RESEARCH

Real Analysis Exchange
Vol. 22(2), 1996-97, pp. 760–765

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ON FUNCTIONS OF TWO VARIABLES EQUICONTINUOUS IN ONE VARIABLE

Abstract

The continuity of some functions of two variables equicontinuous in one variable is considered.

Let $\mathbb{R}$ be the set of all reals and let E denote $\mathbb{R}$ or $\mathbb{R}\times\mathbb{R}$. For $x \in E$ and for a positive real r let $K(x,r)$ denote the open ball with center x and radius r, i.e. $K(x,r) = \{t \in X : |t-x| < r\}$. Moreover, let μ_e (μ) be the outer Lebesgue measure (the Lebesgue measure) in E.

Denote by

$$d_u(A,x) = \limsup_{h\to 0^+} \mu_e(A\cap K(x,h))/\mu(K(x,h))$$

$$(d_l(A,x) = \liminf_{h\to 0^+} \mu_e(A\cap K(x,h))/\mu(K(x,h)))$$

the upper (lower) outer density of a set $A \subset E$ at a point x. A point $x \in E$ is called a density point of a set $A \subset E$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$. The family $\mathcal{T}_d = \{A \subset E; A \text{ is measurable and every point } x \in A \text{ is a density point of } A\}$ is a topology called the density topology [1, 2, 3].

Moreover, let $\mathcal{T}_e$ denote the Euclidean topology in E.

Some examples of functions $f : E \to \mathbb{R}$ having continuous sections $f_x(t) = f(x,t)$ and $f^y(t) = f(t,y)$, $t \in \mathbb{R}$, whose sets of discontinuity points are of positive measure are well known [5]. On the other hand, if all sections f_x of a function $f : E \to \mathbb{R}$ are equicontinuous at a point y (i.e. for every positive real η there is a positive real δ such that for every point v with $|v-y| < \delta$

Key Words: $\mathcal{I}$-almost everywhere continuity, equicontinuity, separate continuity, density topology, functions of two variables
Mathematical Reviews subject classification: 26A15, 26B05, 54C08, 54C30
Received by the editors December 6, 1996

and for every real x we obtain $|f(x,v) - f(x,y)| < \eta$) and if the section f^y is continuous at a point u, then f is continuous (as a function from $(E, \mathcal{T}_e)$ to $(\mathbb{R}, \mathcal{T}_e)$) at the point (u, y). From this we obtain immediately the following remarks.

Remark 1. *Suppose that all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at each point. Then f is continuous at a point (u, v) if and only if the section f^v is continuous at u.*

Remark 2. *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals of subsets $\mathbb{R}$ and $\mathbb{R}^2$ respectively, such that every F_σ set $A \subset \mathbb{R}^2$ having all sections $A^y = \{x; (x,y) \in A\}$, $y \in \mathbb{R}$, belonging to $\mathcal{I}$ is in $\mathcal{J}$. If all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at each point and if all sections f^y, $y \in \mathbb{R}$, are $\mathcal{I}$-almost everywhere continuous (i.e. the sets $D(f^y)$ of all discontinuity points of f^y belong to $\mathcal{I}$), then the function f is $\mathcal{J}$-almost everywhere continuous.*

As particular cases of the last remark we obtain the following.

Corollary 1. *If all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at each point and if all sections f^y, $y \in \mathbb{R}$, are such that $\mu(D(f^y)) = 0$ (all $D(f^y)$ are of the first category) then $\mu(D(f)) = 0$ ($D(f)$ is of the first category).*

We will show some stronger theorems.

Theorem 1. *(a) Let $\mathcal{J}$ and $\mathcal{I}$ be some σ-ideals of subsets of $\mathbb{R}^2$ and of $\mathbb{R}$ respectively such that the vertical and horizontal projections of sets which are in $2^{\mathbb{R}^2} \setminus \mathcal{J}$ do not belong to $\mathcal{I}$. Suppose that there is a set $A \in \mathcal{I}$ such that all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at each point $y \in \mathbb{R} \setminus A$ and for every point y the set $D(f^y)$ of all discontinuity points of the section f^y is in $\mathcal{I}$. Then the set $D(f)$ of all discontinuity points of f belongs to $\mathcal{J}$.*

(b) Let $\mathcal{I}$ and $\mathcal{J}$ be some σ-ideals of subsets of $\mathbb{R}$ and of $\mathbb{R}^2$ respectively such that the vertical projections of sets which are in $2^{\mathbb{R}^2} \setminus \mathcal{J}$ do not belong to $\mathcal{I}$. Suppose that all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at every point and that all sections f^y, $y \in \mathbb{R}$, are $\mathcal{I}$-almost everywhere continuous (i.e. the sets $D(f^y)$ belong to $\mathcal{I}$). Then the set $D(f) \in \mathcal{J}$.

PROOF. Suppose, to the contrary, that the set $D(f)$ of all discontinuity points of f is not in $\mathcal{J}$. Consequently, there is a positive real η such that the set

$$B = \{(x,y); \operatorname{osc} f(x,y) \geq \eta\}$$

is not in $\mathcal{J}$. Since all section f_x, $x \in \mathbb{R}$, are equicontinuous at each point $y \in \mathbb{R} \setminus A$, for every point $(x, y) \in B_1 = B \setminus (\mathbb{R} \times A)$ there is an open interval $I(x, y)$ with rational endpoints such that $y \in I(x, y)$ and $|f(x, u) - f(x, y)| < \eta/4$ for all $u \in I(x, y)$ and $x \in \mathbb{R}$. The set B_1 is not in $\mathcal{J}$; so there is an open interval I such that $B_2 = \{(x, y) \in B_1; I(x, y) = I\}$ is not in $\mathcal{J}$. Fix a point $(x, y) \in B_2$ and consider the section f^y. For every point $(t, u) \in B_2$ we have $|f(t, u) - f(t, y)| < \eta/4$. But $B_2 \subset B$, so for each point $(t, u) \in B_2$ there is a sequence of points $(v_n(t, u), w_n(t, u))$ such that $w_n(t, u) \in I$,

$$|f(v_n(t, u), w_n(t, u)) - f(t, u)| > 3\eta/4$$

for each positive integer n and

$$\lim_{n\to\infty} (v_n(t, u), w_n(t, u)) = (t, u).$$

For each positive integer n and for each $(t, u) \in B_2$ we obtain

$$|f(t, y) - f(v_n(t, u), y)| \geq |f(t, u) - f(v_n(t, u), w_n(t, u))| -$$

$$|f(t, u) - f(t, y)| - |f(v_n(t, u), w_n(t, u)) - f(v_n(t, u), y)| > 3\eta/4 - \eta/4 - \eta/4 = \eta/4.$$

Since $\lim_{n\to\infty} v_n(t, u) = t$, the section f^y is not continuous at any point of the set $F = \{t :$ there is u such that $(t, u) \in B_2\}$ which is not in $\mathcal{I}$. So, the set $D(f^y)$ of discontinuity points of the section f^y is not in $\mathcal{I}$, a contradiction. This contradiction finishes the proof of (a). The proof of the part (b) is analogous. □

Corollary 2. *If we suppose that $\mathcal{I}$ is the family of all subsets of $\mathbb{R}$ of measure zero (of the first category) [which are countable] and that $\mathcal{J}$ is the family of all subsets of $\mathbb{R}^2$ whose vertical projections belong to $\mathcal{I}$, then there is a set $A \in \mathcal{I}$ such that the set $D(f)$ of all discontinuity points of the function f considered in Theorem 1(b) is contained in $A \times \mathbb{R}$.*

Theorem 1 is not true for ideals. For example, if I is the ideal of all finite subsets of $\mathbb{R}$ and if $\mathcal{J}$ is the ideal of all subsets $\mathbb{R}^2$ whose vertical projections are finite, then there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ having equicontinuous sections f_x, $x \in \mathbb{R}$, and such that for every $y \in \mathbb{R}$ the set $D(f^y)$ is finite and $D(f)$ is not in $\mathcal{J}$. For a construction of such a function we denote by $\mathbb{N}$ the set of positive integers, by $E(y)$ the greatest integer which is $\leq y$ and define

$$f(x, y) = \begin{cases} \inf\{|y - v|; v \in \mathbb{N}\} & \text{if } x \leq E(y) \\ 0 & \text{otherwise on } \mathbb{R}^2. \end{cases}$$

We will prove that Theorem 1(b) is true for the ideals of nowhere dense subsets of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. In the proof of the next theorem we will apply the following lemma.

Lemma 1. *Suppose that the sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at each point. If the function f is not continuous at a point (u, v) then there is an open interval I containing v such that f is not continuous at any point (u, t), with $t \in I$.*

PROOF. Since f is not continuous at the point (u, v), there is a positive real η such that $\operatorname{osc} f(u, v) \geq \eta$. From the equicontinuity of the sections f_x, $x \in \mathbb{R}$, at the point v it follows that there is an open interval I containing v such that $|f(x, t) - f(x, v)| < \eta/8$ for all $t \in I$ and $x \in \mathbb{R}$. There is a sequence of points (u_n, v_n) such that $\lim_n (u_n, v_n) = (u, v)$ and $|f(u_n, v_n) - f(u, v)| > \eta/2$ for $n = 1, 2, \ldots$. Since the section f_u is continuous, we can assume that $u_n \neq u$ and $v_n \in I$ for $n = 1, 2, \ldots$. Observe that for each point $t \in I$ and for all $n = 1, 2, \ldots$ we have

$$\begin{aligned} |f(u_n, v_n) - f(u, t)| &\geq |f(u_n, v_n) - f(u, v)| - |f(u, v) - f(u, t)| \\ &> \eta/2 - \eta/8 = 3\eta/8, \\ |f(u_n, v_n) - f(u_n, t)| &< |f(u_n, v_n) - f(u_n, v)| + |f(u_n, t) - f(u_n, v)| \\ &< \eta/8 + \eta/8 = \eta/4 \end{aligned}$$

and

$$\begin{aligned} |f(u_n, t) - f(u, t)| &\geq |f(u_n, v_n) - f(u, t)| - |f(u_n, v_n) - f(u_n, t)| \\ &> 3\eta/8 - \eta/4 = \eta/8. \end{aligned}$$

Since $\lim_n u_n = u$, we obtain that $\operatorname{osc} f(u, t) \geq \eta/8$ and f is not continuous at the point (u, t). □

Theorem 2. *If all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous at each point and if for every $y \in \mathbb{R}$ the set $D(f^y)$ of all discontinuity points of the section f^y is nowhere dense, then the set $D(f)$ is nowhere dense.*

PROOF. Suppose, by way of contradiction, that the set $D(f)$ is dense in an open rectangle $I \times J$, where I, J are open intervals. Enumerate all open intervals with rational endpoints contained in I in a sequence $I_1, \ldots, I_n, \ldots$. Denote by $D(f)$ the set of all discontinuity points of f. Let $(u_1, v_1) \in I_1 \times J$ be a discontinuity point of f. By Lemma 1 there is a closed interval $J_1 \subset J$ such that every point (u_1, t) with $t \in J_1$ belongs to $D(f)$. Next, by induction in the n^{th} step $(n > 1)$ we find a point $(u_n, v_n) \in (I_n \times \operatorname{int}(J_{n-1})) \cap D(f)$ (int denotes the interior operation) and a closed interval $J_n \subset \text{int}(J_{n-1})$ such that for every point $t \in J_n$ the point (u_n, t) is a discontinuity point of f. There is a point $w \in \bigcap_n J_n$. Since the section f^w is not continuous at any point u_n, $n = 1, 2, \ldots$, and the set $\{u_n; n \geq 1\}$ is dense in the open interval I, we obtain a contradiction. So, the set $D(f)$ is nowhere dense. □

Denote by $\mathcal{I}_G$ and by $\mathcal{J}_G$ the ideals of all subsets of $\mathbb{R}$ and of $\mathbb{R}^2$ respectively, which are nowhere dense in every set belonging to $\mathcal{T}_d$.

Problem 1. Is Theorem 1(b) true for the ideals $\mathcal{I}_G$ and $\mathcal{J}_G$?

Theorem 3. *Suppose that all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous and for every $y \in \mathbb{R}$ the set $D(f^y) \in \mathcal{I}_G$. Then for all nonempty sets $K, L \in \mathcal{T}_d$ the set $D(f) \cap (K \times L)$ is nowhere dense in $K \times L$.*

PROOF. We can repeat the proof of Theorem 2. Suppose, by way of contradiction that there are linear nonempty sets $K, L \in \mathcal{T}_d$ such that the set $D(f) \cap (K \times L)$ is dense in $K \times L$. Let $I_1, \ldots, I_n, \ldots$ be an enumeration of all open intervals with rational endpoints for which $I_n \cap K \neq \emptyset$, $n = 1, 2, \ldots$. Let $(u_1, v_1) \in (I_1 \cap K) \times L$ be a discontinuity point of the function f. By Lemma 1 there is an open interval J_1 containing v_1 such that every point (u_1, t), where $t \in J_1$, belongs to $D(f)$. Next, in the n^{th} step $(n > 1)$ we find a point $(u_n, v_n) \in (I_n \cap K) \times (J_{n-1} \cap L)$ belonging to $D(f)$ and a closed interval $J_n \subset \text{int}(J_{n-1})$ containing v_n such that every point (u_n, t), where $t \in J_n$, belongs to $D(f)$. Let $w \in \bigcap_n J_n$. Then the section f^w is discontinuous at each point of the set $\{u_n, n = 1, 2, \ldots\}$, which is dense in $K \in \mathcal{T}_d$. So the set $D(f^w)$ is not in $\mathcal{I}_G$. □

A function $f : E \to \mathbb{R}$ has property $\mathcal{A}$ at a point x ($f \in \mathcal{A}(x)$) ([4]) if for every positive η and for every set $U \in \mathcal{T}_d$ such that $x \in U$ there is a nonempty open set V such that $V \cap U \neq \emptyset$, $D(f) \cap U \cap V = \emptyset$ and $|f(t) - f(x)| < \eta$ for all points $t \in U \cap V$.

Evidently, if $f \in \mathcal{A}(x)$ for all $x \in E(= \mathbb{R}$ or $\mathbb{R}^2)$, then $D(f) \in \mathcal{I}_G$ or resp. $D(f) \in \mathcal{J}_G$.

Theorem 4. *Suppose that all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous and that for every point $(x, y) \in \mathbb{R}^2$ the relation $f^y \in \mathcal{A}(x)$. Then for every positive real η, for every point (u, v) and for all nonempty linear sets $K, L \in \mathcal{T}_d$ with $(x, y) \in K \times L$ there is an open set U such that $D(f) \cap (K \times L) \cap U \neq \emptyset$, $(K \times L) \cap U = \emptyset$ and $|f(s, t) - f(u, v)| < \eta$ for each point $(s, t) \in (K \times L) \cap U$.*

PROOF. Fix a positive real η, a point (u, v) and sets $K, L \in \mathcal{T}_d$ with $(u, v) \in K \times L$. Since $f^v \in \mathcal{A}(u)$, there is an open interval I such that

$$I \cap K \neq \emptyset,\ I \cap K \cap D(f^v) = \emptyset \text{ and } |f(s, v) - f(u, v)| < \eta/2$$

for all points $s \in I \cap K$. From the equicontinuity of the sections f_x, $x \in \mathbb{R}$, it follows that there is an open interval J containing v such that

$$|f(x,t) - f(x,v)| < \eta/2$$

for all points $x \in \mathbb{R}$ and $t \in J$. Evidently, $J \cap L \neq \emptyset$ and consequently,

$$(I \cap K) \times (J \cap L) = (I \times J) \cap (K \times L) \neq \emptyset.$$

For all points $(s,t) \in (I \cap K) \times (J \cap L)$ we obtain

$$|f(s,t) - f(u,v)| \leq |f(s,t) - f(s,v)| + |f(s,v) - f(u,v)| < \eta/2 + \eta/2 = \eta.$$

Since the sets $I \cap K$ and $J \cap L$ belong to $\mathcal{T}_d$, by Theorem 3 there is an open set U such that

$$U \cap ((I \cap K) \times (J \cap L)) \neq \emptyset$$

and

$$D(f) \cap U \cap ((I \cap K) \times (J \cap L)) = \emptyset. \qquad \square$$

Problem 2. Suppose that all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are equicontinuous and that $f^y \in \mathcal{A}(x)$ for each point $(x,y) \in \mathbb{R}^2$. Is it true that $f \in \mathcal{A}(x,y)$ for each point $(x,y) \in \mathbb{R}^2$?

References

[1] A. M. Bruckner, *Differentiation of Integrals*, Amer. Math. Monthly, **78** Part II (1971), 1–51.

[2] A. M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math. 659 (1978), Springer–Verlag.

[3] C. Goffman, C. J. Neugebauer and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–506.

[4] Z. Grande, *On strong quasi-continuity of functions of two variables*, Real Analysis Exch. **21** (1995–96), 236–243.

[5] R. Kershner, *The continuity of functions of many variables*, Trans. Amer. Math. Soc. **53** (1943), 83–100.

[6] T. Neubrunn, *Quasi–continuity*, Real Analysis Exch. **14** (1988–89), 259–306.