1)$$

$$= O(1)\sum_{\nu=1}^{\infty}\nu^{k-1}|\Delta\sigma_{\nu-1}|^k + O(1) = O(1)\,.$$

Thus $|C,1|_k \Rightarrow |R, 1/(n+1)|_k$, $k \geq 1$, whereas $|C,1|_k \not\Rightarrow |\overline{N}, 1/(n+1)|_k$ as it is evident from the following special case of a result of Sarigol [12]:
$|C,1|_k \Rightarrow |\overline{N}, p_n|_k$ *iff*

(i) $np_n = O(P_n)$

(ii) $\sum_1^m |(\nu+1)\Delta(P_{\nu-1}) + P_\nu|^{k'}/\nu+1 = O(P_m^{k'})$.

Here (ii) does not hold if $p_n = 1/(n+1)$.
This shows that extension of the summability $|\overline{N}, p_n|$ to index k in the sense of (1.3) is more appropriate.

2. Bor and Thorpe in [5] gave a generalization of the result (1.12) in following way. Suppose $\{p_n\}$ and $\{q_n\}$ are two sequences of positive constants such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ and $Q_n = q_o + q_1 + \cdots + q_n \to \infty$ as $n \to \infty$. If

$$\frac{p_n}{P_n} \asymp \frac{q_n}{Q_n} \tag{2.1}$$

then

$$|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_k, \quad k \geq 1 \tag{2.2}$$

and so

$$|\overline{N}, q_n|_k \Rightarrow |\overline{N}, p_n|_k, \quad k \geq 1\,. \tag{2.3}$$

Thus under (2.1)

$$|\overline{N}, p_n|_k \Leftrightarrow |\overline{N}, q_n|_k, \quad k \geq 1\,. \tag{2.4}$$

For the necessity part Bor and Thorpe [6] showed that if $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_k$, then

$$\frac{q_n}{Q_n} = O\left(\frac{p_n}{P_n}\right)\,. \tag{2.5}$$

Consequently if $|\overline{N}, q_n|_k \Rightarrow |\overline{N}, p_n|_k$, then

$$\frac{p_n}{P_n} = O\left(\frac{q_n}{Q_n}\right)\,. \tag{2.6}$$

It is clear that if (2.6) holds then

$$\sum_1^n \frac{p_\nu Q_\nu}{P_\nu} = O\left(\sum_1^n q_\nu\right) = O(Q_n)\,. \tag{2.7}$$

We find that it is possible to relax the condition (2.1). We will show that if (2.5) and (2.7) hold then $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_k$, $k \geq 1$. This generalizes a result of Bor and Thorpe [5]. Writing

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^n p_\nu s_\nu \quad\text{and}\quad T_n = \frac{1}{Q_n}\sum_{\nu=0}^n q_\nu s_\nu\,,$$

then as shown by Bor and Thorpe [5] (pp. 147-148),

$$\begin{aligned}\Delta T_{n-1} &= \frac{q_n P_n \Delta t_{n-1}}{p_n Q_n} - \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=1}^{n-1} \frac{P_\nu}{p_\nu} q_\nu \Delta t_{\nu-1} \\ &\quad + \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=1}^{n-1} Q_\nu \Delta t_{\nu-1} \\ &= T_{n,1} + T_{n,2} + T_{n,3}\,, \quad \text{say}\,.\end{aligned}$$

In view of Minkowski's inequality, to prove that

$$\sum_1^\infty \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta T_{n-1}|^k < \infty\,,$$

it is enough to prove that

$$\sum_1^\infty \left(\frac{Q_n}{q_n}\right)^{k-1} |\Delta T_{n,r}|^k < \infty\,, \quad r = 1, 2, 3\,. \tag{2.8}$$

The proof of (2.8) for $r = 1, 2$ is the same as in [5], since these involve the condition (2.5) only. To prove (2.8) for $r = 3$ we have in view of (2.7)

$$\begin{aligned}\sum_1^\infty \left(\frac{Q_n}{q_n}\right)^{k-1} |T_{n,3}|^k &= \sum_1^\infty \left(\frac{Q_n}{q_n}\right)^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left|\sum_{\nu=1}^{n-1} Q_\nu \Delta t_{\nu-1}\right|^k \\ &\le \sum_1^\infty \frac{q_n}{Q_n Q_{n-1}^k} \left(\sum_{\nu=1}^{n-1} \left(\frac{P_\nu}{p_\nu}\right)^{\frac{1}{k'}} \left(\frac{p_\nu}{P_\nu}\right)^{\frac{1}{k'}} Q_\nu^{\frac{1}{k'}} \cdot Q_\nu^{\frac{1}{k}} |\Delta t_{\nu-1}|\right)^k \\ &\le \sum_1^\infty \frac{q_n}{Q_n Q_{n-1}^k} \left(\sum_{\nu=1}^{n-1} \left(\frac{P_\nu}{p_\nu}\right)^{k-1} Q_\nu |\Delta t_{\nu-1}|^k\right) \left(\sum_{\nu=1}^{n-1} \frac{p_\nu}{P_\nu} Q_\nu\right)^{k-1} \\ &= O(1) \sum_1^\infty \frac{q_n}{Q_n Q_{n-1}^k} Q_{n-1}^{k-1} \cdot \sum_{\nu=1}^{n-1} \left(\frac{P_\nu}{p_\nu}\right)^{k-1} Q_\nu |\Delta t_{\nu-1}|^k \\ &= O(1) \sum_{\nu=1}^\infty \left(\frac{P_\nu}{p_\nu}\right)^{k-1} Q_\nu |\Delta t_{\nu-1}|^k \sum_{n=\nu+1}^\infty \frac{q_n}{Q_n Q_{n-1}} \\ &= O(1) \sum_{\nu=1}^\infty \left(\frac{P_\nu}{p_\nu}\right)^{k-1} |\Delta t_{\nu-1}|^k < \infty\,.\end{aligned}$$

This completes the proof of our result.

Interchanging p_n, P_n with q_n, Q_n respectively we conclude that if

$$\frac{p_n}{P_n} = O\left(\frac{q_n}{Q_n}\right) \quad \text{and} \quad \sum_{1}^{n} \frac{q_\nu P_\nu}{Q_\nu} = O(P_n) \tag{2.9}$$

then $|\overline{N}, q_n|_k \Rightarrow |\overline{N}, p_n|_k$, $k \geq 1$. Taking $p_n = 1$ we get generalizations of (1.10) and (1.11).

Remark. It may be observed that (2.5) is a necessary condition while (2.7) is only sufficient. A special case of Sarigol's result [12] states:

Necessary and sufficient conditions for $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_k$, are

$$(I) \qquad \frac{q_n}{Q_n} = O\left(\frac{p_n}{P_n}\right)$$

$$(II) \qquad \sum_{1}^{m} \left|\frac{P_\nu}{p_\nu}(\Delta Q_{\nu-1}) + Q_\nu\right|^{k'} \frac{p_\nu}{P_\nu} = O(Q_m^{k'}), \tag{2.10}$$

where $1/k + 1/k' = 1$.

Here (I) is the same as (2.5). Using (2.5) and (2.7) we have

$$\sum_{1}^{m} \left|\frac{P_\nu}{p_\nu}(-q_\nu) + Q_\nu\right|^{k'} \frac{p_\nu}{P_\nu}$$

$$= O(1) \sum_{1}^{m} Q_\nu^{k'} \frac{p_\nu}{P_\nu} = O(Q_m^{k'-1}) \sum_{1}^{m} \frac{Q_\nu p_\nu}{P_\nu} = O(Q_m^{k'}).$$

Hence (2.5) and (2.7) $\Rightarrow$ (2.10).

Thus our result can be deduced from Sarigol's result. However we include the proof of our result as the condition (2.7) is simpler than (2.10) and our proof is not based on functional analysis method as is the case with that of Sarigol.

3. With a view to generalize the result (1.11), Bor [4] recently proved that if $np_n \asymp P_n$ and

$$\sum_{1}^{\infty} \left(\frac{P_n}{p_n}\right)^{(2-\alpha)k-1} |\Delta T_{n-1}|^k < \infty \quad 0 < \alpha \leq 1, \quad k \geq 1, \tag{3.1}$$

where T_n denotes the $(\overline{N}, p_n)$ mean of a series $\sum a_n$, then the series $\sum a_n$ is summable $|C,\alpha|_k$.

It is clear that if $np_n \asymp P_n$ then (3.1) is equivalent to the condition

$$\sum_{1}^{\infty} n^{(2-\alpha)k-1} |\Delta T_{n-1}|^k < \infty \,. \tag{3.2}$$

Thus Bor's result can be stated as: *If* $np_n \asymp P_n$ *and* (3.2) *holds then* $\sum a_n$ *is summable* $|C,\alpha|_k$, $0<\alpha\le 1$, $k\ge 1$. Analyzing the proof of Bor we observe that the following result holds: *If* $P_n = O(np_n)$ *and* (3.2) *holds, then* $\sum a_n$ *is summable* $|C,\alpha|_k$, $0<\alpha\le 1$, $k\ge 1$. This shows that the condition $np_n = O(P_n)$ in Bor's result is superfluous.

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