Yôto Kubota, Department of Mathematical Science, Ibaraki University, Mito, Ibaraki, 310, Japan

NOTES ON ABSOLUTELY CONVERGENT INTEGRATION

Abstract

We show that there exists an absolutely convergent integral which properly includes the Lebesgue integral and that any such integral does not have positive property as a functional.

Let $I = [a, b]$ be a fixed compact interval on R. We consider only real valued measurable functions defined on I.

Definition 1. *Let T be a functional operation by which there corresponds to each closed interval $J \subset I$ a linear class $K(T, J)$ of measurable functions defined on J, and to each function f of $K(T, J)$ a real number $T(f, J)$. A functional operation T is called an absolutely convergent integral if the following conditions are fulfilled:*

(i) *The operation $T(f, J)$ is linear on $K(T, J)$.*

(ii) *If $f \in K(T, J)$, $J' \subset J$ then $f \in K(T, J')$.*

(iii) *If J_1 and J_2 are abutting intervals and if $f \in K(T, J_1) \cap K(T, J_2)$ then $f \in K(T, J_1 \cup J_2)$ and*

$$T(f, J_1 \cup J_2) = T(f, J_1) + T(f, J_2).$$

(iv) *If $f \in K(T, J)$, $J = [c, d]$ then $F(x) = T(f, [c, x])$ is continuous on J.*

(v) *If $f \in K(T, J)$ then $|f| \in K(T, J)$.*

Definition 2. *Given two absolutely convergent integrals T_1 and T_2, we shall say that T_2 includes T_1 if $f \in K(T_1, J)$ then $f \in K(T_2, J)$ and $T_1(f, J) = T_2(f, J)$ for any interval $J \subset I$, and write $T_1 \subset T_2$.*

The Lebesgue integral is an absolutely convergent integral. The following problem arises: Is there any absolutely convergent integral which properly includes the Lebesgue integral? We shall show that the answer to this question is "yes", however any such integral has no longer positive property as a functional.

Theorem 1. *There is an absolutely convergent integral which properly includes the Lebesgue integral.*

To prove the theorem, we need the following lemma.

Lemma 1. *There is a measurable function $\psi : I \to (0, \infty)$ such that $(L) \int_J \psi = +\infty$ for any subinterval J of I.*

The construction of such a function will be given later.

PROOF OF THEOREM 1. For $J = [c, d]$, let

$$K(T, J) = \{g : g = f + p\psi, f \in L(J), p \in R\},$$

where ψ is the function obtained in Lemma, and define

$$T(g, J) = (L) \int_J f + p(d - c).$$

Then $K(T, J)$ is a linear class and we can see easily that T satisfies conditions (i)~(iv). To show that if $f + p\psi \in K(T, J)$ then $|f + p\psi| \in K(T, J)$, it is sufficient to prove $|f + \psi| \in K(T, J)$ $(f \in L(J))$.

Let $A = \{x \in J : f(x) + \psi(x) \geq 0\}$ and $B = \{x \in J : f(x) + \psi(x) < 0\}$. Then the sets A and B are measurable. Let

$$g(x) = \begin{cases} f(x) & (x \in A), \\ -f(x) - 2\psi(x) & (x \in B). \end{cases}$$

We can write

$$g(x) = f(x)\chi_A(x) + (-f(x) - 2\psi(x))\chi_B(x),$$

and for $x \in B$, $0 < \psi(x) < -f(x)$. Consequently $\psi(x)\chi_B(x)$ is L-integrable on J and hence $g \in L(J)$. On the other hand, by the definition of g, we have $|f + \psi| = g + \psi$. It follows that $|f + \psi| \in K(T, J)$. Since $K(T, J)$ includes $L(J)$ strictly, the proof is complete. $\square$

PROOF OF LEMMA 1. For simplicity, we assume $[a, b] = [0, 1]$. Let $\{t_k\}$ be a sequence with $0 < t_k < \frac{1}{2}$.

We define the sequence of sets as follows:

$$I_1^1 = [0, t_1], \quad I_1^2 = [1 - t_1, 1]$$
$$E_1 = I_1^1 \cup I_1^2$$
$$I_2^1 = [0, t_1 t_2], I_2^2 = [t_1 - t_1 t_2, t_1], I_2^3 = [1 - t_1, 1 - t_1 + t_1 t_2], I_2^4 = [1 - t_1 t_2, 1]$$
$$E_2 = I_2^1 \cup I_2^2 \cup I_2^3 \cup I_2^4$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Repeating this process we can define I_k^j and $E_k = \cup_{j=1}^{2^k} I_k^j$. Then the set $E = \cap_{k=1}^{\infty} E_k$ is perfect and

$$m(I_k^j) = t_1 t_2 \cdots t_k, \qquad m(E) = \lim_{k \to \infty} 2^k t_1 t_2 \cdots t_k.$$

If we choose $t_k = \frac{1}{2} - \frac{1}{2(k+1)^2}$ $(k = 1, 2, \dots)$ then $m(E) = \frac{1}{2}$ by the formula $\Pi_{k=2}^{\infty} (1 - \frac{1}{k^2}) = \frac{1}{2}$. Change the notations E and I_k^j for E^1 and I_{1k}^j respectively, that is,

$$E^1 = \cap_{k=1}^{\infty} E_{1k}, \qquad E_{1k} = \cup_{j=1}^{2^k} I_{1k}^j.$$

Next we set $[0, 1] - E_{1k} = \cup_{j=1}^{2^{k-1}} I_{2k}^j$, where I_{2k}^j are complementary closed intervals of E_{1k}, and construct a perfect set E_{2k}^j on each I_{2k}^j such that $m(E_{2k}^j) = \frac{1}{2} \cdot m(I_{2k}^j)$. Putting $E^2 = \cup_k \cup_j E_{2k}^j$, we have

$$m(E^2) = m(E_{21}^1) + \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} m(E_{2k}^{2^k})$$
$$= \frac{1}{2} |I_{21}^1| + \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2} |I_{2k}^j|$$
$$= \frac{1}{2} (1 - 2t_1) + \sum_{k=1}^{\infty} 2^{k-1} t_1 t_2 \cdots t_k (1 - 2t_{k+1})$$
$$= \frac{1}{2^3} + \sum_{k=1}^{\infty} \frac{1}{2^2} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) = \frac{1}{2^2}.$$

Similarly we can define the set E^3 on $[0, 1] - E^1 \cup E^2$ such that $m(E^3) = \frac{1}{2^3}$ and so on. The sets E^n thus obtained are non-overlapping and $m(\cup_{n=1}^{\infty} E^n) = 1$, that is, $m([0, 1] - \cup_{n=1}^{\infty} E^n) = 0$.

Now we define the function $\psi : [0, 1] \to (0, \infty)$ as follows:

$$\psi(x) = \begin{cases} \dfrac{1}{t_1 t_2 \cdots t_k} & (x \in E^k), \\ 1 & (x \in [0, 1] - \cup_{n=1}^{\infty} E^n), \end{cases}$$

where $t_k = \frac{1}{2} - \frac{1}{2(k+1)^2}$. Since ψ is positive and measurable, it remains to show that $(L) \int_J \psi = +\infty$ for any subinterval J of $[0,1]$.

For any J, there is an interval $I_{nm}^j \subset J$. Since

$$
\begin{aligned}
\int_{I_{nm}^j} \psi &= \sum_{k=n}^{\infty} \int_{E^k \cap I_{nm}^j} \psi \\
&= \sum_{k=n}^{\infty} \frac{m(E^k \cap I_{nm}^j)}{t_1 t_2 \cdots t_k} > \sum_{k=n}^{\infty} \frac{1}{2^k} |I_{nm}^j| 2^k \\
&= \sum_{k=n}^{\infty} |I_{nm}^j| = +\infty,
\end{aligned}
$$

we have $\int_J \psi = +\infty$. $\qquad\square$

Theorem 2. *If T is any absolutely convergent integral which includes strictly the Lebesgue integral, then T is not positive.*

PROOF. Suppose that if $f \in K(T, J)$ and $f \geq 0$ then $T(f, J) \geq 0$.

For any $f \in K(T, J)$ with $f \geq 0$, we set for each n

$$
f_n(x) = \begin{cases} f(x), & \text{if } 0 \leq f(x) \leq n, \\ n, & \text{if } f(x) > n. \end{cases}
$$

Then $f_n \in L(J)$, $f_n \leq f_{n+1}$ for every n and $f_n \to f$ $(n \to \infty)$ on J. Since $L(J) \subset K(T, J)$ and T is positive, we have $L(f_n, J) \leq L(f_{n+1}, J) \leq T(f, J)$. By the monotone convergence theorem of the Lebesgue integral, we have $f \in L(J)$. Thus we have showed that if $f \in K(T, J)$ and $f \geq 0$ then $f \in L(J)$.

Let $f \in K(T, J)$. Then by (v), $|f| \in K(T, J)$, and hence $|f| \in L(J)$ that is, $f \in L(J)$ which implies $K(T, J) \subset L(J)$. This is a contradiction. $\qquad\square$

References

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