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A NEW VARIANT OF BLUMBERG'S THEOREM

Abstract

We prove that for every real function f defined on a separable, complete and dense in itself metric space X there exists a c-dense set $W \subset X$ such that $f \restriction W$ is super quasi-continuous.

Our terminology is standard. We shall consider only real-valued functions defined on topological spaces. No distinction is made between a function and its graph. Symbol $\mathrm{card}\,(X)$ will stand for the cardinality of a set X. The cardinality of $\mathbb{R}$ is denoted by 2^ω. For a cardinal number κ we will write $\mathrm{cf}(\kappa)$ for the cofinality of κ. For a metric space X, $x \in X$ and $\varepsilon > 0$ we denote by $B(x, \varepsilon)$ the open ball in X centered at x and with the radius ε. The set of all points at which a function $f\colon X \to \mathbb{R}$ is continuous (discontinuous) will be denoted by C_f (D_f). The class of all continuous functions defined on X will be denoted by $\mathrm{C}(X)$.

Recall also the following definitions (X is a topological space):

- $f\colon X \to \mathbb{R}$ is a *pointwise discontinuous* (shortly, $f \in \mathrm{PWD}(X)$) if the set C_f in dense in X;
- $f\colon X \to \mathbb{R}$ is *cliquish* (shortly, $f \in \mathrm{CLIQ}(X)$) if for each $x_0 \in X$, $\varepsilon > 0$ and a neighborhood W of x_0 there is a non-empty open set $W_0 \subset W$ such that $\mathrm{osc} f \restriction W_0 < \varepsilon$;
- $f\colon X \to \mathbb{R}$ is *quasi-continuous* (shortly, $f \in \mathrm{QC}(X)$) if for each $x_0 \in X$, $\varepsilon > 0$ and a neighborhood W of x_0 there is a non-empty open set $W_0 \subset W$ such that $|f(x_0) - f(x)| < \varepsilon$ for $x \in W_0$;

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- $f: X \to \mathbb{R}$ is *super quasi-continuous* (shortly, $f \in \mathrm{QC}^*(X)$) if $f \restriction C_f$ is dense in f.

The relationships between those classes are well-known. (See, e.g., [6].) In particular, for a topological space the following inclusions hold:

$$\begin{array}{ccccccc} & & & \nearrow & \mathrm{QC}(X) & \searrow & \\ \mathrm{C}(X) & \longrightarrow & \mathrm{QC}^*(X) & & & & \mathrm{CLIQ}(X) \\ & & & \searrow & \mathrm{PWD}(X) & \nearrow & \end{array}$$

Generally, all those inclusions are proper. Nevertheless, for a complete metric space X we have the following relations:

$$\mathrm{C}(X) \to \mathrm{QC}^*(X) = \mathrm{QC}(X) \to \mathrm{CLIQ}(X) = \mathrm{PWD}(X).$$

In 1922 H. Blumberg proved the following theorem:

Theorem 1 [1] *If X is a complete metric space, then for every $f: X \to \mathbb{R}$ there exists a dense set $D \subset X$ such that $f \restriction D \in \mathrm{C}(D)$.*

This theorem was extended in many directions and by many authors. (See [3] or [4] for the history of this study.) For example, it is known that the set D in Blumberg's construction is countable, and, generally, one cannot increase the size of this set. (See [9].) In 1971 J. Brown proved the following strengthened form of Blumberg's theorem.

Theorem 2 [2] *If X is a complete metric dense in itself space, then for every $f: X \to \mathbb{R}$ there exists a c-dense set $X_0 \subset X$ such that $f \restriction X_0 \in \mathrm{PWD}(X_0)$.*

Brown's theorem yields to the following result.

Corollary 1 *If X is a complete metric dense in itself space, then for every $f: X \to \mathbb{R}$ there exists a c-dense set $X_0 \subset X$ such that $f \restriction X_0$ is cliquish.*

Following Brown, in this note we improve his result in the class of separable metric spaces by showing that for every $f: X \to \mathbb{R}$ there exists a c-dense set $X_0 \subset X$ such that $f \restriction X_0 \in \mathrm{QC}^*(X_0)$.

Suppose that M is a subset of a metric space X and κ is a cardinal number. We say that M is a κ-Lusin set if M has no nowhere dense subsets of cardinality κ. Usually, ω_1-Lusin sets and 2^ω-Lusin sets are called Lusin sets and c-Lusin sets, respectively. It is well known that each Lusin set is of the second category. (See e.g., [7] or [5].) Every Lusin set is also $\mathfrak{c}$-Lusin. Moreover, if Continuum Hypothesis (CH) holds, then every $\mathfrak{c}$-Lusin set is also a Lusin set. However, it is consistent that these notions are not equivalent. Indeed, e.g., under Martin's Axiom (MA) and the failure of CH there are $\mathfrak{c}$-Lusins sets on $\mathbb{R}$ which are not Lusin [5]. Then, for each cardinal $\kappa \leq 2^\omega$ with $\mathrm{cf}(\kappa) > \omega$ there are κ-Lusin sets in $\mathbb{R}$ which are not Lusin. (Indeed, under MA every set of reals with cardinality less than 2^ω is meager [8], so it is enough to take a subset L_0 of L with $\mathrm{card}\,(L_0) = \kappa$.)

Moreover, recall some topological notions, which were introduced in [2].

- M is of a *first κ-type* iff M is the union of a first category set and a κ-Lusin set;
- M is of a *second κ-type* if M is not of a first κ-type;
- if G is an open subset of X, then the statement that M is *κ-typically dense* in G means that if T is a non-empty open subset of G, then $T \cap M$ is of a second κ-type;
- M is *κ-typically dense at a point* $x_0 \in X$ iff $M \cap U$ is of a second κ-type for every neighborhood U of x_0.
- M is *κ-typically dense in itself* iff M is κ-typically dense at every $x \in M$.[1]

Lemma 1 *Assume that κ is a cardinal number with uncountable cofinality. Then the family of all sets of a first κ-type forms a σ-ideal.* □

Lemma 2 *Assume that κ is a cardinal number such that $\omega < \mathrm{cf}(\kappa) \leq \kappa \leq 2^\omega$, X is a separable metric space which is κ-typically dense in itself, $N \subset X$ is κ-typically dense in X and $f\colon X \to \mathbb{R}$. Then there exists a κ-typically dense in X set $N_0 \subset N$ which satisfies the following condition:*

(∗) *for every open set $W \subset \mathbb{R}$ the set $N_0 \cap f^{-1}(W)$ is κ-typically dense in itself.*

Proof. Let $(B_n)_{n=1}^\infty$ and $(R_n)_{n=1}^\infty$ be countable bases of X and $\mathbb{R}$, respectively. For each positive integers n and k put $D_{n,k} = N \cap B_n \cap f^{-1}(R_k)$. Let D be the union of such $D_{n,k}$ that are of a first κ-type. By Lemma 1, D is also of a first κ-type. Set $N_0 = N \setminus D$. Then N_0 is κ-typically dense in X and it satisfies the condition (∗). □

[1] Note that the empty set is always κ-typically dense in itself.

Lemma 3 *Assume that $a < b$, κ is a cardinal number with $\omega < \mathrm{cf}(\kappa) \le \kappa \le 2^\omega$, f is a real valued function a domain of which is a κ-typically dense subset M of an open subset G of a separable metric space X and $f(x) \in (a,b)$ for each $x \in M$. Then there is a subset N of M such that N satisfies the condition $(*)$ (therefore N is κ-typically dense in G) and $f\restriction N$ is continuous at some element of N.*

Proof. In the same way as in the proof of Lemma 8 in [2] we can prove that there exists a subset N of M such that N is κ-typically dense in G and $f\restriction N$ is continuous at some $x_0 \in N$. By Lemma 2 we may assume that N satisfies the condition $(*)$. □

Lemma 4 [2, Lemma 1] *Assume that Φ is a property and every open subset of a metric space X has an open subset with property Φ. Then there exists a collection $\mathcal{G}$ of pairwise disjoint open subsets of X such that $\bigcup\mathcal{G}$ is dense in X and every set in $\mathcal{G}$ has property Φ.*

Theorem 3 *Assume that κ is a cardinal number such that $\omega < \mathrm{cf}(\kappa) \le \kappa \le 2^\omega$ and X is a separable metric space which is κ-typically dense in itself. Then for every function $f\colon X \to \mathbb{R}$ there exists a κ-dense subset W of X such that $f\restriction W$ is super quasi-continuous. Therefore, $f\restriction W$ is quasi-continuous.*

Proof. Let $\mathcal{R}$ be a countable base of $\mathbb{R}$. By Lemma 4, there exists a family $\mathcal{G}_1$ of pairwise disjoint open subsets of X such that $\bigcup\mathcal{G}_1$ is dense in X and for each $G \in \mathcal{G}_1$ there is an $R_G \in \mathcal{R}$ such that $\mathrm{diam}\,(R_G) < 1$ and $M_G = f^{-1}(R_G) \cap G$ is κ-typically dense in G. Now we define inductively an infinite sequence of steps such that each step involves four stages:

Step A1. Let $\mathcal{G}_1$ be the collection described above and for each $G \in \mathcal{G}_1$ let R_G and M_G be as described above.

Step B1. For each $G \in \mathcal{G}_1$, let N_G be a subset of M_G described in Lemma 3, which satisfies the condition $(*)$ from Lemma 2 and let $x_G \in N_G \cap C_{f\restriction N_G}$.

Step C1. For $G \in \mathcal{G}_1$, let H_G be a nowhere dense subset of N_G such that $x_G \in H_G$, $\mathrm{card}\,(H_G) \ge \kappa$ and let $\mathcal{K}_G$ be a collection of open balls such that

- **(i)** $\mathrm{diam}\,(B) < 1$ for each $B \in \mathcal{K}_G$;
- **(ii)** sets in $\mathcal{K}_G$ are pointwise disjoint;
- **(iii)** $\bigcup\mathcal{K}_G \subset G \setminus H_G$ and $\bigcup\mathcal{K}_G$ is dense in G;

(iv) for each $B \in \mathcal{K}_G$ there exists $R_B \in \mathcal{R}$ such that $R_B \subset R_G$, $\operatorname{diam}(R_B) < \frac{1}{2}$ and the set $B \cap N_G \cap f^{-1}(R_B)$ is κ-typically dense in B;

(v) for every $x \in H_G$ and for each open neighborhood $W \subset X \times \mathbb{R}$ of $(x, f(x))$ there exists a $B \in \mathcal{K}_G$ such that $B \times R_B \subset W$.

The construction of $\mathcal{K}_G$. Let $\mathcal{U}$ be a countable base of X and let $(U_n \times R_n)_n$ be a sequence of **all** products $U \times R$ where $U \in \mathcal{U}$, $R \in \mathcal{R}$ and $(f \restriction H_G) \cap (U \times R) \neq \emptyset$. Inductively choose a ball B_n such that $\operatorname{cl}(B_n) \subset U_n \setminus (\operatorname{cl}(H_G) \cup \bigcup_{m<n} \operatorname{cl}(B_m))$ and $f^{-1}(R_n) \cap B_n \cap N_G$ is κ-typically dense in B_n. (It is possible because $N_G \cap f^{-1}(R_n)$ is non-empty and, by $(*)$, κ-typically dense in itself, and $U_n \setminus (\operatorname{cl}(H_G) \cup \bigcup_{m<n} \operatorname{cl}(B_m))$ is an open neighborhood of some $x \in N_G \cap f^{-1}(R_n)$.)

Let $\mathcal{K}'_G = \{B_n : n \in \mathbb{N}\}$ and $R_B = R_n$ for $B = B_n$. Then the conditions (i)–(iv) are evident except the statement $\bigcup \mathcal{K}'_G$ is dense in G. By Lemma 4 this family can be extended to a family $\mathcal{K}_G$ what satisfies statements (i)–(iv). Now we shall verify (v). Fix $x \in H_G$ and an open set $W \subset X \times \mathbb{R}$ such that $(x, f(x)) \in W$. Then there exists $n \in \mathbb{N}$ such that $(x, f(x)) \in U_n \times R_n \subset W$. Thus $B_n \times R_n \subset W$. □

Step D1. For $G \in \mathcal{G}_1$ and for each $B \in \mathcal{K}_G$, put $M_B = N_G \cap B \cap f^{-1}(R_B)$.

Now, for each $n > 1$, steps An, Bn, Cn and Dn are defined as follows:

Step An. Let $\mathcal{G}_n = \bigcup \{\mathcal{K}_G : G \in \mathcal{G}_{n-1}\}$.

Step Bn. The same as step B1, except "$\mathcal{G}_n$" replaces "$\mathcal{G}_1$".

Step Cn. The same as step C1, except "$\mathcal{G}_n$" replaces "$\mathcal{G}_1$" and "$\frac{1}{n}$" replaces "1".

Step Dn. The same as step D1, except "$\mathcal{G}_n$" replaces "$\mathcal{G}_1$".

Now, set $W = \bigcup_{n=1}^{\infty} \bigcup_{G \in \mathcal{G}_n} H_G$ and $C = \{x_G : G \in \bigcup_{n=1}^{\infty} \mathcal{G}_n\}$. As in [2] we can observe that W is κ-dense in X and C is dense in X. Indeed, for $x_0 \in X$ and $\varepsilon > 0$ let n be a positive integer such that $\frac{1}{n} < \frac{\varepsilon}{3}$. Since $\bigcup \mathcal{G}_n$ is dense in X, there exists $G \in \mathcal{G}_n$ such that $G \cap B(x_0, \frac{1}{n}) \neq \emptyset$. Since $\operatorname{diam}(G) < \frac{1}{n}$, $G \subset B(x_0, \varepsilon)$ and H_G is a subset of $W \cap B(x_0, \varepsilon)$ with $\operatorname{card}(H_G) \geq \kappa$. Moreover, $x_G \in C \cap B(x_0, \varepsilon)$.

Now, suppose that $x \in C$. There exist $n \in \mathbb{N}$ and $G \in \mathcal{G}_n$ such that $x = x_G$. Then $W \cap G \subset N_G \cap G$ and $f \restriction N_G$ is continuous at x_G, so $f \restriction W$ is continuous at x.

To verify that $f\restriction C$ is dense in $f\restriction W$, fix $x_0 \in W$ and $\varepsilon > 0$. There exists $n \in \mathbb{N}$ and $G \in \mathcal{G}_n$ such that $x_0 \in H_G$. By the statement (v) of Step Cn, there is $B \in \mathcal{K}_G$ such that $B \subset B(x_0, \varepsilon)$ and $f(N_B) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Then $x_B \in C \cap B(x_0, \varepsilon)$ and $f(x_B) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, which completes the proof. □

Because any complete metric space which is dense in itself is 2^ω-typically dense in itself [2, Corollary] and $\mathrm{cf}(2^\omega) > \omega$, we have the following:

Corollary 2 *If X is a separable, complete, dense in itself metric space, then for every function $f\colon X \to \mathbb{R}$ there exists a c-dense set $W \subset X$ such that $f\restriction W$ is super quasi-continuous.*

Now we shall consider metric spaces for which Corollary 2 does not hold, even in a weaker form.

Lemma 5 *Assume that $\mathrm{cf}(\kappa) > \omega$, $L \subset X$ is a κ-Lusin set and $f\colon L \to \mathbb{R}$ is cliquish. Then the set D_f has cardinality less than κ.*

Proof. Recall that $D_f = \bigcup_{n=1}^{\infty} D_{f,n}$, where $D_{f,n} = \left\{x \in L\colon\ \mathrm{osc}f(x) \geq \frac{1}{n}\right\}$. Because every set $D_{f,n}$ is closed in L, so either $D_{f,n}$ is nowhere dense and $\mathrm{card}(D_{f,n}) < \kappa$ or $\mathrm{int}_L(D_{f,n}) \neq \emptyset$. The second case is impossible, because f is cliquish. Thus $\mathrm{card}(D_{f,n}) < \kappa$ for each n, so $\mathrm{card}\,(D_f) < \kappa$. □

Theorem 4 *Let X be a separable metric space. If X is not 2^ω-typically dense in itself, then there exists a function $f\colon X \to \mathbb{R}$ such that $f\restriction W$ is cliquish for no 2^ω-dense in X set W.*

Proof. We can assume that every open subset of X has cardinality at least 2^ω.

Let G be a non-empty open subset of X such that G is of first 2^ω-type. Then there are: a 2^ω-Lusin set L and a family of pairwise disjoint nowhere dense sets $\{M_i\}_i$ such that $L \cap \bigcup_i M_i = \emptyset$ and $G = L \cup \bigcup_i M_i$. As in the proof of Theorem 2 in [2] we consider two cases: if G is of the first category (or L has cardinality less than 2^ω), and if there exists an open subset T of G such that L is dense in T. In both those cases we define f in the same way as in [2]. In the last part of the proof of the second case we use Lemma 5 to observe that the supposition that f is cliquish implies the continuity of f on a set of size 2^ω, which is impossible. □

Corollary 3 *Assume that X is a separable dense in itself metric space. Then for $\kappa = 2^\omega$ the following conditions are equivalent:*

(i) *for each function* $f\colon X \to \mathbb{R}$ *there exists a* κ*-dense set* $W \subset X$ *such that* $f\restriction W \in \mathrm{QC}^*(W)$*;*

(ii) *for each function* $f\colon X \to \mathbb{R}$ *there exists a* κ*-dense set* $W \subset X$ *such that* $f\restriction W \in \mathrm{PWD}(W)$*;*

(iii) *for each function* $f\colon X \to \mathbb{R}$ *there exists a* κ*-dense set* $W \subset X$ *such that* $f\restriction W \in \mathrm{QC}(W)$*;*

(iv) *for each function* $f\colon X \to \mathbb{R}$ *there exists a* κ*-dense set* $W \subset X$ *such that* $f\restriction W \in \mathrm{CLIQ}(W)$*;*

(v) X *is* κ*-typically dense in itself.*

Questions.

1. Does there exist a metric space X and a cardinal κ for which the conditions (i)—(iv) are not equivalent?
2. Assume that X is a separable dense in itself metric space. Are the conditions (i)—(v) equivalent for $\kappa \in (\omega, 2^\omega)$?

Obviously, if CH is true then the notions of typically dense and c-typically dense are the same. So one can suppose that if X is κ-typically dense in itself for some uncountable cardinal κ then X is c-typically dense in itself. The next proposition shows that this hypothesis is not true.

Proposition 1 *Assume that MA is true and CH fails. Then there exists a subspace* $X \subset \mathbb{R}$ *that is* κ*-typically dense in itself for each* $\kappa < 2^\omega$ *with* $\mathrm{cf}(\kappa) > \omega$*, but not* 2^ω*-typically dense.*

Proof. Let X be a c-Lusin set that is c-dense in $\mathbb{R}$. (See, e.g., [5].) Then X is not 2^ω-typically dense. We shall verify that it is κ-typically dense for a fixed $\kappa < 2^\omega$ with uncountable cofinality. Suppose that there exists a non-empty open in X set G that is a first κ-type set in X. So, in X there are a κ-Lusin set L and a meager set A such that $G = L \cup A$. Then L is a c-Lusin set and A is meager in $\mathbb{R}$. Thus $\mathrm{card}\,(A) < 2^\omega$ and consequently, $\mathrm{card}\,(L) = 2^\omega$. Since X is c-dense in itself, every meager in $\mathbb{R}$ subset B of L is also meager in X. Therefore every set $B \subset L$ with $\mathrm{card}\,(B) = \kappa$ is meager in X (cf., [8]), contrary to the definition of κ-Lusin set. □

Corollary 4 *Assuming* MA + ¬CH *and* 2^ω *is not the successor of* κ *with* $\mathrm{cf}(\kappa) = \omega$*, there exists a subspace* $X \subset \mathbb{R}$ *such that:*

1. *for each* $\kappa < 2^\omega$ *and* $f\colon X \to \mathbb{R}$ *there exists a* κ*-dense in* X *set* W_κ *such that* $f\restriction W_\kappa \in \mathrm{QC}^*(W_\kappa)$*;*

2. *there is* $f\colon X \to \mathbb{R}$ *such that* $f\restriction W \in \mathrm{CLIQ}(W)$ *for no* $\mathfrak{c}$*-dense in* X *set* W*.*

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