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SUMS OF QUASICONTINUOUS FUNCTIONS DEFINED ON PSEUDOMETRIZABLE SPACES

Abstract

It is shown that each real cliquish function f defined on a pseudometrizable space is the sum of two quasicontinuous functions. If moreover f is bounded (in the Baire class α), then we can take the summands with this property.

1 Introduction

It is easy to see that the sum of two quasicontinuous functions need not be quasicontinuous. However it must be cliquish, as the sum of two cliquish functions is cliquish.

The sums of quasicontinuous functions were first examined by Z. Grande [5] in 1985. He proved that each cliquish function f defined on $\mathbb{R}$ can be expressed as the sum of four quasicontinuous functions and in the case f is locally, bounded as the sum of three quasicontinuous functions. E. Strońska has shown that every real cliquish function f defined on $\mathbb{R}^n$ is the sum of six quasicontinuous functions [11] and later that every cliquish function f defined on a separable metrizable Baire space without isolated points is the sum of four quasicontinuous functions [12]. In [4] it is shown that each real cliquish function f defined on $\mathbb{R}^n$ is the sum of two simply continuous functions, each of which can be written as the sum of two quasicontinuous functions. In [2] it is shown that it is sufficient to assume that the domain of f is a separable metrizable space and in this case every cliquish function is the sum of three

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quasicontinuous functions. In [3] it is shown that we can omit the assumption of the separability of the domain. The principal problem was solved by Z. Grande [6], where it is proved that every cliquish function f defined on $\mathbb{R}$ is the sum of two quasicontinuous functions and by A. Maliszewski in [8] (see also [7]), where it is shown that two quasicontinuous functions are sufficient for a cliquish function defined on $\mathbb{R}^n$. In this paper we generalize this result for a cliquish function defined on a pseudometrizable space. Our proof is quite different from those in [6], [8]. (In these proofs properties of intervals in $\mathbb{R}$, resp. in $\mathbb{R}^n$, are essential.)

2 Preliminaries

In what follows, X denotes a topological space. For a subset A of X denote by $\mathrm{Cl}\ A$ and $\mathrm{Int}\ A$ the closure and the interior of A, respectively. If $\mathcal{A}$ is a family of sets in X, then $\mathrm{Cl}\ \mathcal{A} = \{\mathrm{Cl}\ A : A \in \mathcal{A}\}$. If d is a pseudometric on X,, then $S(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$, $S(A,\varepsilon) = \{y \in X : \mathrm{dist}\ (y,A) < \varepsilon\}$ and $\mathrm{diam}\ A = \sup\{d(a,b) : a,b \in A\}$ for each $x \in X$, $\varepsilon > 0$ and $A \subset X$, where $\mathrm{dist}\ (y,A) = \inf\{d(y,a) : a \in A\}$. The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ stand for the set of real, rational and natural numbers, respectively.

We recall that a function $f\colon X \to \mathbb{R}$ is said to be quasicontinuous (cliquish) at $x \in X$ if for every neighborhood U of x and every $\varepsilon > 0$ there is a nonempty open set $G \subset U$ such that $|f(x) - f(y)| < \varepsilon$ for each $y \in G$ ($|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$) (see e.g. [10]). A function is said to be quasicontinuous (cliquish) if it is such at each point. A function $f\colon X \to \mathbb{R}$ is simply continuous if $f^{-1}(V)$ is a simply open set in X for each open set V in $\mathbb{R}$. A set A is simply open if it is the union of an open set and a nowhere dense set [1].

If $f\colon X \to \mathbb{R}$ is a function, then by $D(f)$ we will denote the set of all discontinuity points of f. Further, the oscillation of a function f is the function from X into $\mathbb{R} \cup \{\infty\}$ given by the formula $\omega_f(x) = \inf\{\mathrm{diam}\ f(U) : U \text{ is a neighbourhood of } x\}$. Denote by $\mathcal{Q}, \mathcal{P}, \mathcal{S}, \mathcal{B}_\alpha$ and b the class of all quasicontinuous, cliquish, simply continuous, in the Baire class α and bounded functions.

3 Basic Lemmata

The following lemma is a key to a construction of quasicontinuous functions on pseudometrizable spaces.

Lemma 3.1 *Let X be a pseudometrizable space. Let F be a nonempty nowhere dense closed set and let G be an open set such that $F \subset \mathrm{Cl}\ G$. Then there is a family $\mathcal{K} = \bigcup_n \mathcal{K}_n$ of nonempty open subsets of X such that*

(i) $\mathrm{Cl}\,K \subset G \setminus F$ *for each* $K \in \mathcal{K}$,

(ii) *for each* $x \in X \setminus F$ *there is a neighbourhood* V *of* x *such that the set* $\{K \in \mathcal{K} : V \cap \mathrm{Cl}\,K \neq \emptyset\}$ *has at most one element,*

(iii) *for each* $x \in F$ *and for each neighbourhood* U *of* x *there is a* $k \in \mathbb{N}$ *such that for each* $n \geq k$ *there is* $K \in \mathcal{K}_n$ *with* $\mathrm{Cl}\,K \subset U$.

PROOF. Let d be a pseudometric which pseudometrizes X. We will construct families $\mathcal{K}_i$ of nonempty open sets in X satisfying for each $i \in \mathbb{N}$:

(1) $\operatorname{diam} K \leq 1/(2i)$ for each $K \in \mathcal{K}_i$,

(2) the set $\bigcup \mathrm{Cl}\,\mathcal{K}_i$ is closed,

(3) $\mathrm{Cl}\,K \subset S(F, 2/i) \cap (G \setminus F)$ for each $K \in \mathcal{K}_i$,

(4) for each $x \in F$ there is $K \in \mathcal{K}_i$ with $S(x, 2/i) \cap K \neq \emptyset$.

Let $n \in \mathbb{N}$. Assume that we have constructed families $\mathcal{K}_i$ of nonempty open sets in X satisfying (1)–(4) for each $i < n$. Put

$$T_n = (G \cap S(F, 1/n)) \setminus \left(F \cup \bigcup_{i<n} \bigcup \mathrm{Cl}\,\mathcal{K}_i \right)$$

and

$$\mathcal{P}_n = \{P \subset T_n : d(x, y) \notin (0, 1/n] \text{ for each } x, y \in P\}.$$

Since F is a nonempty nowhere dense subset of $\mathrm{Cl}\,G$, by (2) and (3) the set T_n is nonempty. According to Zorn's lemma there is a maximal element S_n of $\mathcal{P}_n$. Put

$$\alpha_n(x) = \frac{1}{4} \cdot \operatorname{dist}\left(x, F \cup (X \setminus G) \cup \bigcup_{i<n} \bigcup \mathrm{Cl}\,\mathcal{K}_i \right)$$

(For $x \in S_n$ we have $\alpha_n(x) > 0$.) and

$$\mathcal{K}_n = \{S(x, \alpha_n(x)) : x \in S_n\}.$$

We shall show that $\mathcal{K}_n$ satisfies (1)–(4).

(1) For $x \in S_n$ we have $\operatorname{dist}(x, F) < 1/n$ and hence $\alpha_n(x) < 1/(4n)$, which implies $\operatorname{diam} S(x, \alpha_n(x)) < 1/(2n)$.

(2) If $u \in \mathrm{Cl}\,S(x, \alpha_n(x))$, $v \in \mathrm{Cl}\,S(y, \alpha_n(y))$, where $x, y \in S_n$ and $d(x, y) > 1/n$, then $1/n < d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \leq 1/(2n) + d(u, v)$. This implies $d(u, v) > 1/(2n)$ and the set $\bigcup \mathrm{Cl}\,\mathcal{K}_n$ is closed.

(3) Let $x \in S_n$, $y \in S(x, \alpha_n(x))$ and let $z \in F$ be such that $d(x,z) < 1/n$. Then $d(y,z) \leq d(x,y) + d(x,z) < \alpha_n(x) + 1/n < 2/n$. Therefore $\mathrm{Cl}\ K \subset S(F, 2/n)$ for each $K \in \mathcal{K}_n$. Evidently $\mathrm{Cl}\ K \subset G \setminus F$.

(4) Let $z \in F$. We shall show that $S(z, 2/n) \cap S_n \neq \emptyset$. Assume that $S(z, 2/n) \cap S_n = \emptyset$. Then by (2) and (3) there is $0 < \delta < 1/n$ such that $S(z,\delta) \cap \bigcup_{i<n} \bigcup \mathrm{Cl}\, \mathcal{K}_i = \emptyset$. Since $S(z,\delta) \cap G \neq \emptyset$, $S(z,\delta) \subset S(F, 1/n)$ and F is nonempty nowhere dense, there is $y \in S(z, 1/n) \cap T_n$. For each $s \in S_n$ we have $2/n \leq d(z,s) \leq d(z,y) + d(y,s) < 1/n + d(y,s)$. This yields $d(y,s) > 1/n$ and $y \in S_n$, contrary to the maximality of S_n. Now $S(t, \alpha_n(t))$ for some $t \in S_n \cap S(z, 2/n)$ is a member of $\mathcal{K}_n$ satisfying (4).

Finally, put $\mathcal{K} = \bigcup_n \mathcal{K}_n$. We shall show that $\mathcal{K}$ satisfies (i)–(iii).

(i) It follows from (3).

(ii) Let $x \in X \setminus F$. Then there is $k \in \mathbb{N}$ with dist $(x, F) > 4/k$. For $n \geq k$ and $K \in \mathcal{K}_n$ we have $\mathrm{Cl}\ K \subset S(F, 2/n) \subset S(F, 2/k)$ and hence $\mathrm{Cl}\ K \cap S(x, 2/k) = \emptyset$. Now, in view of (2), it is sufficient to put $V = S(x, 2/k) \setminus (\bigcup \mathrm{Cl}\, \mathcal{K} \setminus \mathrm{Cl}\ K)$ if $x \in \mathrm{Cl}\ K$ for some $K \in \mathcal{K}$, and $V = S(x, 2/k) \setminus \bigcup \mathrm{Cl}\, \mathcal{K}$ otherwise.

(iii) Let $x \in F$ and let U be a neighbourhood of x. Then there is $k \in \mathbb{N}$ with $S(x, 3/k) \subset U$. Let $n \geq k$. Then by (4) there is $K \in \mathcal{K}_n$ such that $S(x, 2/n) \cap K \neq \emptyset$. According to (1) we have diam $K < 1/(2n)$ and hence $\mathrm{Cl}\ K \subset S(x, 3/k) \subset U$.

□

Remark 3.2 *Condition (ii) implies that* $\mathrm{Cl}\ K \cap \mathrm{Cl}\ L = \emptyset$ *for* $K, L \in \mathcal{K}$, $K \neq L$.

Remark 3.3 *From (ii) we obtain that* $\bigcup \mathrm{Cl}\, \mathcal{L}$ *is a closed set in* $X \setminus F$ *whenever* $\mathcal{L} \subset \mathcal{K}$.

The following lemma is proved in [8] for $X = \mathbb{R}^n$. The proof for a pseudometrizable space is the same.

Lemma 3.4 *Let X be a pseudometrizable space, let G be a subset of X and let $f \colon X \to \mathbb{R}$ be a function. Let ε be such that $\omega_f(x) < \varepsilon$ for each $x \in G$. Then there is a continuous function $g \colon G \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon/2$ for each $x \in G$.*

4 Result

We shall show that $\mathcal{Q} + \mathcal{Q} = \mathcal{P}, b\mathcal{Q} + b\mathcal{Q} = b\mathcal{P}, \mathcal{B}_\alpha\mathcal{Q} + \mathcal{B}_\alpha\mathcal{Q} = \mathcal{B}_\alpha\mathcal{P}, b\mathcal{B}_\alpha\mathcal{Q} + b\mathcal{B}_\alpha\mathcal{Q} = b\mathcal{B}_\alpha\mathcal{P}$ for a pseudometrizable space X.

Theorem 4.1 *Let X be a pseudometrizable space. Then every cliquish function $f\colon X \to \mathbb{R}$ is the sum of two quasicontinuous functions f_1 and f_2. Moreover, $D(f_1) \cup D(f_2) \subset D(f)$ and if f is bounded (in the Baire class α), then the summands can be taken bounded (in the Baire class α).*

PROOF. Let $A_n = \{x \in X : \omega_f(x) \geq 2^{-n}\}$. Since f is cliquish, the sets A_n are nowhere dense and closed in X and $D(f) = \bigcup_n A_n$. If f is continuous, then the case is trivial. In the opposite case, without loss of generality we may assume that $A_1 \neq \emptyset$. Set $G_n = X \setminus A_n$. According to Lemma 3.4 for each $n \in \mathbb{N}$ there is a continuous function $g_n\colon G_n \to \mathbb{R}$ such that $|f(x) - g_n(x)| < 2^{-n-1}$ for each $x \in G_n$. Define a function $s_n\colon X \to \mathbb{R}$ by

$$s_n(x) = \begin{cases} f(x) & \text{for } x \in A_n, \\ g_n(x) & \text{for } x \in G_n. \end{cases}$$

and a function $\beta\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $\beta(m,k) = 2^{k-1} \cdot (2m-1)$. Then β is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ and

$$\beta(m, k+2) = 4 \cdot \beta(m,k).$$

Further, let $\mathbb{Q} = \{q_1^1, q_2^1, \dots, q_m^1, \dots\}$ and $\mathbb{Q} \cap [-\frac{1}{2^n}, \frac{1}{2^n}] = \{q_1^n, q_2^n, \dots, q_m^n, \dots\}$ for $n > 1$ be one-to-one sequences such that $q_1^n = 0$ for each $n \in \mathbb{N}$. Let $\mathbb{N}_1$ be the set of all even numbers and let $\mathbb{N}_2$ be the set of all odd numbers from $\mathbb{N}$.

We will construct families $\mathcal{K}^i = \bigcup_j \mathcal{K}_j^i$ of nonempty open sets in X and functions $f_i^s\colon X \to \mathbb{R}$, $s \in \{1,2\}$, such that for each $i \in \mathbb{N}$:

(a) $\mathcal{K}^i = \bigcup_j \mathcal{K}_j^i$ satisfies conditions (i)–(iii) of Lemma 3.1 for $F = A_i$ and $G = X$,

(b) for each $x \in A_i$, for each neighbourhood U of x, for each $\varepsilon > 0$ and each $s \in \{1,2\}$ there are $m \in \mathbb{N}$, $k \in \mathbb{N}_s$ and $K \in \mathcal{K}^i_{\beta(m,k)}$ such that $\mathrm{Cl}\, K \subset U$ and $|f_i^s(x) - f_i^s(y)| < \varepsilon$ for each $y \in K$,

(c) f_i^s are continuous on G_i,

(d) $f_i^1 + f_i^2 = s_i$,

(e) $\sup\{|f_{i+1}^s(x) - f_i^s(x)| : x \in X, s \in \{1,2\}\ \} < 2^{1-i}$.

By Lemma 3.1, for $G = X$ and $F = A_1$ there is a family $\mathcal{K}^1 = \bigcup_j \mathcal{K}_j^1$ of nonempty open sets in X satisfying (i)–(iii). Let $p_1\colon \bigcup \mathrm{Cl}\, \mathcal{K}^1 \to \mathbb{R}$ be a function defined by

$$p_1(x) = (-1)^k (g_1(x)/2 - q_m^1)$$

for $x \in \bigcup \mathrm{Cl}\ \mathcal{K}^1_{\beta(m,k)}$. By (i) and (ii), p_1 is continuous on $\bigcup \mathrm{Cl}\ \mathcal{K}^1$ and by Remark 3.3 there is a continuous function $r_1 \colon G_1 \to \mathbb{R}$ such that $p_1(x) = r_1(x)$ for $x \in \bigcup \mathrm{Cl}\ \mathcal{K}^1$. Now define functions $w_1, f_1^1, f_1^2 \colon X \to \mathbb{R}$ by $w_1(x) = r_1(x)$ for $x \in G_1$, $w_1(x) = 0$ for $x \in A_1$ and

$$f_1^s = s_1/2 + (-1)^s w_1$$

for $s \in \{1, 2\}$.

Now let $n > 1$. Assume that for each $i < n$ we have constructed families $\mathcal{K}^i = \bigcup_j \mathcal{K}^i_j$ of nonempty open sets in X and functions $f_i^s \colon X \to \mathbb{R}$, $s \in \{1,2\}$, satisfying (a)–(e). Denote $S_n = G_{n-1} \setminus \bigcup \mathrm{Cl}\ \mathcal{K}^{n-1}$. By (a)(ii), S_n is an open set in X. Put $\mathcal{D}^n = \{B \in \mathcal{K}^{n-1} \cup \{S_n\} : A_n \cap \mathrm{Cl}\ B \neq \emptyset\}$ and $\mathcal{E}^n = \mathcal{K}^{n-1} \setminus \mathcal{D}^n$. According to Lemma 3.1, for each $D \in \mathcal{D}^n$ (for $F = A_n \cap \mathrm{Cl}\ D$ and $G = D$) there is a family $\mathcal{K}^{n,D} = \bigcup_j \mathcal{K}^{n,D}_j$ of nonempty open sets in D satisfying (i)–(iii). Put $\mathcal{K}^n = \mathcal{E}^n \cup \bigcup_{D \in \mathcal{D}^n} \mathcal{K}^{n,D}$ and define families $\mathcal{K}^n_j$ for $j \in \mathbb{N}$ as follows: K belongs to $\mathcal{K}^n_j$ if at least one of the following conditions is fulfilled:

- $K \in \mathcal{E}^n \cap \mathcal{K}^{n-1}_j$ or
- $K \in \mathcal{K}^{n,S_n}_j$ (if $S_n \in \mathcal{D}^n$) or
- $K \in \mathcal{K}^{n,D}_{4j}$, where $D \in \mathcal{K}^{n-1} \setminus \mathcal{E}^n$ or
- $K \in \mathcal{K}^{n,D}_t$, where $D \in \mathcal{K}^{n-1}_j \setminus \mathcal{E}^n$ and $t \not\equiv 0 \pmod 4$.

Then it is easy to see that $\mathcal{K}^n = \bigcup_j \mathcal{K}^n_j$. We shall show that $\mathcal{K}^n = \bigcup_j \mathcal{K}^n_j$ satisfies (a):

(i) If $K \in \mathcal{K}^{n,D}$ for some $D \in \mathcal{D}^n$ then $\mathrm{Cl}\ K \subset D \setminus A_n \subset X \setminus A_n$. If $K \in \mathcal{E}^n$, then $A_n \cap \mathrm{Cl}\ K = \emptyset$ and $\mathrm{Cl}\ K \subset X \setminus A_n$.

(ii) It is easy to see.

(iii) Let $x \in A_n$ and let U be a neighbourhood of x.

If $x \in A_n \setminus A_{n-1}$, then $x \in G_{n-1}$ and hence $x \in \mathrm{Cl}\ D$ for some $D \in \mathcal{D}^n$. Now $\exists k \in \mathbb{N}\, \forall p \geq k\ \exists L \in \mathcal{K}^{n,D}_p : \mathrm{Cl}\ L \subset U$. If $D \in \mathcal{K}^{n-1}$, then for each $p \geq k$ there is $K \in \mathcal{K}^{n,D}_{4p}$ with $\mathrm{Cl}\ K \subset U$ and $K \in \mathcal{K}^n_p$. The case $D = S_n$ is obvious. If $x \in A_{n-1}$, then by assumptions $\exists k \in \mathbb{N}\, \forall p \geq k\ \exists L \in \mathcal{K}^{n-1}_p : \mathrm{Cl}\ K \subset U$. If $L \in \mathcal{E}^n$, then for $K = L$ we have $K \in \mathcal{K}^n_p$ and $\mathrm{Cl}\ K \subset U$. If $L \subset \mathcal{K}^{n-1} \setminus \mathcal{E}^n$, then $\exists r \in \mathbb{N}\, \forall t \geq r\ \exists M \in \mathcal{K}^{n,L}_t : \mathrm{Cl}\ M \subset U$. For $K \in \mathcal{K}^{n,L}_{4r+1}$ with $\mathrm{Cl}\ K \subset U$ we have $K \in \mathcal{K}^n_p$.

Define a function $h_n \colon X \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} \frac{1}{2}(g_n(x) - g_{n-1}(x)) & \text{for } x \in G_n, \\ \frac{1}{2}(f(x) - g_{n-1}(x)) & \text{for } x \in A_n \setminus A_{n-1}, \\ 0 & \text{for } x \in A_{n-1}. \end{cases}$$

Since g_n and g_{n-1} are continuous on G_n, the function h_n is also continuous on G_n. Moreover, for each $x \in X$ we have

$$|h_n(x)| \le |g_n(x) - f(x)|/2 + |g_{n-1}(x) - f(x)|/2 < 2^{-n-2} + 2^{-n-1} < 2^{-n}.$$

By Remark 3.2 we can define a function $p_n \colon \bigcup \mathrm{Cl}\, \mathcal{K}^n \to \mathbb{R}$ by

$$p_n(x) = \begin{cases} (-1)^k \cdot (h_n(x) - q_m^n), & \text{if } x \in \mathrm{Cl}\, K, \text{ where } K \in \mathcal{K}^{n,D}_{\beta(m,k)} \\ & \text{for some } D \in \mathcal{D}^n \text{ and } m, k \in \mathbb{N}, \\ (-1)^k \cdot h_n(x), & \text{if } x \in \mathrm{Cl}\, K, \text{ where } K \in \mathcal{K}^{n-1}_{\beta(m,k)} \cap \\ & \mathcal{E}^n \text{ for some } m, k \in \mathbb{N}. \end{cases}$$

The continuity of h_n on G_n and the condition (a)(ii) imply that p_n is continuous on $\bigcup \mathrm{Cl}\, \mathcal{K}^n$. Moreover, for each $x \in \bigcup \mathrm{Cl}\, \mathcal{K}^n$ we have $|p_n(x)| \le q_m^n + |h_n(x)| < 2^{-n} + 2^{-n} = 2^{-n+1}$. Hence by Remark 3.3 there is a continuous function $r_n \colon G_n \to [-2^{-n+1}, 2^{-n+1}]$ such that $p_n(x) = r_n(x)$ for $x \in \bigcup \mathrm{Cl}\, \mathcal{K}^n$. Now let

$$w_n(x) = \begin{cases} r_n(x) & \text{for } x \in G_n, \\ 0 & \text{for } x \in A_n. \end{cases}$$

Then $|w_n(x)| \le 2^{-n+1}$ for each $x \in X$. Now finally define functions $f_n^s \colon X \to \mathbb{R}$, $s \in \{1, 2\}$, by

$$f_n^s = f_{n-1}^s + h_n + (-1)^s \cdot w_n.$$

We shall show that $\mathcal{K}^n$ and f_n^s satisfy (a)–(e). It is easy to see that (a)–(d) is satisfied for $n = 1$.

Let $n > 1$. We have already shown (a).

(b) Let $x \in A_n$, let U be a neighbourhood of x, let $\varepsilon > 0$ and $s \in \{1, 2\}$.

1. Let $\in A_{n-1}$. Then $f_n^s(x) = f_{n-1}^s(x)$ and by assumptions there is $t \in \mathbb{N}$, $r \in \mathbb{N}_s$ and $L \in \mathcal{K}^{n-1}_{\beta(t,r)}$ such that $\mathrm{Cl}\, L \subset U$ and $|f_{n-1}^s(x) - f_{n-1}^s(y)| < \varepsilon$ for each $y \in L$.

If $L \in \mathcal{E}^n$, then it is sufficient to take $m = t$, $k = r$ and $K = L$. Now $k \in \mathbb{N}_s$, $K \in \mathcal{K}^n_{\beta(m,k)}$, $\mathrm{Cl}\, K \subset U$ and for each $y \in K$ we have $p_n(y) = (-1)^k \cdot h_n(y) = (-1)^{s+1} \cdot h_n(y)$ and hence

$$|f_n^s(x) - f_n^s(y)| \le |f_{n-1}^s(x) - f_{n-1}^s(y)| + |h_n(y) + (-1)^s \cdot p_n(y)| < \varepsilon.$$

If $L \in \mathcal{K}^{n-1} \setminus \mathcal{E}^n$, then $\exists l \in \mathbb{N} \forall p \geq l \, \exists M \in \mathcal{K}_p^{n,L}$ with Cl $M \subset U$. Now it is sufficient to take $m = 1$, $k \in \mathbb{N}_s$ such that $\beta(1,k) \geq l$ and $K \in \mathcal{K}^{n,L}_{\beta(1,k+2)}$ such that Cl $K \subset U$. Then $K \in \mathcal{K}^{n,L}_{4\beta(1,k)}$ and thus $K \in \mathcal{K}^n_{\beta(1,k)}$. Further, for each $y \in K$ we have $p_n(y) = (-1)^{k+2} \cdot (h_n(y) - q_1^n) = (-1)^{s+1} \cdot h_n(y)$ and therefore

$$|f_n^s(x) - f_n^s(y)| \leq |f_{n-1}^s(x) + f_{n-1}^s(y)| + |h_n(y) + (-1)^s \cdot p_n(y)| < \varepsilon.$$

2. Let $x \in A_n \setminus A_{n-1}$. Then $f_n^s(x) = f_{n-1}^s(x) + h_n(x)$. By the assumption (c) f_{n-1}^s is continuous at $x \in G_{n-1}$ and hence there is a neighbourhood $V \subset U$ of x such that $|f_{n-1}^s(x) - f_{n-1}^s(z)| < \varepsilon/2$ for each $z \in V$. Since $|h_n(x)| \leq 2^{-n}$, there is $m \in \mathbb{N}$ such that $|q_m^n - h_n(x)| < \varepsilon/2$. Since $x \in G_{n-1} \cap A_n$ there is $D \in \mathcal{D}^n$ such that $x \in$ Cl D. By assumptions $\exists l \in \mathbb{N} \, \forall p \geq l \, \exists M \in \mathcal{K}_p^{n,D}$ with Cl $M \subset V$. Let $k \in \mathbb{N}_s$ be such that $\beta(m,k) \geq l$ and let $K \in \mathcal{K}^{n,D}_{\beta(m,k+2)}$ be such that Cl $K \subset V$. For each $y \in K$ we have $p_n(y) = (-1)^{s+1}(h_n(y) - q_m^n)$ and therefore

$$|f_n^s(x) - f_n^s(y)| \leq |f_{n-1}^s(x) - f_{n-1}^s(y)| +$$
$$|h_n(x) - h_n(y) - (-1)^s(-1)^{s+1}(h_n(y) - q_m^n)| < \varepsilon/2 + |h_n(x) - q_n^m| < \varepsilon.$$

Further, if $D = S_n$, then $K \in \mathcal{K}^n_{\beta(m,k+2)}$ and $k+2 \in \mathbb{N}_s$ and if $D \neq S_n$, then $K \in \mathcal{K}^n_{\beta(m,k)}$.

(c) Since h_n and w_n are continuous on G_n and by assumptions f_{n-1}^s, $s \in \{1,2\}$, are continuous on G_{n-1} the functions f_n^s are continuous on G_n.

(d) For $x \in G_n$ we have $s_{n-1}(x) + 2h_n(x) = g_{n-1}(x) + g_n(x) - g_{n-1}(x) = g_n(x)$, for $x \in A_{n-1}$ we have $s_{n-1}(x) + 2h_n(x) = f(x) + 0 = f(x)$ and for $x \in A_n \setminus A_{n-1}$ we have $s_{n-1}(x) + 2h_n(x) = g_{n-1}(x) + f(x) - g_{n-1}(x) = f(x)$. Therefore for each $x \in X$ we have $s_{n-1}(x) + 2h_n(x) = s_n(x)$. This yields

$$f_n^1 + f_n^2 = f_{n-1}^1 + f_{n-1}^2 + 2h_n = s_{n-1} + 2h_n = s_n.$$

(e) For each $x \in X$ and $s \in \{1,2\}$ we have

$$|f_n^s(x) - f_{n-1}^s(x)| = |h_n(x) + (-1)^s \cdot w_n(x)| \leq |h_n(x)| + |w_n(x)| < 2^{-n} + 2^{-n+1}.$$

This implies

$$\sup\{|f_n^s(x) - f_{n-1}^s(x)| : \; x \in X, s \in \{1,2\}\} \leq 2^{-n} + 2^{-n+1} < 2^{2-n}.$$

For $s \in \{1,2\}$ define a function $f_s(x) : \; X \to \mathbb{R}$ by $f_s(x) = \lim_{n\to\infty} f_n^s(x)$. By (e), f_s is the uniform limit of the sequence $\{f_n^s\}_n$. By (b) and (c), all functions

f_n^s are quasicontinuous and hence f_s as the uniform limit of quasicontinuous functions is quasicontinuous.

If $x \in A_k$ for some $k \in \mathbb{N}$, then $s_n(x) = f(x)$ for each $n \geq k$. If $x \notin \bigcup_n A_n$, then $x \in \bigcap_n G_n$ and $s_n(x) = g_n(x)$ for each $n \in \mathbb{N}$. Now the inequality $|f(x) - g_n(x)| < 2^{-n-1}$ implies $\lim_{n\to\infty} s_n(x) = f(x)$. Therefore by (d) for each $x \in X$ we have

$$f(x) = \lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} (f_n^1(x) + f_n^2(x)) = f_1(x) + f_2(x).$$

If f is continuous at x, then $x \in \bigcap_n G_n$. By (c) f_n^s are continuous at x for each $n \in \mathbb{N}$ and $s \in \{1, 2\}$ and therefore f_s as the uniform limit is also continuous at x. Thus $D(f_1) \cup D(f_2) \subset D(f)$.

All functions w_n are in the Baire class one and if f is bounded, then also w_1 can be bounded. If f is bounded (in the Baire class α), then also functions h_n are such. Then also f_n^s, $s \in \{1, 2\}$, are such and therefore also f_s as the uniform limit is bounded (in the Baire class α). $\square$

Remark 4.2 *There is a normal (of course, not T_1) second countable space X such that every quasicontinuous function defined on X is constant but there are nonconstant cliquish functions on X [2].*

Remark 4.3 *In [4] it is shown that if X is a Baire second countable T_3 space such that the family of all open sets is a π-base for X then every cliquish function $f\colon X \to \mathbb{R}$ is the sum of two simply continuous functions. Since every quasicontinuous function is simply continuous Theorem 4.1 implies that this is true for each pseudometrizable space. Therefore for a pseudometrizable space we have $\mathcal{P} \subset \mathcal{S} + \mathcal{S}$ and similarly $b\mathcal{P} \subset b\mathcal{S} + b\mathcal{S}, \mathcal{B}_\alpha\mathcal{P} \subset \mathcal{B}_\alpha\mathcal{S} + \mathcal{B}_\alpha\mathcal{S}$ and $b\mathcal{B}_\alpha\mathcal{P} \subset b\mathcal{B}_\alpha\mathcal{S} + b\mathcal{B}_\alpha\mathcal{S}$. If X is also Baire, then we have the equalities.*

References

[1] N. Biswas, *On some mappings in topological spaces,* Bull. Calcutta Math. Soc. **61** (1969), 127–135.

[2] J. Borsík, *Sums of quasicontinuous functions,* Math. Bohem. **118** (1993), 313–319.

[3] J. Borsík, *Algebraic structures generated by real quasicontinuous functions,* Tatra Mt. Math. Publ. (to appear).

[4] J. Borsík and J. Doboš, *A note on real cliquish functions,* Real Analysis Exch. **18** (1992–93), 139–145.

[5] Z. Grande, *Sur les fonctions cliquish*, Časopis Pěst. Mat. **110** (1985), 225–236.

[6] Z. Grande, *On some representations of a. e. continuous functions,* Real Analysis Exch. **21** (1995–96), 175–180.

[7] A. Maliszewski, *On the sums and the products of quasi-continuous functions,* Real Analysis Exch. **20** (1994–95), 418–421.

[8] A. Maliszewski, *Sums and products of quasi-continuous functions,* Real Analysis Exch. **21** (1995–96), 320–329.

[9] A. Maliszewski, *Darboux Property and Quasi-Continuity. A Uniform Approach,* WSP, Slupsk, 1996.

[10] T. Neubrunn, *Quasi-continuity,* Real Analysis Exch. **14** (1988–89), 259–306.

[11] E. Strońska, *L'espace linéaire des fonctions cliquées sur* $\mathbb{R}^n$ *est généré par les fonctions quasi-continues,* Math. Slovaca **39** (1989), 155-164.

[12] E. Strońska, *On the group generated by quasi continuous functions,* Real Analysis Exch. **17** (1991–92), 577-589.