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SUPERPOROSITY IN A CLASS OF NON-NORMABLE SPACES

Abstract

Let $\mathcal{M}$ stand for the space of all S-measurable real functions on the infitite σ-finite measure space (X, S, μ) endowed with the (metrizable but non-normable) topology of convergence in measure on sets of finite measure. Some natural subsets (including the L_p-spaces) are proved to be sigma-superporous in $\mathcal{M}$. The possibility of finding non-sigma-porous meager sets in this non-normable setting is discussed.

1 Introduction

The concept of a porous set was introduced by Dolženko in [2]. Since then it has been thoroughly investigated and diversely generalized (see [14] or [11] for a survey). It is possible to define several notions concerning porosity also in metric spaces (see [14],[11]). It is known that in Banach spaces the ideal of meager sets is strictly wider than that of the σ-porous sets ([14]). It is true also in closed non-locally compact convex subsets of a separable Banach space ([1]). Recently it has been established in dense in itself completely metrizable spaces as well (cf. [16]).

The primary goal of the research presented in this paper is in the line of the above results, i.e. to compare σ-porous and meager sets, respectively in some non-normable spaces. Such an attempt was made in [12] where the space s of all real sequences endowed with the Fréchet metric

$$\rho_F(\{a_n\}_n, \{b_n\}_n) = \sum_n 2^{-n} \frac{|a_n - b_n|}{1 + |a_n - b_n|} \text{ where } \{a_n\}_n, \{b_n\}_n \in s$$

was scrutinized in this respect. This space is non-normable ([8, Exercise 276]) and it was shown in [12] e.g. that the set $\{\{a_n\}_n \in s; \sum_n \Phi(a_n) \text{ converges }\}$ is

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σ-superporous in s for a residual family of functions Φ in the space of all real functions furnished with the uniform topology.

It is the purpose of this paper to carry on these investigations generalizing results of [12] for the space $\mathcal{M}$ of all measurable functions on an infinite σ-finite measure space (X, S, μ) endowed with the (metrizable) topology of convergence in measure on sets of finite measure (see [4]). We will show that results quite analogous to those of exposed in [12] for s hold in this generality as well. For instance, the set $A(\Phi) = \{f \in \mathcal{M}; \int_X^* |\Phi \circ f| \, d\mu^* < +\infty\}$ is σ-superporous in $\mathcal{M}$ for a broad class of functions $\Phi : \mathbb{R} \to \mathbb{R}$, where μ^* is the outer measure induced by μ and $\int_X^* h \, d\mu^*$ stands for the μ^*-upper integral of the function $h : X \to \mathbb{R}$ (see [3, Section 2.4]).

Further we show that $A(\chi_{\mathbb{R}\setminus M})$ is σ-superporous in $\mathcal{M}$ for every σ-very porous set $M \subset \mathbb{R}$ ($\chi_{\mathbb{R}\setminus M}$ is the characteristic function of $\mathbb{R} \setminus M$) and that $A(\chi_{\mathbb{R}\setminus M})$ is meager in $\mathcal{M}$ if M is meager at some point of $\mathbb{R}$. In particular, $A(\chi_{\mathbb{R}\setminus M})$ is meager in (s, ρ_F) if and only if M is meager at some point of $\mathbb{R}$. This could provide a method for relating meager non-σ-porous subsets of $\mathbb{R}$ to meager non-σ-porous subsets of $\mathcal{M}$ (resp. s) if the porosity of $A(\chi_{\mathbb{R}\setminus M})$ in $\mathcal{M}$ (resp. s) could be characterized in terms of $M \subset \mathbb{R}$.

It is worth noticing here that a more familiar metrization of $\mathcal{M}$ by the metric

$$m(f, g) = \inf\{\varepsilon > 0; \mu(\{x \in X; |f(x) - g(x)| \geq \varepsilon\}) < \varepsilon\} \ (f, g \in \mathcal{M})$$

which coincides with the topology of convergence in measure on X (cf. [3, Section 2.3.8]), yields a setting where our considerations are not feasible even for continuous Φ's. This question was studied in [17].

2 Preliminaries

In the sequel (X, S, μ) will be an infinite σ-finite measure space and μ^* the outer measure induced by μ. Without loss of generality we may suppose that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint, S-measurable sets such that $2 < \mu(X_n) < +\infty$ for each $n \in \mathbb{N}$.

Denote by $\mathcal{M}$ (resp. $\mathcal{M}_n$) the set of all S-measurable functions that are finite almost everywhere (abbr. a.e.) on X (on X_n). We will identify members of $\mathcal{M}$ provided they equal a.e. on X. If the sequence $f_k \in \mathcal{M}$ ($k \in \mathbb{N}$) converges in measure to $f \in \mathcal{M}$, write $f_k \xrightarrow{\mu} f$.

Denote by $\mathcal{F}_m$ the space of all functions $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi \circ f \in \mathcal{M}$ for all $f \in \mathcal{M}$. It is known that $\mathcal{F}_m$ contains the class of Borel-measurable functions. Observe that $\mathcal{F}_m$ is a closed subspace of the complete metric space

$(\mathcal{F}, d)$, where $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$ and

$$d(\Phi, \Psi) = \min\{1, \sup_{t \in \mathbb{R}} |\Phi(t) - \Psi(t)|\} \quad (\Phi, \Psi \in \mathcal{F}).$$

Indeed, if a sequence $\Phi_n \in \mathcal{F}_m$ $(n \in \mathbb{N})$ d-converges to $\Phi \in \mathcal{F}$, then $\Phi_n \circ f \in \mathcal{M}$ converges pointwise to $\Phi \circ f$ (for all $f \in \mathcal{M}$), thus $\Phi \circ f \in \mathcal{M}$ and consequently $\Phi \in \mathcal{F}_m$. It follows that $(\mathcal{F}_m, d)$ is a complete metric space.

For $\Phi \in \mathcal{F}$ and $p \in \mathbb{N}$ define

$$A(\Phi) = \left\{ f \in \mathcal{M}; \int_X^* |\Phi \circ f| \, d\mu^* < +\infty \right\} \text{ and}$$
$$A_p(\Phi) = \left\{ f \in \mathcal{M}; \int_X^* |\Phi \circ f| \, d\mu^* \leq p \right\},$$

where $\int_X^* f \, d\mu^*$ is the upper integral of f with respect to μ^* (see [3, Section 2.4]).

For $f, g \in \mathcal{M}$ and $n \in \mathbb{N}$ define

$$\rho_n(f, g) = \int_{X_n} \frac{|f - g|}{1 + |f - g|} \, d\mu$$
$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(X_n)} \rho_n(f, g).$$

For $i, j \in \mathbb{N}$ and $M \subset \mathbb{R}$ denote

$$\tilde{A}_{i,j}(M) = \left\{ f \in \mathcal{M}; \mu^*(f^{-1}(M) \cap X_i) \geq \frac{\mu(X_i)}{j} \right\},$$
$$A_{i,j}(M) = \left\{ f|_{X_i}; f \in \tilde{A}_{i,j}(M) \right\} \text{ and}$$
$$A_{i,0}(M) = \left\{ f \in \mathcal{M}_i; \mu^*(f^{-1}(M)) = \mu(X_i) \right\}.$$

It is not hard to see that ρ (resp. ρ_n) is a metric on $\mathcal{M}$ (resp. $\mathcal{M}_n$). It can be shown similarly as for (s, ρ_F) that $(\mathcal{M}, \rho)$ is non-normable (see [18]).

Convergence in measure implies ρ-convergence and the converse holds if and only if the underlying measure space is finite. More precisely we have:

Lemma 1. *Let $f_k, f \in \mathcal{M}$ $(k \in \mathbb{N})$. The following are equivalent:*

(i) $f_k \xrightarrow{\rho} f$*;*

(ii) $f_k \xrightarrow{\mu} f$ *on every S-measurable set of finite measure;*

(iii) $f_k|_{X_n} \xrightarrow{\rho} f|_{X_n}$ *for all* $n \in \mathbb{N}$.

PROOF. For (i)⇔(ii) see [4, Theorem 3]. The equivalence (i)⇔(iii) follows easily from [7, Theorem 14, p. 122]. □

Remark 1. *Observe that* $(\mathcal{M}_n, \rho_n)$ *is a complete metric space for each* $n \in \mathbb{N}$ *and the* ρ_n*-convergence of sequences from* $\mathcal{M}_n$ *coincides with their convergence in measure on* X_n *([5, Problem 42(4)]). Further the equivalence (i)⇔(iii) in the previous lemma actually yields that* $(\mathcal{M}, \rho)$ *and the Tychonoff product* $\Pi_n(\mathcal{M}_n, \rho_n)$ *are homeomorphic.*

Lemma 2. (cf. [4]) $(\mathcal{M}, \rho)$ *is a complete metric space.*

Denote by $B_d(y, r)$ the open ball about $y \in Y$ with radius $r > 0$ in the metric space (Y, d). By $B(x, r)$ we will denote the interval $(x - r, x + r)$, where $x \in \mathbb{R}$. For $E \subset Y, y \in Y$ and $r > 0$ define

$$\gamma(y, r, E) = \sup\{r' > 0; \exists y' \in Y \ B_d(y', r') \subset B_d(y, r) \setminus E\}.$$

We say that E is *porous (very porous)* at y if

$$\limsup_{r \to 0^+} \frac{\gamma(y, r, E)}{r} > 0 \quad (\liminf_{r \to 0^+} \frac{\gamma(y, r, E)}{r} > 0).$$

Further E is said to be *superporous* at $y \in Y$ (see [14], [15]), if $E \cup F$ is porous at y whenever $F \subset Y$ is porous at y.

A set $E \subset Y$ is said to be *globally very porous* if there exist constants $0 < a_E < 1$ and $r_0 > 0$ such that $\gamma(y, r, E) > a_E r$ for every $y \in E$ and $0 < r < r_0$ ([14]).

We say that E is superporous (very porous) if it is superporous (very porous) at each of its points, further E is σ-superporous (σ-very porous) if it is a countable union of superporous (very porous) sets. Superporosity was defined in [15] in connection with the $\mathcal{I}$-density topology of Wilczynski and others (cf. [13]).

Note that superporosity implies very porosity as observed in [15] (see [11, Corollary 8.15] as well) and σ-superporosity is equivalent to σ-very porosity which is further equivalent to σ-globally very porosity ([11, Corollary 8.17]).

We will denote by $\operatorname{card} Y$ and $\mathcal{P}(Y)$ the cardinality and the power set, respectively of the set Y, further c will stand for the power of the continuum. Denote by $|I|$ the length of the interval $I \subset \mathbb{R}$.

3 Main Results

Lemma 3. *Let $\{I_q; q \in \mathbb{N}\}$ be an enumeration of the open intervals with rational endpoints. Let $\Phi_{pq} = p\chi_{I_q}$ for $p, q \in \mathbb{N}$. Then $A_p(\Phi_{pq})$ is superporous in $(\mathcal{M}, \rho)$ for every $p, q \in \mathbb{N}$.*

Proof. Choose $p, q \in \mathbb{N}$ and denote by t_q the midpoint of I_q. Let $f \in A_p(\Phi_{pq})$. Suppose that $F \subset \mathcal{M}$ is an arbitrary set porous at f. Then there exist sequences $r_n, r'_n > 0$ $(n \in \mathbb{N})$ and $\alpha > 0$ such that $\alpha r_n < r'_n < r_n < 2^{-n}$, further we get an $f_n \in \mathcal{M}$ such that

$$B_\rho(f_n, r'_n) \subset B_\rho(f, r_n) \setminus F. \tag{1}$$

Define $p_n = \min\{k \in \mathbb{N}; 2^{-k} < r'_n\} + 1$ and $\varepsilon_n = 2^{-p_n+1}$ for all $n \in \mathbb{N}$. Then we have

$$r'_n > \varepsilon_n \geq \frac{r'_n}{2}. \tag{2}$$

Denote $E_{n1} = X_{p_n} \cap f_0^{-1}((t_q - \frac{1}{8}|I_q|, t_q + \frac{1}{8}|I_q|))$ and $E_{n2} = X_{p_n} \setminus E_{n1}$ and define $g_n = f_n\chi_{X \setminus X_{p_n}} + t_q\chi_{E_{n2}} + (t_q + \frac{1}{4}|I_q|)\chi_{E_{n1}} \in \mathcal{M}$. It is clear that

$$|f_n(x) - g_n(x)| \geq \frac{1}{8}|I_q| \quad \text{for all } x \in X_{p_n}. \tag{3}$$

Since $\rho(f_n, g_n) = \frac{1}{2^{p_n}\mu(X_{p_n})} \int_{X_{p_n}} \frac{|f_n - g_n|}{1+|f_n - g_n|}\, d\mu$, by the definition of ε_n, X_{p_n} and (3), respectively we get

$$\rho(f_n, g_n) < \frac{\varepsilon_n}{2}, \tag{4}$$

$$\rho(f_n, g_n) > \frac{|I_q|}{8 + |I_q|} \cdot \frac{\varepsilon_n}{2}. \tag{4'}$$

Put $\delta_n = \frac{|I_q|}{16+2|I_q|}\rho(f_n, g_n)$ and pick an arbitrary $h_n \in B_\rho(g_n, \delta_n)$. Define

$$D_n = \left\{x \in X_{p_n}; |h_n(x) - g_n(x)| < \frac{4\delta_n}{\varepsilon_n - 4\delta_n}\right\} \text{ and } D_{n0} = X_{p_n} \setminus D_n.$$

Observe that D_n is well-defined, since by (4) $\delta_n = \frac{|I_q|}{8+|I_q|} \cdot \frac{\rho(f_n,g_n)}{2} < \frac{\varepsilon_n}{4}$.

Then we have

$$\begin{aligned}\delta_n > \rho(h_n, g_n) &\geq \frac{1}{2^{p_n}\mu(X_{p_n})} \int_{D_{n0}} \frac{|h_n - g_n|}{1 + |h_n - g_n|}\, d\mu \\ &\geq \frac{\varepsilon_n}{2\mu(X_{p_n})} \int_{D_{n0}} \frac{4\delta_n}{\varepsilon_n}\, d\mu = \frac{2\delta_n\mu(D_{n0})}{\mu(X_{p_n})},\end{aligned}$$

thus $\mu(D_{n0}) < \frac{1}{2}\mu(X_{p_n})$ hence $\mu(D_n) \geq \frac{1}{2}\mu(X_{p_n}) > 1$.

In view of (4) we get $|h_n(x) - g_n(x)| < \frac{4\delta_n}{\varepsilon_n - 4\delta_n} < \frac{1}{8}|I_q|$ for every $x \in D_n$, so $h_n(D_n) \subset (t_q - \frac{3}{8}|I_q|, t_q + \frac{3}{8}|I_q|)$ (see the definition of g_n). Then $\int_X^* |\Phi_{pq} \circ h_n|\,d\mu^* \geq \int_{D_n}^* |\Phi_{pq} \circ h_n|\,d\mu^* \geq p\mu(D_n) > p$, so

$$h_n \in \mathcal{M} \setminus A_p(\Phi_{pq}). \tag{5}$$

Using (4) we get $\varepsilon_n - \rho(f_n, g_n) > \frac{\varepsilon_n}{2} > \frac{\varepsilon_n}{2} \cdot \frac{|I_q|}{8+|I_q|} > \delta_n$, therefore $B_\rho(g_n, \delta_n) \subset B_\rho(f_n, \varepsilon_n) \subset B_\rho(f_n, r'_n)$. Then in virtue of (5) and (1) there holds

$$B_\rho(g_n, \delta_n) \subset B_\rho(f_n, r'_n) \setminus A_p(\Phi_{pq}) \subset B_\rho(f, r_n) \setminus (F \cup A_p(\Phi_{pq})).$$

From (4') and (2) we get

$$\begin{aligned}\gamma(f, r_n, F \cup A_p(\Phi_{pq})) \geq \delta_n &\geq \left(\frac{|I_q|}{8+|I_q|}\right)^2 \frac{\varepsilon_n}{2} \geq \left(\frac{|I_q|}{8+|I_q|}\right)^2 \frac{r'_n}{4} \\ &> \left(\frac{|I_q|}{8+|I_q|}\right)^2 \frac{\alpha}{4} r_n,\end{aligned}$$

thus $\limsup\limits_{r\to 0^+} \frac{\gamma(f,r,F\cup A_p(\Phi_{pq}))}{r} \geq (\frac{|I_q|}{8+|I_q|})^2 \frac{\alpha}{4} > 0$, which proves the porosity of $F \cup A_p(\Phi_{pq})$ at f. □

Theorem 1. *Let $\Phi \in \mathcal{F}$ be a function for which there exists $t_0 \in \mathbb{R} \cup \{\pm\infty\}$ such that*

$$\liminf_{t \to t_0} |\Phi(t)| > 0. \tag{6}$$

Then $A(\Phi)$ is σ-superporous in $(\mathcal{M}, \rho)$.

Proof. In view of (6) there exists $\beta > 0$ and a bounded open interval I such that

$$|\Phi(t)| \geq \beta \text{ for all } t \in I. \tag{7}$$

Let $\{J_k; k \in \mathbb{N}\}$ be a partition of I consisting of open intervals. Choose an $f \in A(\Phi)$. Then by (7) we have

$$\beta \sum_{k\in\mathbb{N}} \mu(f^{-1}(J_k)) = \beta\mu(f^{-1}(I)) \leq \int_X^* |\Phi \circ f|\,d\mu^* < p$$

for some $p \in \mathbb{N}$. Thus $\mu(f^{-1}(J_k)) \leq 1$ for some $k \in \mathbb{N}$ and hence $\mu(f^{-1}(I_q)) \leq 1$ for some open interval $I_q \subset J_k$ with rational endpoints. Consequently,

$$\int_X^* |\Phi_{pq} \circ f|\,d\mu^* = p\mu(f^{-1}(I_q)) \leq p,$$

so $f \in A_p(\Phi_{pq})$, whence $A(\Phi) \subset \bigcup_{p,q\in\mathbb{N}} A_p(\Phi_{pq})$, which concludes the proof by Lemma 3. $\square$

As the following results show, there are also functions Φ, not necessarily satisfying (6), for which $A(\Phi)$ is still σ-superporous (cf. Theorem 2):

Lemma 4. *Let $M \subset \mathbb{R}$ be a globally very porous set. Then $\tilde{A}_{i,j}(M)$ is superporous in $(\mathcal{M}, \rho)$ for each $i, j \in \mathbb{N}$.*

PROOF. According to the assumption on M there exist $0 < a_M < 1$ and $r_0 > 0$ such that

$$\gamma(x, r, M) > a_M r \quad \text{for all} \quad x \in M \cup (\mathbb{R} \setminus \overline{M}) \text{ and all } 0 < r < r_0. \tag{8}$$

Choose $f \in \tilde{A}_{ij}(M)$ and a set $F \subset \mathcal{M}$ which is porous at f. Then there exist $\alpha > 0$, sequences $r_n, r'_n > 0$ and $f_n \in \mathcal{M}$ such that $r_n \searrow 0, \alpha r_n < r'_n < r_n < 2^{-i+1} \cdot \frac{3r_0}{1+r_0}$ and

$$B(f_n, r'_n) \subset B(f, r_n) \setminus F. \tag{9}$$

It is not hard to find $b_{nk} \in \mathbb{R}$ ($1 \le k \le m_n$, where $m_n \in \mathbb{N}$) and a partition $\{D_{nk}; 1 \le k \le m_n\}$ of X_i such that for $g_{n0} = f_n \chi_{X\setminus X_i} + \sum_{k=1}^{m_n} b_{nk}\chi_{D_{nk}} \in \mathcal{M}$ there holds

$$\rho(f_n, g_{n0}) < \frac{r'_n}{4}. \tag{10}$$

We can actually choose $b_{nk} \in M \cup (\mathbb{R} \setminus \overline{M})$ for every $1 \le k \le m_n$.

Put $\eta_n = \frac{2^i r'_n}{6-2^i r'_n}$. Then $\eta_n < r_0$, so it follows from (8) that for each $1 \le k \le m_n$ there exists $b'_{nk} \in \mathbb{R}$ and $r_{nk} > 0$ such that

$$a_M \eta_n \le r_{nk} < \eta_n \text{ and } B(b'_{nk}, r_{nk}) \subset B(b_{nk}, \eta_n) \setminus M. \tag{11}$$

Define $g_n = g_{n0}\chi_{X\setminus X_i} + \sum_{k=1}^{m_n} b'_{nk}\chi_{D_{nk}} \in \mathcal{M}$. Then by (11) we have

$$\begin{aligned}
\rho(g_{n0}, g_n) &\le \frac{1}{2^i \mu(X_i)} \sum_{k=1}^{m_n} \Big(\int_{D_{nk}} |b_{nk} - b'_{nk}|\, d\mu\Big) \\
&= \frac{1}{2^i \mu(X_i)} \sum_{k=1}^{m_n} |b_{nk} - b'_{nk}| \mu(D_{nk}) \le \frac{1}{2^i \mu(X_i)} \eta_n \sum_{k=1}^{m_n} \mu(D_{nk}) \\
&= \frac{r'_n}{6 - 2^i r'_n} \le \frac{r'_n}{4},
\end{aligned}$$

thus in view of (10)

$$\rho(f_n, g_n) \le \rho(f_n, g_{n0}) + \rho(g_{n0}, g_n) \le \frac{r'_n}{2}. \tag{12}$$

We have $0 < a_M < 1 < 3j$, thus $\frac{r'_n}{2} > \frac{a_M r'_n}{6j}$. Then putting $\delta_n = \frac{a_M r'_n}{6j}$ we get by (12) that $r'_n - \rho(f_n, g_n) \geq \frac{r'_n}{2} > \delta_n$, so

$$B_\rho(g_n, \delta_n) \subset B_\rho(f_n, r'_n). \tag{13}$$

Choose $h \in \tilde{A}_{i,j}(M)$ arbitrarily. According to (11) we have

$$\begin{aligned}\rho(h, g_n) &\geq \frac{1}{2^i \mu(X_i)} \int^*_{h^{-1}(M)\cap X_i} \frac{|h - g_n|}{1 + |h - g_n|}\, d\mu^* \\ &\geq \frac{1}{2^i \mu(X_i)} \mu^*(h^{-1}(M) \cap X_i) \cdot \frac{\min\limits_{1\leq k\leq m_n} r_{nk}}{1 + \min\limits_{1\leq k\leq m_n} r_{nk}} \\ &\geq \frac{1}{2^i \mu(X_i)} \cdot \frac{\mu(X_i)}{j} \cdot \frac{a_M \eta_n}{1 + a_M \eta_n} > \frac{1}{2^i j} \cdot \frac{a_M \eta_n}{1 + \eta_n} = \delta_n.\end{aligned}$$

It means by (13) that $B_\rho(g_n, \delta_n) \subset B_\rho(f_n, r'_n) \setminus \tilde{A}_{i,j}(M)$. Then in virtue of (9) we get $B_\rho(g_n, \delta_n) \subset B_\rho(f, r_n) \setminus (F \cup \tilde{A}_{i,j}(M))$. Consequently

$$\gamma(f, r_n, F\cup \to \widetilde{A}_{i,j}(M)) \geq \delta_n > \frac{a_M \alpha r_n}{6j},$$

which justifies the porosity of $F \cup \tilde{A}_{i,j}(M)$ at f. □

Theorem 2. *Let M be a σ-very porous set. Then $A(\chi_{\mathbb{R}\setminus M})$ is σ-superporous in $(\mathcal{M}, \rho)$.*

PROOF. We may already suppose that $M = \bigcup_{k=1}^\infty M_k$, where M_k is globally very porous and $a_{M_k} < 1$ for all $k \in \mathbb{N}$.

Choose $f \in A(\chi_{\mathbb{R}\setminus M})$. Then we have

$$\begin{aligned}+\infty = \mu(X) - \int^*_X |\chi_{\mathbb{R}\setminus M} \circ f|\, d\mu^* &= \mu(X) - \mu^*(f^{-1}(\mathbb{R} \setminus M)) \\ \leq \mu^*(f^{-1}(M)) &\leq \sum_{i,k\in\mathbb{N}} \mu^*(f^{-1}(M_k) \cap X_i),\end{aligned}$$

thus $\mu^*(f^{-1}(M_k)\cap X_i) > 0$ for some $i, k \in \mathbb{N}$. It suffices now to pick $j \in \mathbb{N}$ such that $\mu^*(f^{-1}(M_k) \cap X_i) \geq \frac{\mu(X_i)}{j}$. Then clearly $f \in \tilde{A}_{i,j}(M_k)$, consequently

$$A(\chi_{\mathbb{R}\setminus M}) \subset \bigcup_{i,j,k\in\mathbb{N}} \tilde{A}_{i,j}(M_k),$$

which concludes the proof by Lemma 4. □

Now we turn to characterizing the meagerness of $A(\chi_{\mathbb{R}\setminus M})$ in $(\mathcal{M}, \rho)$ in terms of properties of M. We will need the following

Lemma 5. *If M is meager at some point of $\mathbb{R}$, then $A_{i,j}(M)$ is meager at some point of $(\mathcal{M}_i, \rho_i)$ for all $i, j \in \mathbb{N}$.*

Proof. In the sequel we will use that the topology induced by ρ_i on $\mathcal{M}_i$ is equivalent with the topology of convergence in measure on X_i, i.e. with the topology induced by the metric $m_i = m|_{\mathcal{M}_i \times \mathcal{M}_i}$ (see [5, Problem 42(4)]).

Suppose that there exists an interval $U = B(t_0, r)$ ($t_0 \in \mathbb{R}, r > 0$) such that $U \cap M = \bigcup_{k=1}^{\infty} M_k$ for some nowhere dense sets $M_k \subset \mathbb{R}$ ($k \in \mathbb{N}$). Without loss of generality we may assume that $M_k \subset M_{k+1}$ for all $k \in \mathbb{N}$. Let $f_0 \equiv t_0$ on X_i and put $V = B_{m_i}(f_0, r)$.

We will show that $V \cap A_{i,j}(M_k)$ is nowhere dense in $(\mathcal{M}_i, m_i)$: take an open ball $B_{m_i}(f, \varepsilon)$ in $\mathcal{M}_i$. We may already suppose that $f \in V$ and f equals a simple function $\sum_{s=1}^{m} b_s \chi_{D_s}$ where $b_1, \ldots, b_m \in U$ and $D_1, \ldots, D_m$ is a measurable partition of X_i.

Then the nowhere density of M_k in $\mathbb{R}$ yields some $b_s' \in \mathbb{R}$ and $0 < \varepsilon_0 < \frac{\mu(X_i)}{j}$ such that

$$B(b_s', \varepsilon_0) \subset B(b_s, \varepsilon) \setminus M_k \text{ for any } 1 \leq s \leq m. \tag{14}$$

Choose $g \in B_{m_i}(f_1, \varepsilon_0)$ where $f_1 = \sum_{s=1}^{m} b_s' \chi_{D_s}$. Then by (14)

$$g^{-1}(M_k) \subset \{x \in X_i; |f_1(x) - g(x)| \geq \varepsilon_0\}.$$

Therefore $\mu^*(g^{-1}(M_k)) \leq \varepsilon_0 < \frac{\mu(X_i)}{j}$, so $g \notin A_{i,j}(M_k)$. On the other hand $f_1 \in B_{m_i}(f, \varepsilon)$; thus,

$$\emptyset \neq B_{m_i}(f, \varepsilon) \cap B_{m_i}(f_1, \varepsilon_0) \subset B_{m_i}(f, \varepsilon) \setminus A_{i,j}(M_k),$$

which justifies the nowhere density of $V \cap A_{i,j}(M_k)$ in $\mathcal{M}_i$.

Finally, denote $V_0 = B_{m_i}(f_0, r_0)$ where $r_0 = \min\{r, \frac{1}{j}\}$. Pick $h \in A_{i,j}(M) \cap V_0$. Then $h^{-1}(M \setminus U) \subset \{x \in X_i; |h(x) - f_0(x)| \geq r_0\}$, so $\mu^*(h^{-1}(M \setminus U)) \leq r_0 \leq \frac{1}{j} < \frac{\mu(X_i)}{2j}$. Furthermore in view of the regularity of μ^* we get (cf. [3, Section 2.1.5(1)])

$$\begin{aligned} \frac{\mu(X_i)}{j} &\leq \mu^*(h^{-1}(M)) \leq \mu^*(h^{-1}(M \cap U)) + \mu^*(h^{-1}(M \setminus U) \\ &< \lim_{k\to\infty} \mu^*(h^{-1}(M_k)) + \frac{\mu(X_i)}{2j}, \end{aligned}$$

hence $\lim_{k\to\infty} \mu^*(h^{-1}(M_k)) > \frac{\mu(X_i)}{2j}$, so $h \in A_{i,2j}(M_k) \cap V_0 \subset A_{i,2j}(M_k) \cap V$ for some $k \in \mathbb{N}$. Therefore

$$A_{i,j}(M) \cap V_0 \subset \bigcup_{k=1}^{\infty} A_{i,2j}(M_k) \cap V$$

which means that $A_{i,j}(M)$ is meager at f_0 in $\mathcal{M}_i$. $\square$

Theorem 3. *If M is meager at some point of $\mathbb{R}$, then $A(\chi_{\mathbb{R}\setminus M})$ is meager in $(\mathcal{M}, \rho)$.*

PROOF. Let $t_0 \in \mathbb{R}$ and $r > 0$ be such that $B(t_0, r) \cap M$ is meager in $\mathbb{R}$. Let $V_i = B_{m_i}(f_0, r_0)$ where $f_0 \equiv t_0$ on X and $0 < r_0 = \min\{r, \frac{1}{2}\}$. Then by Lemma 5 $A_{i,2}(M) \cap V_i$ is meager in $(\mathcal{M}_i, \rho_i)$ for all $i \in \mathbb{N}$.

Choose $f \in A(\chi_{\mathbb{R}\setminus M})$. Then $\mu^*(f^{-1}(\mathbb{R}\setminus M)) < +\infty$ and by the regularity of μ^* there exists a μ^*-hull B of $f^{-1}(\mathbb{R}\setminus M)$ (see [3, Section 2.1.4]). Consequently, $\mu(B \cap X_i) = \mu^*(f^{-1}(\mathbb{R} \setminus M) \cap X_i) = \mu^*(X_i \setminus (X_i \cap f^{-1}(M)))$; thus,

$$+\infty > \mu^*(f^{-1}(\mathbb{R}\setminus M)) = \mu(B) = \sum_{i=1}^{\infty} \mu(B \cap X_i) = \sum_{i=1}^{\infty} \mu^*(X_i \setminus (X_i \cap f^{-1}(M))).$$

Then for all $i \geq m$ ($m \in \mathbb{N}$) we have

$$\frac{\mu(X_i)}{2} > 1 > \mu^*(X_i \setminus (X_i \cap f^{-1}(M))) \geq \mu(X_i) - \mu^*(X_i \cap f^{-1}(M)),$$

hence $f|_{X_i} \in A_{i,2}(M)$ for all $i \geq m$. Accordingly,

$$A(\chi_{\mathbb{R}\setminus M}) \subset \bigcup_{m=1}^{\infty} P_m \text{ where } P_m = \Pi_{i=1}^{m-1} \mathcal{M}_i \times \Pi_{i=m}^{\infty} A_{i,2}(M) \text{ for each } m \in \mathbb{N}.$$

It suffices now to show by Remark 1 that P_m is meager in $P = \Pi_{i=1}^{\infty} \mathcal{M}_i$ for every $m \in \mathbb{N}$: Let $\mathbf{U} = \Pi_{i=1}^{n} U_i \times \Pi_{n+1}^{\infty} \mathcal{M}_i$ be any basic open set of the product topology on P such that $n \geq m$. Denote by $\mathbf{V}$ the open set $\Pi_{i=1}^{n} U_i \times V_{n+1} \times \Pi_{n+2}^{\infty} \mathcal{M}_i \subset P$. Then $\mathbf{V} \subset \mathbf{U}$ and $\mathbf{V} \cap P_m \subset \Pi_{i=1}^{n} U_i \times (V_{n+1} \cap A_{n+1,2}(M)) \times \Pi_{i=n+2}^{\infty} A_{i,2}(M)$, which is meager in P. It means by Theorem 1.7. in [6] that P_m is meager in P. $\square$

Corollary 1. *$A(\chi_{\mathbb{R}\setminus M})$ is meager in (s, ρ_F) if and only if M is meager at some point of $\mathbb{R}$.*

PROOF. The sufficiency follows from the previous theorem by putting $X = \mathbb{N}$, $S = \mathcal{P}(\mathbb{N})$ and the counting measure on $\mathbb{N}$ for μ.

Conversely, suppose that M is non-meager everywhere in $\mathbb{R}$. Then M with the relative topology is a dense Baire subspace of $\mathbb{R}$. Then the product $E = M^{\mathbb{N}}$ is a Baire space which is clearly dense in s ([6, Lemma 5.6]). Therefore E is non-meager in s and hence $A(\chi_{\mathbb{R}\setminus M}) \supset E$ is non-meager in s. $\square$

Remark 2. *In connection with the Corollary a question arises if a similar characterization of $A(\chi_{\mathbb{R}\setminus M})$ is possible also in $\mathcal{M}$. Mimicking the above proof and using Remark* 1 *it would be sufficient to prove that non-meagerness of M everywhere in $\mathbb{R}$ implies non-meagerness of $A_{i,0}$ everywhere in $\mathcal{M}_i$ for each $i \in \mathbb{N}$, further that $\mathcal{M}_i$ is separable for each $i \in \mathbb{N}$. This last condition is needed for the theorem on product of Baire spaces* ([6, Lemma 5.6].*), thus we may consider the question only for separable measure spaces (X, S, μ) (see* [5, §41]*).*

It is not hard to show that this is really the case if each X_i is a finite disjoint sum of atoms, however in general the answer is not known to me.

Remark 3. *Another question here arises in connection with finding necessary conditions for σ-porosity of $A(\chi_{\mathbb{R}\setminus M})$ in $\mathcal{M}$ (or at least in s). If we want to use some argument similar to that of in the Corollary, we would need some "porosity-Baire" product theorem as the mentioned result of Oxtoby* ([9], [6])*. This ultimately breaks down to proving a porosity version of the well-known Kuratowski-Ulam theorem on sections of nowhere dense subsets of the product space* ([10, Theorem 15.1]*). More precisely, the questions are as follows:*

(i) If X and Y are separable metric spaces and E is a porous subset of $X \times Y$ with (say) the box metric, then are the x-sections E_x of E porous in Y except for a σ-porous set in X?

(ii) Call a metric space Z p-Baire if every nonempty open subset of Z is non-σ-porous. Is the property of being separably p-Baire (countably) productive?

The preceding theorems provide sufficient background for investigating the class

$$\mathcal{U} = \{\Phi \in \mathcal{F};\ A(\Phi) \text{ is } \sigma\text{-superporous in } (\mathcal{M}, \rho)\}.$$

Theorem 4. *We have*

(i) $\operatorname{card}(\mathcal{U} \cap \mathcal{F}_m) = \operatorname{card}\mathcal{U} = 2^c$

(ii) $\operatorname{card}(\mathcal{F} \setminus \mathcal{U}) = 2^c$ *for* (s, ρ_F).

Proof. (i) Every subset of the Cantor's ternary set C is very porous therefore in view of Theorem 2 $\chi_{\mathbb{R}\setminus E} \in \mathcal{U} \cap \mathcal{F}_m$ for every $E \subset C$, further $\chi_{\mathbb{R}\setminus E} \neq \chi_{\mathbb{R}\setminus E'}$ provided $E \neq E'$. Consequently $\operatorname{card}(\mathcal{U} \cap \mathcal{F}_m) \geq \operatorname{card}\mathcal{P}(C) = 2^c$. Further clearly $\operatorname{card}\mathcal{U} \leq \operatorname{card}\mathcal{F} \leq \operatorname{card}(\mathbb{R}^{\mathbb{R}}) = 2^c$.

(ii) If we restrict ourselves to (s, ρ_F) only, then $\chi_E \notin \mathcal{U}$ for each $E \subset C$ since $A(\chi_E) = s \setminus A(\chi_{\mathbb{R}\setminus E})$ and (s, ρ_F) is a nonmeager space by Lemma 2. Thus again $2^c = \operatorname{card}\mathcal{P}(C) \leq \operatorname{card}(\mathcal{F} \setminus \mathcal{U}) \leq \operatorname{card}\mathcal{F} \leq 2^c$. □

Further we have

Theorem 5. *$\mathcal{U}$ is residual in $\mathcal{F}$.*

Proof. See [12, Lemma 2] and our Theorem 1. □

Remark 4. *It is worth noticing that if we restrict our investigations onto $\mathcal{F}_m$ only, then similar results hold. Actually, Lemma 3–4 and Theorem 1–2 hold without change, we need only to replace μ^* by μ and the upper integral by integral, respectively in the proofs.*

We can also prove the analogue of Tóth's Theorem (Theorem 5) for $\mathcal{F}_m$:

Theorem 5' *$\mathcal{U} \cap \mathcal{F}_m$ is residual in $(\mathcal{F}_m, d)$.*

Proof. See Lemma 2 in [12]. The only difference is in proving the density of $\mathcal{U}_0 = \{\Phi \in \mathcal{F}_m;\ \Phi$ satisfies (6) for some $t_0 \in \overline{\mathbb{R}}\}$ in $(\mathcal{F}_m, d)$, more precisely in proving that $\Psi = \Phi\chi_M + \frac{\varepsilon}{4}\chi_{\mathbb{R}\setminus M} \in \mathcal{F}_m$, where $\Phi \in \mathcal{F}_m, \varepsilon > 0$ and $M = \{t \in \mathbb{R};$ either $t \notin (0,1)$ or $t \in (0,1)$ and $|\Phi(t)| \geq \frac{\varepsilon}{4}\}$.

To show this pick $f \in \mathcal{M}, c \in \mathbb{R}$ arbitrarily and observe that

$$(\Psi \circ f)^{-1}([c, +\infty)) = \begin{cases} \Phi \circ f)^{-1}([c, +\infty)), & \text{if } c > \frac{\varepsilon}{4} \\ (\Phi \circ f)^{-1}([c, +\infty)) \cup (f^{-1}((0,1)) & \\ \cap(\Phi \circ f)^{-1}((-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}))), & \text{if } c \leq \frac{\varepsilon}{4} \end{cases}$$

thus $\Psi \circ f \in \mathcal{M}$. □

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