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Richard G. Gibson, Department of Mathematics, Columbus State University, Columbus, Georgia, 31907, USA, email: gibson_richard@colstate.edu
Tomasz Natkaniec, Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland, email: mattn@ksinet.univ.gda.pl

DARBOUX LIKE FUNCTIONS

Contents

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1 Historical background

A function $f : R \to R$ is said to have the *intermediate value property* provided that if p and q are real numbers such that $p \neq q$ and $f(p) < f(q)$, then for every $y \in (f(p), f(q))$ there exists a number x between p and q with $f(x) = y$. In 1875, G. Darboux showed that there exist functions with the intermediate value property that are not continuous [37]. Because of his work with functions having the intermediate value property, these functions are called Darboux functions.

In 1907, J. Young [122] studied real-valued functions defined on an interval with the following property: for every $x \in R$ there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \nearrow x$, $y_n \searrow x$, and both $f(x_n)$ and $f(y_n)$ converge to $f(x)$. In [122], J. Young showed that for Baire class 1 functions, Darboux functions and functions having this property of Young are equivalent. In more general spaces, functions having the property of Young are said to be peripherally continuous. (See [59, 60, 121].)

K. Kuratowski and W. Sierpiński, in 1922, showed that for real-valued Baire class 1 functions defined on an interval, Darboux functions and functions with a connected graph are equivalent [79].

I. Maximoff, in 1936, showed that for real-valued Baire class 1 functions defined on an interval, Darboux functions and functions with a perfect road are equivalent [86].

J. Stallings, in 1959, defined almost continuous functions in the sense of Stallings and extendable functions [114].

In [14], J. Brown showed that for real-valued Baire class 1 functions defined on an interval, Darboux functions and almost continuous functions in the sense of Stallings are equivalent. In [16], J. Brown, P. Humke, and M. Laczkovich showed that for real-valued Baire class 1 functions defined on an interval, Darboux functions and extendable functions are equivalent.

From the preceding we see that for Baire class 1 functions $f : R \to R$, Darboux functions have been characterized in several ways. For further information concerning these functions see the first three chapters of the book [17] by A. M. Bruckner.

2 Basic definitions

Our terminology is standard. We consider only real-valued functions of one real variable. No distinction is made between a function and its graph. By R and $\mathbb{I}$ we denote the set of all reals and the interval $[0,1]$, respectively. The family of all subsets of a set X is denoted by $\mathcal{P}(X)$. The family of all functions from a set X into Y is denoted by Y^X. By C and Const we denote the families of all continuous functions and all constant functions. The symbol $|X|$ stands for the cardinality of a set X. The cardinality of R is denoted by $\mathfrak{c}$. For the cardinal number κ we write $[X]^\kappa$ to denote the family of all subsets Y of X with $|Y| = \kappa$. In particular, $[X]^1$ stands for the family of all singletons in X and $[X]^2$ for the family of all doubletons in X. By a Cantor set we mean any non-empty perfect nowhere dense subset of R. Moreover, we say that a set $A \subset R$ is Cantor dense in a set $X \subset R$, if $A \cap J$ contains a Cantor set whenever J is a non-empty open interval J with $J \cap X \neq \emptyset$. By (a,b) we denote an open interval with end-points a and b, i.e., the set of all $x \in R$ such that $\min\{a,b\} < x < \max\{a,b\}$.

The following is a list of the definitions of the different types of functions that will be investigated. Note that we have abbreviated these classes of functions with letters on the left.

Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then:

D – f is a *Darboux function* if $f(C)$ is connected whenever C is connected in X;

PC – f is *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset $W \subset U$ such that $x \in W$ and $f(\mathrm{bd}\,(W)) \subset V$, where $\mathrm{bd}\,(W)$ denotes the boundary of W;

Conn – f is a *connectivity function* if the graph of f restricted to C, denoted by $f{\restriction}C$, is connected in $X \times Y$ whenever $C \subset X$ is connected;

ACS – f is an *almost continuous function* in the sense of Stallings, if U is an open subset of $X \times Y$ containing the graph of f, then U contains the graph of a continuous function $g : X \to Y$ [114];

Ext – f is an *extendable function* if there exists a connectivity function $g : X \times \mathbb{I} \to Y$ such that $f(x) = g(x, 0)$ for all $x \in X$ [114].

The class of all Darboux functions from X to Y we shall denote by $\mathrm{D}(X, Y)$, or shortly, by D, when X and Y will be clear from a context (usually, $X = Y = R$). Similarly, for other Darboux like classes.

The next definitions concern only real-valued functions defined on R (or, on subspaces of R). Then:

PR – f has a *perfect road* if for every $x \in R$, there exists a perfect set P having x as a bilateral limit point such that $f \restriction P$ is continuous at x [86];

WCIVP – *Weak Cantor Intermediate Value Property*: $f \in \mathrm{WCIVP}$ if for all $p, q \in R$ with $p < q$ and $f(p) \neq f(q)$, there exists a Cantor set $C \subset (p, q)$ such that $f(C)$ is between $f(p)$ and $f(q)$ [48];

CIVP – *Cantor Intermediate Value Property*: $f \in \mathrm{CIVP}$ if for all $p, q \in R$ with $p \neq q$ and $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$ [47];

SCIVP – *Strong Cantor Intermediate Value Property*: $f \in \mathrm{SCIVP}$ if for all $p, q \in R$ with $p \neq q$ and $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$ and $f \restriction C$ is continuous [102];

PB – *Property B*: $f \in \mathrm{PB}$ if for all pairs of open intervals I and J, if $I \cap f^{-1}(J)$ is uncountable, then $I \cap f^{-1}(J)$ contains a non-empty perfect set [44].

Theorem 2.1. *If $f : R \to R$, then*

- *$f \in \mathrm{D}$ if and only if f has the intermediate value property;*
- *f is a connectivity function if and only if the entire graph of f is connected;*
- *f is peripherally continuous if and only if it satisfies the Young condition:*

  for every $x \in R$ there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \nearrow x$, $y_n \searrow x$, and both $f(x_n)$ and $f(y_n)$ converge to $f(x)$.

- Conn $\subset$ D $\subset$ PC *and* PR $\subset$ PC

The preceding is a brief survey of some of the known facts concerning the families of functions defined.

3 Darboux like functions in the class R^R

Stallings defined almost continuous functions in the sense of Stallings (ACS) in [114] and proved the implications

$$\text{Ext} \subset \text{ACS} \quad \text{and} \quad \text{ACS} \subset \text{Conn}.$$

Thus in the class of all functions R^R, we have the following implications.

$$\begin{tikzcd}
\text{C} \arrow[r] & \text{Ext} \arrow[r] \arrow[drr] & \text{ACS} \arrow[r] & \text{Conn} \arrow[r] & \text{D} \arrow[r] & \text{PC} \\
 & & & \text{PR} \arrow[urr] & &
\end{tikzcd}$$

Chart 1

Note that all inclusions in Chart 1, denoted by $\to$, are proper. (See [16].)

The following contains a survey of research that has been done concerning these properties. In this investigation a series of questions were asked by the first of the Authors in a lecture given at the Banach Center in November of 1989 [43]. We will now give an update on the status of those questions. The first question:

Question 3.1. *What is both a necessary and sufficient condition for an almost continuous function in the sense of Stallings to be an extendable function?*

is one on which the first Author has spent a great amount of time. This question also led to most of the others questions.

F. Roush and R. Gibson in their investigation of the characterization of extendable functions defined the properties WCIVP, CIVP, and SCIVP. (See [45, 47, 48, 102].) These properties were given the following names due to their similarity to the intermediate value property. Hence they are Darboux like functions.

In reference to the question

"Does ACS $\subset$ Ext?"

asked by Stallings in [114], Gibson and Roush in [48], defined the weak Cantor intermediate value property(WCIVP), and proved the following results.

Theorem 3.2. (Gibson, Roush [48]) *In the class* $\mathbb{I}^{\mathbb{I}}$ *we have*

- *if* $f \in \mathrm{Ext}$*, then* $f \in \mathrm{WCIVP}$*,*
- *there exists a function* $f : \mathbb{I} \to \mathbb{I}$ *such that* $f \in \mathrm{ACS}$ *and* $f \notin \mathrm{WCIVP}$*.*

Hence in the class of all real functions defined on subintervals of R we have

Corollary 3.3. $\mathrm{Ext} \subset (\mathrm{ACS} \cap \mathrm{WCIVP})$ *and* $\mathrm{ACS} \neq \mathrm{Ext}$.

Theorem 3.4. (Gibson, Roush [51]) *In the class* $\mathbb{I}^{\mathbb{I}}$ *we have*

- *if* $f \in \mathrm{Ext}$ *then* $f \in \mathrm{PR}$*;*
- *there exists a function* $f : \mathbb{I} \to \mathbb{I}$ *such that* $f \in \mathrm{ACS}$ *but* $f \notin \mathrm{PR}$*.*

Hence in the class of all real functions defined on subintervals of R we have

Corollary 3.5. $\mathrm{Ext} \subset \mathrm{ACS} \cap \mathrm{PR}$.

In [51], Gibson and Roush posed the following question:

> "Does there exists a function $f \colon \mathbb{I} \to \mathbb{I}$ such that $f \in \mathrm{ACS} \cap \mathrm{PR}$ but $f \notin \mathrm{Ext}$?".

Rosen, Gibson, and Roush in [102] gave an affirmative answer to this question by proving the following theorem

Theorem 3.6. (Rosen, Gibson, Roush [102]) *In the class* $\mathbb{I}^{\mathbb{I}}$ *we have*

- *if* $f \in \mathrm{Ext}$*, then* $f \in \mathrm{SCIVP}$*;*
- *there is a function* $f \in (\mathrm{ACS} \cap \mathrm{PR}) \setminus \mathrm{CIVP}$*.*

Since $\mathrm{SCIVP} \subset \mathrm{CIVP}$, hence in the class of all real functions defined on subintervals of R we have

Corollary 3.7. $\mathrm{Ext} \neq (\mathrm{ACS} \cap \mathrm{PR})$.

It was stated in [51] that $\mathrm{CIVP} \subset \mathrm{PR}$ but it was not proved. The proof of that statement now follows.

Theorem 3.8. *If* $f \colon R \to R$ *is a function and* $f \in \mathrm{CIVP}$*, then* $f \in \mathrm{PR}$*.*

Proof. Select any $x \in R$. Assume that there exists $\varepsilon > 0$ such that f is constant on no subinterval of $[x-\varepsilon, x]$ having x as a right endpoint.

Let x_n be a increasing sequence in $(x-\varepsilon, x)$ such that $x_n \to x$, $f(x_n) \neq f(x)$ and $f(x_n) \to f(x)$. Select any Cantor set K_n between $f(x)$ and $f(x_n)$ such that K_n is a subset of $(f(x)-1/n, f(x)+1/n)$. Since $f \in \text{CIVP}$, there exists a Cantor set C_n between x_n and x such that $f(C_n)$ is a subset of K_n. Let A be the union of all C_n and $\{x\}$. Then A is a perfect set and $f\restriction A$ is continuous at x (from the left). In a similar way we can construct a perfect set B such that $f\restriction B$ is continuous at x (from the right).

If f is constant on $[x-\varepsilon, x]$ or $[x, x+\varepsilon]$ for some $\varepsilon > 0$, let $A = [x-\varepsilon, x]$ or $B = [x, x+\varepsilon]$. Now if $P = A \cup B$, then P is a perfect set with x as a bilateral limit point and $f\restriction P$ is continuous at x. $\square$

For real-valued functions $f : R \to R$ we have only the following implications among the classes of functions defined above. This is an expansion of the previous diagram. (See the papers by Brown, Humke and Laczkovich, [16]; Rosen, Gibson and Roush [102]; and Banaszewski and Natkaniec, [8].)

$$\begin{tikzcd}
 & & \text{ACS} \arrow[r] & \text{Conn} \arrow[r] & \text{D} \arrow[dr] & \\
\text{C} \arrow[r] & \text{Ext} \arrow[r] \arrow[dr] & \text{ACS} \cap \text{PR} \arrow[u] \arrow[r] \arrow[drr] & \text{WCIVP} & & \text{PC} \\
 & & \text{SCIVP} \arrow[r] & \text{CIVP} \arrow[r] \arrow[u] & \text{PR} \arrow[ur] \arrow[dr] & \\
 & & & & & \text{PB}
\end{tikzcd}$$

Chart 2

Note that in the class R^R all inclusions in Chart 2 are proper.

We now recall some additional questions from [43] and give the answer, if it is known.

Question 3.9. *If* $f \in \text{CIVP}$*, is* $f \in \text{SCIVP}$*?*

Answer: **No.** K. Banaszewski and T. Natkaniec in [8] constructed a Sierpiński-Zygmund (SZ) function $f \colon R \to R$ having the CIVP and observed that every function in SZ does not have the SCIVP. (See Theorem 4.8.) Thus $f \in \text{CIVP} \setminus \text{SCIVP}$.

Question 3.10. *If* $f \in (\text{ACS} \cap \text{CIVP})$*, is* $f \in \text{Ext}$*?*

Answer: **No.** If the real line R is not a union of less than continuum many of its meager subsets, K. Banaszewski and T. Natkaniec in [8] constructed $f \in (\text{ACS} \cap \text{CIVP} \cap \text{SZ})$. (See Theorem 4.10.) Since $f \in \text{SZ}$, $f \notin \text{SCIVP}$. Therefore $f \notin \text{Ext}$. Moreover, quite recently, K. Ciesielski [28] constructed in ZFC an example $f \in (\text{ACS} \cap \text{CIVP})$ that is continuous on no perfect subset. Thus $f \notin \text{SCIVP}$ and consequently, $f \notin \text{Ext}$.

The next question remains **open**[1].

Question 3.11. *If* $f \in (\mathrm{ACS} \cap \mathrm{SCIVP})$*, is* $f \in \mathrm{Ext}$*?*

In [45] it was shown that if $f : [a, b] \to R$ and $f \in \mathrm{D}$, then

$$\mathrm{WCIVP} = \mathrm{PB} = \mathrm{PR}.$$

Also we discussed the following questions which have negative answers.

Question 3.12. *If* $f \in (\mathrm{ACS} \cap \mathrm{PR})$*, is* $f \in \mathrm{Ext}$*?*

Answer: **No.** See Corollary 3.7.

Question 3.13. *If* $f \in (\mathrm{Conn} \cap \mathrm{PR})$*, is* $f \in \mathrm{ACS}$*?*

Answer: **No.** See [16] and [45].

Question 3.14. *If* $f \in \mathrm{D} \cap \mathrm{PR}$*, is* $f \in \mathrm{Conn}$*?*

Answer: **No.** See [16] and [45].

Each of the functions defined in the answers to Questions 3.13 and 3.14 has a graph that is a G_δ set, and hence is Borel measurable. Thus they satisfy the SCIVP. (See subsection 3.2.)

However **we left open** the following question.

Question 3.15. *If* $f \in \mathrm{ACS}$ *and has a* G_δ *graph, is* $f \in \mathrm{Ext}$*?*

3.1 Darboux like functions in the first class of Baire

Darboux like functions that belong to the first class of Baire were studied in many papers. (See also the survey of J. Ceder and T. L. Pearson [26].)

Theorem 3.16. (See [17].) *In Baire class one the following properties are equivalent:*

$$\mathrm{Conn} = \mathrm{D} = \mathrm{PR} = \mathrm{PC}.$$

Theorem 3.17. (Brown [14]) *In Baire class one,*

$$\mathrm{ACS} = \mathrm{Conn}.$$

Theorem 3.18. (Brown, Humke, Laczkovich [16]) *In Baire class one,*

$$\mathrm{Ext} = \mathrm{ACS}.$$

[1]Recently H. Rosen proved under CH that there exists $f \in \mathrm{ACS} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}$ [107]. Actually, his proof works under assumption that the union of less than $\mathfrak{c}$ many meger sets is meger.

Corollary 3.19. *In Baire class one the following properties are equivalent*

$$\text{Ext} = \text{ACS} = \text{Conn} = \text{D} = \text{SCIVP} = \text{CIVP} = \text{PR} = \text{PC}.$$

Now we will state the relations that hold in the first class of Baire between Darboux functions and PB and WCIVP functions.

First, because every Borel measurable function has the property PB, so

$$\text{PC} \cup \text{WCIVP} \subset \text{PB}.$$

On the other hand, the characteristic function of the halfline $(0, \infty)$ belongs to $\text{PB} \setminus (\text{PC} \cup \text{WCIVP})$. Thus

$$\text{PB} \not\subset \text{PC} \quad \text{and} \quad \text{PB} \not\subset \text{WCIVP}.$$

3.2 Darboux like functions that are Borel measurable

For Borel measurable function, Brown, Humke and Laczkovich proved the following theorem.

Theorem 3.20. (Brown, Humke, Laczkovich [16]) *In the class of Borel measurable functions the following implications hold*

$$\text{Ext} \Rightarrow \text{ACS} \Rightarrow \text{Conn} \Rightarrow \text{D} \Rightarrow \text{PR} \Rightarrow \text{PC}.$$

Moreover, those implications are not reversible except for possibly Ext $\Rightarrow$ ACS*.*

Thus we have the following **open question**. (See also Question 3.15.)

Question 3.21. *If $f : \mathbb{I} \to \mathbb{I}$ is a Borel measurable function and $f \in$ ACS, is $f \in$ Ext?*

The next example is strictly connected with the Question 3.21.

Example. (Cesáro) Let $\varphi : \mathbb{I} \to \mathbb{I}$ be defined by

$$\varphi(x) = \overline{\lim}_{n\to\infty} \frac{a_1 + \ldots + a_n}{n}$$

where a_i are given by the unique nonterminating binary expansion of the number $x = (0.a_1 a_2 \ldots)$.

The function φ is called the *Cesáro-Vietoris* function. Note that φ belongs to the second class of Baire [17]. Vietoris proved in 1921 that φ is connected: $\varphi \in$ Conn [117]. In 1975, J. Brown proved that $\varphi \in$ ACS [15]. The following problem remains **open**:

Question 3.22. *Does the Cesáro-Vietoris function φ belong to* Ext*?*

Note that the solution of Question 3.22 in the negative implies also the negative answer to the Question 3.21.

Note also that in the class of Borel measurable functions,

$$\mathrm{D} \subset \mathrm{SCIVP} = \mathrm{CIVP}.$$

Thus, $\mathrm{ACS} \cap \mathrm{PR} = \mathrm{ACS}$.

3.3 Darboux like functions that are Lebesgue measurable

Theorem 3.23. (Brown, Humke, Laczkovich [16]) *In the class of all Lebesgue measurable functions the following relations hold:*

$$\begin{tikzcd} \mathrm{C} \arrow[r] & \mathrm{Ext} \arrow[r] \arrow[drr] & \mathrm{ACS} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[r] & \mathrm{PC} \\ & & & \mathrm{PR} \arrow[urr] & & \end{tikzcd}$$

Moreover, all those inclusions are proper.

3.4 Darboux like functions that are Marczewski measurable

Recall that a function $f\colon X \to Y$ is said to have *property-(s)* or to be *(s)-measurable* provided that

(s) – for each non-void perfect subset P of X there exists a non-void perfect subset Q of P such that the restriction $f \restriction Q$ is continuous.

Marczewski defined property (s) for sets in [85] and showed that the class of (s)-measurable (*Marczewski measurable*) functions and the class of functions (functions with property (s)) studied by Sierpiński in [108] were the same. Note that each Borel measurable function is Marczewski measurable.

Theorem 3.24. (Gibson, Roush [52]) *There exists a connectivity function $g\colon \mathbb{I}^2 \to \mathbb{I}$ and $p \in \mathbb{I}$ such that the extendable function $f\colon \mathbb{I} \to \mathbb{I}$ given by $f(x) = g(x,p)$ does not have property (s). Thus f is not Marczewski measurable.*

J. B. Brown, P. Humke, and M. Laczkovich in [16] stated the problem that can be formulated as follows:

Question 3.25. *How are the Darboux like properties related within the function classes:*

U – *universally measurable functions;*

$\mathbf{B}_w$ – *functions with the Baire property in wide sense;*

$\mathbf{B}_r$ – *functions with the Baire property in restricted sense;*

(s) – *Marczewski measurable functions?*

See [78] for definitions and discussion. Generally, this problem remains **open**. However, we know that the answer depends on some additional set theoretical assumptions. Namely, concerning this problem, I. Recław and R. G. Gibson, [55], proved the following theorems.

Theorem 3.26. (Gibson, Recław [55]) *For functions $f\colon R \to R$, the following are equivalent:*

(i) $\mathrm{U} \cap \mathrm{D} \subset \mathrm{PR}$*;*

(ii) $\mathrm{U} \cap \mathrm{ACS} \subset \mathrm{PR}$*;*

(iii) there is no universally null set of size of the continuum on the real line.

Theorem 3.27. (Gibson, Recław [55]) *For functions $f\colon R \to R$, the following are equivalent:*

(i) $\mathrm{B}_r \cap \mathrm{D} \subset \mathrm{PR}$*;*

(ii) $\mathrm{B}_r \cap \mathrm{ACS} \subset \mathrm{PR}$*;*

(iii) there is no always of the first category set of size of the continuum on the real line.

4 Darboux like properties in the class of Sierpiński-Zygmund functions

The next theorems are connected with a theorem of Blumberg from 1922.

Theorem 4.1. (Blumberg [9]) *For every $f\colon R \to R$ there exists a dense subset D of R such that the restriction $f{\restriction}D$ of f to D is continuous.*

The set D constructed by Blumberg is countable. In a quest whether it can be chosen any bigger Sierpiński and Zygmund proved in 1923 the following theorem. This theorem shows that we cannot prove in ZFC a version of the Blumberg theorem in which the set D is uncountable.

Theorem 4.2. (Sierpiński, Zygmund [110]) *There exists a function $f\colon R \to R$ whose restriction $f{\restriction}X$ is discontinuous for any subset X of R of cardinality $\mathfrak{c}$.*

Every function that satisfies the assertions of Theorem 4.2 is called to be a *Sierpiński-Zygmund* function (shortly, SZ-function):

SZ – f is SZ-function if the restriction $f\restriction X$ is discontinuous for any subset X of R of cardinality $\mathfrak{c}$.

In 1981, J. Ceder constructed an example of connectivity SZ function.

Theorem 4.3. (Ceder [24]) *Assume the Continuum Hypothesis (CH). Then*

$$\mathrm{SZ} \cap \mathrm{Conn} \neq \emptyset.$$

This result was improved by K. Kellum.

Theorem 4.4. (Kellum [74]) *Assume the Continuum Hypothesis (CH). Then*

$$\mathrm{SZ} \cap \mathrm{ACS} \neq \emptyset.$$

On the other hand, it is easy to observe that

$$\mathrm{SZ} \cap \mathrm{SCIVP} = \emptyset.$$

Thus

Theorem 4.5. $\mathrm{SZ} \cap \mathrm{Ext} = \emptyset$.

The PR functions in the class SZ were considered by Darji in 1993.

Theorem 4.6. (Darji [38]) *There exists* $f \in \mathrm{SZ} \cap \mathrm{PR}$.

Answering a question posed by Darji, in 1996 Balcerzak, Ciesielski and Natkaniec proved the following theorem

Theorem 4.7. (Balcerzak, Ciesielski, Natkaniec [1])

(a) If R is not a union of less than continuum many of its meager subsets (thus under CH and MA) then there exists an $f \in \mathrm{SZ} \cap \mathrm{PR} \cap \mathrm{ACS}$.

(b) There is a model of ZFC in which every Darboux function $f\colon R \to R$ is continuous on some set of cardinality continuum.

In particular, in this model we have $\mathrm{SZ} \cap \mathrm{ACS} = \mathrm{SZ} \cap \mathrm{D} = \emptyset$.

K. Banaszewski and T. Natkaniec replaced PR property in Theorem 4.6 by CIVP.

Theorem 4.8. (K. Banaszewski, Natkaniec [8]) *There exists* $f \in \mathrm{SZ} \cap \mathrm{CIVP}$.

Thus we obtain the following.

Corollary 4.9. SCIVP $\neq$ CIVP.

Similarly, part (a) of Theorem 4.7 is improved as follows.

Theorem 4.10. (K. Banaszewski, Natkaniec [8]) *If R is not a union of less than continuum many of its meager subsets, then there exists an $f \in \mathrm{SZ} \cap \mathrm{CIVP} \cap \mathrm{ACS}$.*

Corollary 4.11. *If R is not a union of less than continuum many of its meager subsets, then*

$$\mathrm{Ext} \neq \mathrm{ACS} \cap \mathrm{CIVP}.$$

5 Darboux like and additive functions

In 1942, F. B. Jones constructed a function $f\colon R \to R$ such that

(1) f is additive, i.e., $f(x+y) = f(x) + f(y)$ for each $x, y \in R$;

(2) f intersects every closed subset P of R^2 with uncountable x-projection $\mathrm{dom}\,(P)$.

Such a function was studied in several papers.

Theorem 5.1. *Let $f\colon R \to R$ be the Jones' function. Then*

(1) f is connectivity; (Jones [68])

(2) f is almost continuous in the sense of Stallings; (Kellum [74])

(3) f does not have the WCIVP*, thus it is not extendable.* (Rosen [103])

Darboux like properties in the class **Add** of additive functions were also considered by J. Smítal [111] and by Z. Grande [56]. Grande in his paper [56] posed the following, very interesting question. It was presented during the Joint US-Polish Workshop in Real Analysis in Łódź, Poland, in July 1994, but still **remains open**[2]. (See also [57].)

Question 5.2. *Does there exist a discontinuous additive almost continuous in the sense of Stallings (or connected) function whose graph is "small" in the sense of measure or category?*

[2] Recently K. Ciesielski and U. Darji find under CH the affirmative answer to this problem.(Private communication.)

Recall that there are discontinuous additive Darboux functions possessing small graph both in the sense of measure and in the sense of category. (See [2].)

Recently Darboux like functions in the class Add were considered by D. Banaszewski in his doctor's thesis. In particular, he proved the following

Theorem 5.3. (D. Banaszewski [2]) *For every $f \in \mathrm{Add}$ the following conditions are equivalent*

(i) $f \in \mathrm{PR}$*;*

(ii) *f has a perfect road at* 0*;*

(iii) *f has a perfect road at some $x \in R$;*

(iv) $f \in \mathrm{WCIVP}$*.*

Theorem 5.4. (D. Banaszewski [2])

(1) There exists $f \in \mathrm{Add} \cap \mathrm{PR}$ such that $f \notin \mathrm{CIVP} \cup \mathrm{D}$.

(2) There exists $f \in \mathrm{Add} \cap \mathrm{ACS}$ such that $f \notin \mathrm{PR}$.

(3) There exists $f \in \mathrm{Add} \cap \mathrm{CIVP}$ such that $f \notin \mathrm{D}$.

(4) There exists $f \in \mathrm{Add} \cap \mathrm{D}$ such that $f \notin \mathrm{Conn}$.

D. Banaszewski posed also the following **open question**.

Question 5.5. *Does there exist $f \in \mathrm{Add} \cap \mathrm{Conn} \setminus \mathrm{ACS}$?*

Moreover, we are unable to construct a discontinuous function $f \in \mathrm{Add} \cap \mathrm{Ext}$. Note that it is easy to construct a discontinuous function $f \in \mathrm{Add} \cap \mathrm{SCIVP} \cap \mathrm{ACS}$. (This holds because there exists a Hamel base which contains a perfect set.)

6 Darboux like functions versus quasi-continuity

We now give some facts that are related to a different kind of discontinuity. In this investigation we will also discuss some relations with the previous classes of functions. Recall the following notions.

Let $f : X \to Y$ be a function. Then:

ACH – f is an *almost continuous function in the sense of Husain*, if for every $x \in X$ and for each open neighborhood V of $f(x)$ in Y, $\mathrm{cl}\,(f^{-1}(V))$ is a neighborhood of x.

CT – f is said to be of the *Cesaro type* if there exist non-empty open sets U and V in X and Y, respectively, such that $U \subset \mathrm{cl}\,(f^{-1}(y))$ for all $y \in V$.

QC – f is said to be *quasi-continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an non-empty open subset $W \subset U$ such that $f(W) \subset V$.

CLIQ – Let Y be a metric space with metric ϱ. Then f is *cliquish* if for every $x \in X$, for each open neighborhood U of x and for every $\varepsilon > 0$ there exists a non-empty open subset $W \subset U$ such that $\varrho(f(y), f(z)) < \varepsilon$, for all $y, z \in W$.

T. Husain defined the notion of almost continuous functions in the sense of Husain in [63]. Note that the function $f\colon [0,1] \to R$ defined by $f(x) = \sin(\frac{1}{x})$ for $x > 0$, and $f(0) = 0$ is a Darboux function of Baire class 1 but is not almost continuous in the sense of Husain. Thus this type of almost continuity is different from the other in a very restrictive class of functions. It should be noted that almost continuity in the sense of Husain was defined earlier by H. Blumberg [9], who used the phrase "densely approached". S. Kempisty defined the notion of quasi continuous function in [77]. (See also [84].) Finally, the notion of cliquishness was introduced by H. P. Thielman in [115].

Clearly, each function with values in a metric space, which is quasicontinuous, is cliquish. Moreover, it is worth to notice that for the real functions defined on a Baire space,

- $f \in \mathrm{QC}$ iff the restriction $f \restriction C(f)$, of f to the set of all points at which f is continuous, is dense in f;
- $f \in \mathrm{Cliq}$ iff f is pointwise discontinuous, i.e., the set $C(f)$ is dense in X.
- Each $f \in \mathrm{Cliq}$ has the Baire property.

Also, there exist quasi-continuous functions $f\colon R \to R$ that are not almost continuous in the sense of Husain nor in the sense of Stallings.

Darboux like functions in the class of quasi-continuous functions were studied in two papers, by R. Gibson and I. Recław in [55], and independently, by T. Natkaniec in [90].

Theorem 6.1. (Gibson, Recław [55])

(1) There exists $f\colon \mathbb{I} \to \mathbb{I}$ such that $f \in \mathrm{PR} \cap \mathrm{QC}$ but $f \notin \mathrm{D}$.

(2) There exists $f\colon \mathbb{I} \to \mathbb{I}$ such that $f \in \mathrm{PR} \cap \mathrm{Cliq}$ but $f \notin \mathrm{QC}$.

(3) There exists $f\colon \mathbb{I} \to \mathbb{I}$ such that $f \in \mathrm{QC} \cap \mathrm{D}$ but $f \notin \mathrm{Conn}$.

(4) *If* $f: R \to R$ *and* $f \in \mathrm{QC}$, *then* $f \in \mathrm{PR}$ *iff* $f \in \mathrm{PC}$.

(5) *There exists* $f: \mathbb{I} \to \mathbb{I}$ *such that* $f \in \mathrm{Cliq} \cap \mathrm{PC}$ *but* $f \notin \mathrm{PR}$.

Gibson and Recław also asked the questions, for functions $f: R \to R$,

Question 6.2. *Does* $\mathrm{QC} \cap \mathrm{Conn} \subset \mathrm{ACS}$*?*

This question is answered in the negative by A. Andryszczak (Nowik) and M. Szyszkowski. (See [55].) They observed that the function f constructed in [65] by J. Jastrzębski has the property that $f \in \mathrm{QC} \cap \mathrm{Conn}$ but $f \notin \mathrm{ACS}$. (See also [90].)

Question 6.3. *Does* $\mathrm{QC} \cap \mathrm{ACS} \subset \mathrm{Ext}$*?*

To answer this question, we prove the following

Theorem 6.4. *There exists a quasi-continuous function* $f: \mathbb{I} \to \mathbb{I}$ *in the class* $\mathrm{ACS} \setminus \mathrm{CIVP}$.

PROOF. Let C be the ternary Cantor set and let $(I_{n,m})_{n,m}$ be the sequence of all components of $\mathbb{I} \setminus C$ such that

- for each n, $\bigcup_m I_{n,m}$ is dense in C.

Let $(q_n)_n$ be a sequence of all rationals. Moreover, let $C_0 = C \setminus \bigcup_{n,m} \mathrm{cl}\,(I_{n,m})$ and let $B \subset C_0$ be a Bernstein set in C, i.e., $B \cap P \neq \emptyset \neq C \setminus P$ for each non-empty perfect set $P \subset C$.

Now let $(F_\alpha)_{\alpha<\mathfrak{c}}$ be a sequence of all minimal blocking sets in $\mathbb{I} \times \mathbb{I}$ such that $\mathrm{dom}\,(F_\alpha) \cap C_0 \neq \emptyset$. Note that $|\mathrm{dom}\,(F_\alpha) \cap B| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$. For each $\alpha < \mathfrak{c}$ choose $(x_\alpha, y_\alpha) \in F_\alpha$ such that

- $x_\alpha \in B$;
- $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$.

Define $f: \mathbb{I} \to \mathbb{I}$ by

$$f(x) = \begin{cases} q_n & \text{for } x \in \bigcup_m \mathrm{cl}\,(I_{n,m}); \\ y_\alpha & \text{for } x = x_\alpha,\ \alpha < \mathfrak{c}; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $f_0 = f \restriction \bigcup_{n,m} I_{n,m}$ is continuous and f_0 is dense in f. Thus f is quasi-continuous.

Now, f is almost continuous. Indeed, if $F \subset \mathbb{I}^2$ is a minimal blocking set then either $\mathrm{dom}\,(F) \subset \mathrm{cl}\,(I_{n,m})$ for some $n, m \in N$ or $F = F_\alpha$ for some $\alpha < \mathfrak{c}$.

(See, e.g., [88].) Therefore either $(x, q_n) \in F$ for some x (because $\operatorname{rng}(F) = R$) or $(x_\alpha, y_\alpha) \in F \cap f$. Thus $f \cap F \neq \emptyset$.

Finally, let K be a Cantor set such that $K \subset \mathbb{I} \setminus \mathbb{Q}$. Then $f^{-1}(K) \subset B$, so it contains no Cantor set. $\square$

Corollary 6.5. $\mathrm{QC} \cap \mathrm{ACS} \setminus \mathrm{Ext} \neq \emptyset$.

6.1 Decomposition of the continuity

Theorem 6.6. (D. B. Smith [113]) *$f : [a, b] \to R$ is continuous if and only if it satisfies the conjunction of the following three conditions:*

(1) $f \in \mathrm{ACS}$;

(2) $f \in \mathrm{ACH}$;

(3) $f \notin \mathrm{CT}$.

With the examples given in the paper [113] and the examples given by R. J. Fleissner [41] and by J. Brown [15] it follows that the three conditions are not redundant. At a real variable conference at Auburn University and at the XV Summer Symposium in Real Analysis held in Smolenice Castle, Smolenice, Czechoslovakia, August 1991, R. Gibson answered the following three questions. (Remember Chart 1! See page 496)

Question 6.7. *In Theorem 6.6, if* $f \in \mathrm{ACS}$ *is replaced with the stronger condition* $f \in \mathrm{Ext}$*, are the three conditions redundant?*

Answer: **No.** See [46].

Question 6.8. *In Theorem 6.6, if the condition* $f \in \mathrm{ACS}$ *is replaced with the weaker condition* $f \in \mathrm{D}$*, is the theorem true?*

Answer: **Yes.** See [46].

In [46], it is shown that if $f : [a, b] \to R$ is almost continuous in the sense of Husain, then f is peripherally continuous. Thus it follows that we can weaken condition (1) of the theorem of B. D. Smith to include all Darboux functions, but we can not weaken condition (1) to include all peripherally continuous functions.

In [112], Smítal and Stanova proved the following theorem.

Theorem 6.9. (Smítal, Stanova [112]) *There exists a function* $h \colon R \to R$ *such that* $h \in \mathrm{ACS}$, $h \notin \mathrm{CT}$ *and* $h \notin \mathrm{QC}$.

This suggests the following problem.

Question 6.10. *Does there exists a function $h\colon R \to R$ such that $h \in \mathrm{Ext}$, $h \notin \mathrm{CT}$, and $h \notin \mathrm{QC}$?*

Answer: **Yes.** Recently H. Rosen remarked that the Croft's function, i.e., Darboux Baire 1 function $h\colon R \to R$ that equals 0 almost everywhere (See [17].) belongs to the class $\mathrm{Ext} \setminus \mathrm{CT} \cup \mathrm{QC}$.

J. Smítal and E. Stanova [112] gave a generalization of the theorem of Smith by proving the following theorem.

Theorem 6.11. (Smítal, Stanova [112]) *Let X be a T_3 locally connected Baire topological space. A function $f : X \to R$ is continuous if and only if it satisfies the conjunction of the following three conditions:*

(1) $f \in \mathrm{ACS}$*;*

(2) $f \in \mathrm{ACH}$*;*

(3) $f \notin \mathrm{CT}$.

Question 6.12. *Is Theorem 6.11 true when condition (1) is replaced with the condition $f \in \mathrm{D}$?*

Answer: **Yes.** See [46].

7 Some characterizations of Darboux like functions

7.1 Extendability and peripheral families

The following question was posed in [43].

Question 7.1. *Is there a "nice condition" that characterizes extendable functions?*

See also Question 3.1. Concerning this question, Gibson and Roush in [53] defined a family of peripheral intervals for a function $f\colon \mathbb{I} \to \mathbb{I}$.

Definition. Let $f\colon \mathbb{I} \to \mathbb{I}$ be a function. A family of *peripheral intervals* (**PI**) for f consists of a sequence of ordered pairs (I_n, J_n) of subintervals of $\mathbb{I}$ such that

(1) I_n is open in $\mathbb{I}$ and the length of I_n converges to 0;

(2) for each $x \in \mathbb{I}$ and for any $\varepsilon > 0$ there exists (I_n, J_n) such that $x \in I_n$, $f(x) \in J_n$, and the length of I_n and J_n are less than ε;

(3) both endpoints of I_n map into J_n;

(4) if I_n and I_m have points in common but neither is a subset of the other, then J_n and J_m have points in common.

Then in the same paper [53], Gibson and Roush proved the following two theorems.

Theorem 7.2. (Gibson, Roush [53]) *Assume that for $f\colon \mathbb{I} \to \mathbb{I}$ there exists a family of* PI*. Then f is the restriction of a connectivity function $F\colon \mathbb{I}^2 \to \mathbb{I}$ such that F is continuous on the complement of $\mathbb{I} \times \{0\}$, where $\mathbb{I}$ is embedded in $\mathbb{I}^2$ as $\mathbb{I} \times \{0\}$.*

Note also that a function F in Theorem 7.2 can be chosen to be constant on intervals $\{0\} \times \mathbb{I}$ and $\{1\} \times \mathbb{I}$.

Theorem 7.3. (Gibson, Roush [53]) *The existence of a family of* PI *is both necessary and sufficient that a function $f\colon \mathbb{I} \to \mathbb{I}$ be an extendable function.*

It should be noted that Theorems 7.2 and 7.3 gives a result that is a generalization of Tietze's extension theorem for closed set $\mathbb{I} \times \{0\}$ in $\mathbb{I}^2$ and for an extendable function defined on $\mathbb{I} \times \{0\}$. The definition of a family of PI is long and difficult to deal with[3]. Thus can this definition of a family of PI be replaced with a "nice condition"?

7.2 Negligible sets

Assume that $\mathcal{K}$ is a class of functions from X into Y and $g \in \mathcal{K}$. A set $M \subset X$ is called *g-negligible* with respect to $\mathcal{K}$, if every function $f\colon X \to Y$ which agrees with g on $X \setminus M$ belongs to $\mathcal{K}$.

In 1970, J. Brown proved the following result.

Theorem 7.4. (Brown [12]) *If $\mathcal{K} = \mathrm{Conn}$ and $g \in \mathbb{I}^{\mathbb{I}} \cap \mathcal{K}$ then the following statements are equivalent:*

(i) g is dense in $\mathbb{I}^2$;

(ii) every nowhere dense subset of $\mathbb{I}$ is g-negligible with respect to $\mathcal{K}$;

(iii) there exists a dense subset of $\mathbb{I}$ which is g-negligible with respect to $\mathcal{K}$.

In [74] K. Kellum showed that Brown's characterization is still valid when $\mathcal{K}$ is replaced by the class ACS. A stronger result for the class Ext was obtained recently by H. Rosen.

[3]Nevertheless it can be useful. See [66].

Theorem 7.5. (Rosen [101]) *If $\mathcal{K} = \mathrm{Ext}$ and $g \in \mathbb{I}^{\mathbb{I}} \cap \mathcal{K}$ then the following statements are equivalent:*

(i) g is dense in $\mathbb{I}^2$;

(ii) every nowhere dense subset of $\mathbb{I}$ is g-negligible with respect to $\mathcal{K}$;

(iii) there exists a dense G_δ subset of $\mathbb{I}$ which is g-negligible with respect to $\mathcal{K}$.

Note that the analogous result holds also in the class of all real functions defined on R [32, 105]. Theorem 7.5 together with examples of extendable functions that are dense in $\mathbb{I}^2$ (See [13, 15].) and extendable functions that are dense in R^2 (See [32, 105].) are the useful instruments to construct extendable functions. (See [31, 32, 91, 105, 106].)

7.3 Darboux like functions that are characterizable by images, preimages and associated sets

Recall the following definitions. For the families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(R)$ we define

$$\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in R^R \colon (\forall A \in \mathcal{A})\,(f(A) \in \mathcal{B})\},$$

and

$$\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} = \{f \in R^R \colon (\forall B \in \mathcal{B})\,(f^{-1}(B) \in \mathcal{A})\}.$$

Also, we say that a family $\mathcal{F}$ of real functions is

- *characterizable by images* of sets if $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(R)$;
- *characterizable by preimages* of sets if $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(R)$;
- *topologized* if $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some <u>topologies</u> $\mathcal{A}, \mathcal{B}$ on R;
- *characterizable by associated sets* if there exists an $\mathcal{A} \subset \mathcal{P}(R)$ such that

  $f \in \mathcal{F}$ if and only if for every $\alpha \in R$, the "associated" sets $E^\alpha(f) = \{x \colon f(x) < \alpha\}$ and $E_\alpha(f) = \{x \colon f(x) > \alpha\}$ belong to $\mathcal{A}$.

  (i.e., $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for $\mathcal{B} = \{(a, \infty) \colon a \in R\} \cup \{(-\infty, a) \colon a \in R\}$.)

Clearly the class C can be defined by preimages of open sets, so it can be topologized and characterized by associated sets. On the other hand, this class cannot be characterized by images of sets [116].

The problem of characterizing some Darboux like functions by associated sets has been studied in several papers.

Theorem 7.6.

- *The class* D *cannot be characterized by associated sets* (Bruckner [18]).
- *The class* Conn *cannot be characterized by associated sets* (Cristian, Tevy [36]).
- *The class* ACS *cannot be characterized by associated sets* (Kellum [74]).
- *The class* Ext *cannot be characterized by associated sets* (Rosen [106]).

The problem of characterizing Darboux like functions by images and by preimages of sets has been recently addressed by Ciesielski and Natkaniec.

Theorem 7.7. (Ciesielski, Natkaniec [31])

(1) The classes: Ext*,* ACS*,* ACS $\cap$ PR*,* Conn*,* D*,* SCIVP*,* CIVP*, and* WCIVP *cannot be characterized by preimages of sets.*

(2) The classes: PR *and* PC *can be characterized by preimages of sets as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ with $\mathcal{B}$ being the natural topology on R. However, they can neither be topologized nor be characterized by associated sets.*

Theorem 7.8. (Ciesielski, Natkaniec [31])

(1) The classes: Ext*,* ACS*,* ACS $\cap$ PR*,* Conn*,* SCIVP*,* PR*, and* PC *cannot be characterized by images of sets.*

(2) The classes: D*,* CIVP *and* WCIVP *can be characterized by images of sets.*

We can complete those results and determine whether the classes ACS$\cap$PR and PB can be characterized by images or by preimages of sets.

Theorem 7.9.

(1) The class ACS$\cap$PR *can neither be characterized by images nor by preimages of sets.*

(2) *The class* PB *cannot be characterized by images. It can be characterized by preimages, however cannot be topologized nor characterized by associated sets.*

Proof. It is known that if $\mathcal{F} \subset R^R$ is characterizable by images and $\text{Ext} \subset \mathcal{F} \subset \text{D}$, then $\mathcal{F} = \text{D}$ [31, Theorem 4]. Thus, $\text{ACS} \cap \text{PR}$ cannot be characterized by images.

Similarly, it is known that if $\mathcal{F} \subset R^R$ is characterizable by preimages and $\text{Ext} \subset \mathcal{F}$, then $\mathcal{F} \not\subset \text{D}$ [31, Theorem 5]. Thus, $\text{ACS} \cap \text{PR}$ cannot be characterized by preimages.

Also, it is known that if $\mathcal{F} \in R^R$ satisfies the following conditions

(1) $\text{Const} \subset \mathcal{F}$;

(2) for every distinct $a, b \in R$ there exists $f \in \mathcal{F}$ with $f(R) = \{f(a), f(b)\} \in [R]^2$;

(3) there exists $Z \subset R$ such that any distinct $a, b \in R$ the "characteristic" function
$$\varphi^Z_{a,b} = \begin{cases} a & \text{if } x \in Z, \\ b & \text{if } x \notin Z \end{cases}$$
does not belong to $\mathcal{F}$,

then $\mathcal{F}$ cannot be characterized by images of sets. (See [31, Corollary 1].) Therefore the class PB cannot be characterized by images.

However, $\text{PB} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, where

- $A \in \mathcal{A}$ iff for each interval $\mathbb{I} \subset R$, if $|A \cap \mathbb{I}| > \omega$ then $A \cap \mathbb{I}$ contains a non-empty perfect set;
- $\mathcal{B}$ is the natural topology on R

To see that PB cannot be topologized recall that if $\mathcal{F}$ is topologized and $\text{PR} \subset \mathcal{F}$ then $\mathcal{F} = R^R$. (See [31, Corollary 3].)

Finally, we shall prove that PB cannot be characterized by associated sets. Assume that $\text{PR} \subset \mathcal{F}$ and $\mathcal{F}$ can be characterized by associated sets. We will prove that $\mathcal{F} \setminus \text{PB} \neq \emptyset$.

Let $\mathcal{A}$ denote the family of all associated sets of $\mathcal{F}$ and let C be the ternary Cantor set. Divide the set $R \setminus C$ onto two sets A_0 and A_1, each Cantor dense in R. Let C_1 be a subset of C such that $|C_1| = \mathfrak{c}$ and C_1 contains no Cantor set. Put $C_0 = C \setminus C_1$. Since the characteristic functions $\chi_{A_0 \cup C_0}, \chi_{A_1} \in \text{PR} \subset \mathcal{F}$, the sets $A_0 \cup C_0$, $A_1 \cup C_1$, A_1, and $A_0 \cup C$ belong to $\mathcal{A}$. Then $f = \chi_{A_0 \cup C_0} - \chi_{A_1} \in \mathcal{F}$. However, $f \notin \text{PB}$, because the set $f^{-1}(-1, 1) = C_1$ is of the size $\mathfrak{c}$ and contains no Cantor set. $\square$

8 Darboux like functions of n variables

First, we should notice that for $n > 1$, Chart 1 is no longer valid. Many of the results that are presented in this survey follow from the following fact.

Theorem 8.1. (Hagan [59], Whyburn [121]) *If $f\colon \mathbb{I}^n \to \mathbb{I}$ and $n > 1$, then* $\mathrm{Conn} = \mathrm{PC}$.

From the paper [114] by Stallings, it follows the following inclusions

Theorem 8.2. (Stallings [114]) *Assume that $n > 1$.*

(1) If $f\colon \mathbb{I}^n \to \mathbb{I}$ is a connectivity function, then f is almost continuous in the sense of Stallings.

(2) If $f\colon \mathbb{I}^n \to \mathbb{I}$ is a connectivity function, then f is a Darboux function.

Thus in the class of real functions of n variables (for $n > 1$) the following inclusions hold

$$\mathrm{Ext} \subset \mathrm{Conn} = \mathrm{PC} \subset \mathrm{D} \cap \mathrm{ACS}.$$

The examples showing that $\mathrm{D} \not\subset \mathrm{ACS}$ and $\mathrm{ACS} \not\subset \mathrm{D}$ can be found in [88, Examples 1.6 and 1.7]. An example of $f\colon \mathbb{I}^2 \to \mathbb{I}$ such that $f \in \mathrm{ACS} \cap \mathrm{D} \setminus \mathrm{Conn}$ is constructed in [102, Example 1]. The following **open** problem is posed by K. Ciesielski and J. Wojciechowski [33].

Question 8.3. *Is the inclusion* $\mathrm{Ext} \subset \mathrm{Conn}$ *proper in the class of all real functions of n variables, when $n > 1$?*

From the paper [54], by Gibson, Rosen and Roush we have that if $n > 1$ and $f\colon \mathbb{I}^n \to \mathbb{I}$ is a connectivity function, then for any $x \in \mathbb{I}^n$ and for any line segment L or a union of two line segment L_1 and L_2 containing x as a limit point from two directions, there exists a perfect set P containing x as a limit point from two direction such that $f{\restriction}P$ is continuous. Thus we can say that if $f\colon \mathbb{I}^n \to \mathbb{I}$ is a connectivity function, then f has a perfect road.

A strengthening of this result have been obtained recently by K. Ciesielski and J. Wojciechowski. (Note that it also implies that PC functions on R^2 have a two-dimensional version of SCIVP.)

Theorem 8.4. (Ciesielski, Wojciechowski [33]) *Let $n > 1$ and $g\colon R^n \to R$ be peripherally continuous. If X is a non-empty connected perfect subset of R^n, then there exists a non-empty perfect subset P of X such that the restriction $g{\restriction}P$ is continuous.*

It is a well-known fact that, if $f\colon \mathbb{I}^n \to \mathbb{I}$, $n > 1$, is continuous and z separates the range of f, then $f^{-1}(z)$ separates $\mathbb{I}^n$. From the paper [102], by Rosen, Gibson, and Roush it follows that the same conclusion holds for a connectivity function $f\colon \mathbb{I}^n \to \mathbb{I}$, $n > 1$. However, this is not true for Darboux functions nor for almost continuous functions in the sense of Stallings, [114].

In particular, Rosen, Gibson and Roush proved in [102] that if $f\colon \mathbb{I}^2 \to \mathbb{I}$ is a connectivity function and z is an interior point of $f(\mathbb{I}^2)$, then any point of $f^{-1}([0, z))$ and any point of $f^{-1}((z, 1])$ lie in different quasi-components of $\mathbb{I}^2 \setminus f^{-1}(z)$. They gave an example that shows that this conclusion is false for Darboux functions.

In [103] H. Rosen proved: if for all $z \in f(\mathbb{I}^2)$, any point of $f^{-1}([0, z))$ and any point of $f^{-1}((z, 1])$ lie in different quasi-components of $\mathbb{I}^2 \setminus f^{-1}(z)$, then f is a Darboux function. In Theorem 1 of [103], H. Rosen (using results from [120]) proved that if $f\colon \mathbb{I}^n \to \mathbb{I}$, $n > 1$, and $f \in \mathrm{PC}$, then the quasi-components and the components of $f^{-1}(z)$ are the same.

For future work we make the following definition.

QCOMP – $f\colon \mathbb{I}^2 \to \mathbb{I}$ has the QCOMP *property* if for every point $z \in f(\mathbb{I}^2)$, any point of $f^{-1}([0, z))$ and any point of $f^{-1}((z, 1])$ lie in different quasi-component.

Clearly, if $f\colon \mathbb{I}^2 \to \mathbb{I}$, then

$$\mathrm{Conn} = \mathrm{PC} \subset \mathrm{QCOMP} \subset D.$$

Define the function $f\colon [-1, 1] \times [0, 1] \to [-1, 1]$ as follows:

$$f(x, y) = \begin{cases} \sin(\frac{1}{y}) & \text{if } y > 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Clearly, the quasi-components and the components of the complement of $f^{-1}(z)$, for all $z \in [-1, 1]$, are the same. Hence $f \in \mathrm{QCOMP} \subset \mathrm{D}$. Also $f \in \mathrm{ACS}$.

However $f \notin \mathrm{Conn}$. Indeed, fix $y_k \in (0, 1]$ such that $y_{k-1} > y_k$ and $\sin(\frac{1}{y_k}) = 1$ for $k = 1, 2, 3, \ldots$. Let $A = (\bigcup_k([-1, 1] \times \{y_k\})) \cup (\{0\} \times [0, 1]) \cup \{(1, 0)\}$. Then A is connected but $f{\restriction}A$ is not connected.

Now, define the function $g\colon [-1, 1] \times [0, 1] \to [-1, 1]$ as follows:

$$g(x, y) = \begin{cases} \sin(\frac{1}{y}) & \text{if } y > 0 \\ x & \text{if } y = 0. \end{cases}$$

Clearly, the quasi-components of the complement of $g^{-1}(0)$ are not connected. Thus $g \notin \mathrm{Conn} = \mathrm{PC}$. However, $g \in \mathrm{D} \cap \mathrm{ACS}$. Thus, $\mathrm{D} \cap \mathrm{ACS} \setminus \mathrm{QCOMP} \neq \emptyset$.

This suggests the following **open question**.

Question 8.5. *If* $f\colon \mathbb{I}^n \to \mathbb{I}$, $n > 1$, *and* $f \in$ QCOMP, *is* $f \in$ ACS*?*

9 Algebraic operations

9.1 Compositions

At the 11th Summer Symposium on Real Analysis at Esztergom, Hungary, R. Gibson asked the following question.

Question 9.1. *If* $f, g \in$ Ext, *is the composition* $g \circ f \in$ Ext*?*

This question remains **open**.

Obviously, if $h\colon X \to Y$ is the composition of connectivity functions, then h must be a Darboux function. On the other hand, we have the following.

Theorem 9.2.

(1) There exist almost continuous functions $f\colon \mathbb{I}^n \to \mathbb{I}^m$ *and* $g\colon \mathbb{I}^m \to \mathbb{I}^n$ *such that* $g \circ f$ *has no fixed point and is not almost continuous.* (Kellum [73])

(2) There exists almost continuous function $f\colon R \to R$ *such that* $f \circ f$ *is not almost continuous.* (Kellum [75])

This suggests the following question. (See [89] or [57].)

Question 9.3. *Is every Darboux function the composition of two (finite many) of* ACS *(or* Conn*) functions?*

Note that the classes SCIVP and CIVP are closed with respect to compositions, so no $f \in \mathrm{D} \setminus \mathrm{SCIVP}$ can be expressed as the composition of extendable functions. In particular, by Theorem 3.6, there are almost continuous functions that cannot be written as the composition of extendable functions.

For functions defined on the plane, Question 9.3 was solved by H. Rosen.

Theorem 9.4. (Rosen [103]) *There exists a Darboux function* $h\colon \mathbb{I}^2 \to \mathbb{I}$ *that is also an almost continuous function, that is not the composite of any two connectivity functions* $f\colon \mathbb{I}^2 \to \mathbb{I}$ *and* $g\colon \mathbb{I} \to \mathbb{I}$.

Generally, for real functions defined on R, Question 9.3 remains **open**. However, there are some partial results in this direction.

Theorem 9.5. (Natkaniec [89]) *Assume that the covering of category is equal to continuum. Then every function with dense level sets can be expressed as the composition of two* ACS *functions.*

We do not know whether this theorem can be proved in ZFC. Question 9.3 has a surprising solution in the class Add of additive function. In fact, D. Banaszewski proved recently the following theorem.

Theorem 9.6. (D. Banaszewski [2]) *Assume that the covering of category is equal to continuum and* $f\colon R \to R$ *is an additive function. Then the following are equivalent:*

(i) $\dim(\ker(f)) \neq 1$*;*

(ii) f *is the composition of two* ACS *additive functions;*

(iii) f *is the composition of two* Conn *additive functions.*

Moreover, K. Ciesielski observed that Theorem 9.6 can be proved in ZFC, without extra set-theoretic assumptions. (See [3].)

Theorem 9.7. (K. Banaszewski [5]) *There exists a* PR*-function* $h\colon R \to R$ *with the following property:*

for every $f\colon R \to R$ *there exists* $f^* \in \mathrm{PR}$ *such that* $f = f^* \circ h$*.*

In particular, every real function can be expressed as the composition of two PR*-functions.*

9.2 Pointwise limits

Theorem 9.8. *Any real-valued function defined on an interval is the pointwise limit of a sequence*

- *of Darboux functions* (Lindenbaum [80])*;*
- *of* Conn *functions* (Phillips [99])*;*
- *of* ACS *functions* (Kellum [80])*;*
- *of* CIVP $\cap$ D *functions* (K. Banaszewski [6])*;*
- *of* Ext *functions* (Rosen [101]).

9.3 Uniform limits

In this subsection we shall deal with the classes of uniform limits of sequences of Darboux like functions. Let $\overline{\mathcal{F}}$ denote the uniform limit of a class $\mathcal{F}$.

Theorem 9.9. *In the class of all functions from* R *into* R*,* *(1) The class* D *is not closed with respect to uniform limits.* (Sierpiński [109])

(2) There exists $f \in \overline{\text{ACS}} \setminus \text{D}$*. Thus the classes* ACS *and* Conn *are not closed with respect to uniform limits.* (Kellum [80])

(3) The class PC *is closed with respect to uniform limits.* (Gibson, Roush [50])

(4) The class PR *is closed with respect to uniform limits.* (K. Banaszewski [4])

(4) The class CIVP *is not closed with respect to uniform limits.* (K. Banaszewski [6])

(5) The class Ext *is not closed with respect to uniform limits.* (Rosen [104])

Moreover, we have the following examples.

Theorem 9.10. *In the class of all functions from R into R,*

(1) There exists $f \in \text{D} \setminus \overline{\text{Conn}}$*.* (Gibson, Roush [50])

(2) There exists $f \in \text{Conn} \setminus \overline{\text{ACS}}$*.* (Jastrzębski [65])

(3) There exists $f \in \text{ACS} \setminus \overline{\text{Ext}}$*.* (Rosen [106])

The class **UL** of all uniform limits of sequences of Darboux functions has been described by A. Bruckner, J. Ceder and M. Weiss [22]. Note that the following inclusions hold:

$$\text{D} \cup \text{CIVP} \subset \text{UL} \subset \text{PC}.$$

The uniform closures of the classes CIVP and SCIVP are described by K. Banaszewski.

Theorem 9.11.

(1) $\overline{\text{CIVP}} = \text{PR} \cap \text{UL} = \text{WCIVP} \cap \text{UL}$ (K. Banaszewski [6])

(2) $\overline{\text{SCIVP}} = \overline{\text{CIVP}}$ (K. Banaszewski, T. Natkaniec [8])

Corollary 9.12. *There is* $f \in \text{PR} \cap \text{UL} \setminus \text{D}$*.*

The answer to the following question is **unknown**.

Question 9.13. *Does there exist* $f \in \text{ACS} \cap \text{PR} \setminus \overline{\text{Ext}}$*?*

The following problems also remain **open**.

Question 9.14. *Characterize the uniform limits of sequences of* Ext *functions (*ACS *functions,* Conn *functions or* WCIVP *functions).*

Note that a partial solution of Question 9.14 for the class ACS is contained in the paper by T. Natkaniec [88]. In [50], Gibson and Roush proved the following theorems.

Theorem 9.15. (Gibson, Roush [50]) *Let X be a metric space. Then the uniform limit f of a sequence $f_m \colon X \to R$ of peripherally continuous functions is peripherally continuous.*

As a corollary to Theorem 9.15 we have the following result.

Corollary 9.16. *Let (f_m) be a sequence of functions from $\mathbb{I}^n$ into $\mathbb{I}$, where $n > 1$. If each f_m is a connectivity function and f_m converges to f uniformly, then f is a connectivity function.*

9.4 Sums

It is known that every function $f \colon R \to R$ is the sum of two:

- Darboux functions (Lindenbaum [80]);
- connectivity functions (Phillips [99]);
- almost continuous functions (Kellum [71]);
- perfect road functions (K. Banaszewski [4]);
- peripherally continuous functions (K. Banaszewski [4]).

At the 11th Summer Symposium on Real Analysis at Esztergom, Hungary, R. Gibson proved that there are extendable functions $f_1, f_2 \in R^R$ such that $f_1 + f_2 \notin \mathrm{PB}$. (See [44].) Thus $\mathrm{Ext} + \mathrm{Ext} \neq \mathrm{Ext}$. Also he asked the following question.

Question 9.17. *If $h \colon R \to R$ is any function, does there exist functions $f, g \in \mathrm{Ext}$ such that $f + g = h$?*

Answer: **Yes.** H. Rosen in [105] proved that an arbitrary function $f \colon R \to R$ can be written as the sum of two extendable functions. Toward this end he gave an example of an extendable function $g \colon R \to R$ whose graph is dense in R^2. In a separate paper [32], K. Ciesielski and I. Recław gave the same results.

The problem, whether every bounded function can be written as the sum of two bounded functions from a fixed class of Darboux like functions has been studied recently in several papers.

Theorem 9.18.

(1) *Every bounded function $f\colon R \to R$ is the sum of two bounded Darboux functions.* (Maliszewski [82])

(2) *Every bounded function $f\colon R \to R$ is the sum of two bounded almost continuous functions.* (Ciesielski, Maliszewski [29])

(3) *There exists a bounded function $f\colon R \to R$ which is not the sum of two bounded functions with perfect road.* (Ciesielski, Maliszewski [29])

In particular, Theorem 9.18 generalizes the result of Darji and Humke that every bounded function can be expressed as the sum of three bounded almost continuous functions [39]. On the other hand, Theorem 9.18 shows that the following result of Natkaniec is sharp.

Theorem 9.19. (Natkaniec [91]) *Every bounded function $f\colon R \to R$ is the sum of three bounded extendable functions.*

Note also a surprising result of K. Ciesielski and J. Wojciechowski.

Theorem 9.20. (Ciesielski, Wojciechowski [33]) *Assume that $n > 1$. Then*

(1) *Every function $f\colon R^n \to R$ is the sum of $n+1$ extendable functions.*

(2) *There exists $f\colon R^n \to R$ that is not the sum of n connectivity functions.*

Note also that quite recently F. Jordan constructed a Baire one function $f\colon R^n \to R$ that is not the sum of n Darboux functions (unpublished).

In 1959, H. Fast proved the following theorem:

Theorem 9.21. (Fast [40]) *For every family $\mathcal{F} \subset R^R$ with $|\mathcal{F}| \le \mathfrak{c}$ there exists $g \in R^R$ such that $g + f \in \mathrm{D}$ for every $f \in \mathcal{F}$.*

In 1974, K. Kellum proved the analogous result for the class ACS of almost continuous functions [71]. On the other hand, there is not $g \in R^R$ such that $g + f \in \mathrm{D}$ for each $f \in R^R$. The problem, for how big families of real functions $\mathcal{F}$ there exists such a "uniform summand" will be study in Subsection 9.7. Here we note that such a g can be found for some regular families $\mathcal{F} \subset R^R$ of the power $2^{\mathfrak{c}}$.

Theorem 9.22. (Natkaniec [88]) *There exists $g \in R^R$ such that $g + f \in \mathrm{ACS}$ for each $f \in R^R$ with the Baire property (or, for each f that is Lebesgue measurable).*

The similar result was obtained recently for the class Ext of extendable functions.

Theorem 9.23. (Natkaniec, Recław [95]) *There exists* $g \in R^R$ *such that* $g + f \in \mathrm{Ext}$ *for each* $f \in R^R$ *with the Baire property (for each* f *that is Lebesgue measurable).*

J. Ceder considered the analogous problem for classes of Borel measurable functions.

Theorem 9.24. (Ceder [23]) *Let* $\mathcal{A}$ *be a countable family of Baire* α *functions. Then there exists a function* f *such that* $f + g$ *is a Darboux function of Baire class* $\max(\alpha + 1, 3)$ *for any* $g \in \mathcal{A}$.

Moreover, Ceder wrote in his paper: *"We do not know, however, whether or not Theorem 9.24 itself can be valid for families* $\mathcal{A}$ *with cardinality* $\mathfrak{c}$*."* This problem was solved quite recently by Recław and Natkaniec.

Theorem 9.25. (Natkaniec, Recław [95]) *For every* $\alpha < \omega_1$ *there is a Borel measurable function* f *such that* $f + g \in \mathrm{ACS}$ *for any* $g \in B_\alpha$.

For a class $\mathcal{F} \subset R^R$ we can consider also the *maximal additive family for* $\mathcal{F}$:

$$\mathrm{M_a}(\mathcal{F}) = \{g \in R^R\colon\ f + g \in \mathcal{F} \text{ for all } f \in \mathcal{F}\}$$

The maximal additive families for Darboux like functions were studied in several papers.

Theorem 9.26.

(1) $\mathrm{M_a}(\mathrm{D}) = \mathrm{Const}$ (Radakovič [100]).

(2) $\mathrm{M_a}(\mathrm{Ext}) = \mathrm{C}$ (Jastrzębski, Natkaniec [66]).

(3) $\mathrm{M_a}(\mathrm{ACS}) = \mathrm{M_a}(\mathrm{Conn}) = \mathrm{C}$ (Jastrzębski, Jędrzejewski, Natkaniec [67]).

(4) $\mathrm{M_a}(\mathrm{PC}) = \mathrm{M_a}(\mathrm{PR}) = \mathrm{C}$ (K. Banaszewski [4]).

(5) If the additivity of category equals $\mathfrak{c}$*, then* $\mathrm{M_a}(\mathrm{CIVP}) = \mathrm{Const}$ (K. Banaszewski [6]).

(6) If the additivity of category equals $\mathfrak{c}$*, then* $\mathrm{M_a}(\mathrm{SCIVP}) = \mathrm{Const}$.

The proof of (6) is essentially the same as Banaszewski's proof of (5). We are unable to prove those equalities in ZFC.

9.5 Products

Theorem 9.27. (K. Banaszewski [6]) *Every function $f\colon R \to R$ can be expressed as the product of two* SCIVP *functions.*

Note also that there are real functions that cannot be written as the product of finite many of Darboux functions. The class of all products of Darboux functions was characterized by J. Ceder.

Theorem 9.28. (Ceder [25]) *A function $f\colon R \to R$ is the product of two Darboux functions iff it satisfies the following condition*

(JC) *f has a zero in each subinterval in which it changes sign.*

The analogous theorem was proved by T. Natkaniec for almost continuous functions.

Theorem 9.29. (Natkaniec [89]) *Assume that the additivity of category is equal to $\mathfrak{c}$. A real function $f\colon R \to R$ is the product of two almost continuous functions iff it satisfies the condition (JC).*

Recently A. Maliszewski shoved that the extra set-theoretical assumption in Theorem 9.29 can be omitted [83]. Thus we have the following corollary.

Corollary 9.30. *Assume that $f\colon R \to R$. The following conditions are equivalent*

(i) f is the product of two almost continuous functions;

(ii) f is the product of two connectivity functions;

(iii) f is the product of two Darboux functions;

(iv) f possesses the property (JC).

Corollary 9.30 suggests the following **open** questions.

Question 9.31. *Assume that $f\colon R \to R$ satisfies the condition (JC). Is it the product of* Ext *(*ACS $\cap$ PR*) functions?*

Question 9.32. *Characterize products of* Ext *(*Conn *or* D*) functions from R^n into R.*

The maximal multiplicative families for Darboux like functions were studied in several papers. Recall that

$$\mathrm{M_m}(\mathcal{F}) = \{g \in R^R\colon\ fg \in \mathcal{F} \text{ for all } f \in \mathcal{F}\}$$

Recall also that the maximal multiplicative family for the class of all Darboux, Baire one functions is equal to the following class M defined by R. Fleissner [42].

M – $f \in \mathrm{M}$ iff it possesses the following property: if x_0 is a right-hand (left-hand) point of discontinuity of f, then $f(x_0) = 0$ and there is a sequence (x_n) converging to x_0 such that $x_n > x_0$ ($x_n < x_0$) and $f(x_n) = 0$.

Theorem 9.33.

(1) $\mathrm{M_m}(\mathrm{D}) = \mathrm{Const}$ (Radakovič [100]).

(2) $\mathrm{M_m}(\mathrm{Ext}) = \mathrm{M}$ (Jastrzębski, Natkaniec [66]),

(3) $\mathrm{M_m}(\mathrm{ACS}) = \mathrm{M_m}(\mathrm{Conn}) = \mathrm{M}$ (Jastrzębski, Jędrzejewski, Natkaniec [67]).

(4) $\mathrm{M_m}(\mathrm{PC}) = \mathrm{M_m}(\mathrm{PR}) = \mathrm{M}$ (K. Banaszewski [4]).

(5) If the additivity of category equals $\mathfrak{c}$*, then* $\mathrm{M_m}(\mathrm{CIVP}) = \mathrm{Const}$ (K. Banaszewski [6]).

(6) If the additivity of category equals $\mathfrak{c}$*, then* $\mathrm{M_m}(\mathrm{SCIVP}) = \mathrm{Const}$.

The proof of (6) is essentially the same as Banaszewski's proof of (5). We are unable to prove those equalities in ZFC.

9.6 Maxima and minima

Theorem 9.34. (Natkaniec [91]) *Every function* $f\colon \mathbb{I} \to \mathbb{I}$ *can be written as* $\min(\max(f_0, f_1), \max(f_2, f_3))$*, where* $f_i \in \mathrm{Ext}$ *for* $i = 0, 1, 2, 3$.

Note that the same result can be proved also for real functions defined on R (c.f., [32, 105]).

Real functions defined on R that are the maximum of Darboux functions were characterized by J. Ceder [21]. Such functions that are the maximum of perfect road functions were described by K. Banaszewski [4]. Thus, notice the following **open** problem.

Question 9.35. *Characterize functions that are the maximum of functions from other Darboux like classes.*

A partial solution of Question 9.35 for the class ACS is contained in [88].

9.7 Cardinal functions

Results from the previous subsections were recently strengthened by the consideration of the following cardinal functions. (This functions were introduced in[88, 92, 32]. See also the survey [27].)

$$\begin{aligned}
\mathrm{a}(\mathcal{F}) &= \min\left\{|\mathcal{H}| : \mathcal{H} \subset R^R \ \&\ \neg\exists g \in R^R\ \forall h \in \mathcal{H}\ g+h \in \mathcal{F}\right\} \cup \{(2^{\mathfrak{c}})^+\} \\
&= \min\left\{|\mathcal{H}| : \mathcal{H} \subset R^R \ \&\ \forall g \in R^R\ \exists h \in \mathcal{H}\ g+h \notin \mathcal{F}\right\} \cup \{(2^{\mathfrak{c}})^+\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{m}(\mathcal{F}) &= \min\left\{|\mathcal{H}| : \mathcal{H} \subset R^R \ \&\ \neg\exists g \in R^R \setminus \{0\}\ \forall h \in \mathcal{H}\ g\cdot h \in \mathcal{F}\right\} \cup \{(2^{\mathfrak{c}})^+\} \\
&= \min\left\{|\mathcal{H}| : \mathcal{H} \subset R^R \ \&\ \forall g \in R^R \setminus \{0\}\ \exists h \in \mathcal{H}\ g\cdot h \notin \mathcal{F}\right\} \cup \{(2^{\mathfrak{c}})^+\}.
\end{aligned}$$

$$\mathrm{c}_{\mathrm{out}}(\mathcal{F}) = \min\left\{|\mathcal{H}| : \mathcal{H} \subset R^R \ \&\ \neg\exists g \in R^R \setminus \mathrm{Const}\ \forall h \in \mathcal{H}\ g\circ h \in \mathcal{F}\right\} \cup \{(2^{\mathfrak{c}})^+\}$$

$$\mathrm{c}_{\mathrm{in}}(\mathcal{F}) = \min\left\{|\mathcal{H}| : \mathcal{H} \subset R^R \ \&\ \neg\exists g \in R^R \setminus \mathrm{Const}\ \forall h \in \mathcal{H}\ h\circ g \in \mathcal{F}\right\} \cup \{(2^{\mathfrak{c}})^+\}$$

$$\mathrm{c}_{\mathrm{r}}(\mathcal{F}) = \min\left\{|\mathcal{G}| : \mathcal{G} \subset R^R \ \&\ \neg\exists h \in \mathcal{F}\ \forall g \in \mathcal{G}\ \exists f \in \mathcal{F}\ f\circ h = g\right\} \cup \{(2^{\mathfrak{c}})^+\}$$

$$\mathrm{c}_{\mathrm{l}}(\mathcal{F}) = \min\left\{|\mathcal{G}| : \mathcal{G} \subset R^R \ \&\ \neg\exists h \in \mathcal{F}\ \forall g \in \mathcal{G}\ \exists f \in \mathcal{F}\ h\circ f = g\right\} \cup \{(2^{\mathfrak{c}})^+\}$$

Values of those cardinal functions for Darboux like classes from R to R were investigated by T. Natkaniec [88, 92, 93], K. Ciesielski and A. W. Miller [30], K. Ciesielski and I. Recław [32], and T. Natkaniec and I. Recław [94]. Known results are listed in the following table.

$\mathcal{F}$	Ext	ACS	Conn	D	SCIVP	CIVP	PR	PC
$\mathrm{a}(\mathcal{F})$	$\mathfrak{c}^+$	κ	κ	κ	$\mathfrak{c}^+$	$\mathfrak{c}^+$	$\mathfrak{c}^+$	$2^{\mathfrak{c}}$
$\mathrm{m}(\mathcal{F})$	2	$\mathrm{cf}(\mathfrak{c})$	$\mathrm{cf}(\mathfrak{c})$	$\mathrm{cf}(\mathfrak{c})$	2	2	2	$\mathfrak{c}$
$\mathrm{c}_{\mathrm{out}}(\mathcal{F})$	1	$\mathrm{cf}(\mathfrak{c})$	$\mathrm{cf}(\mathfrak{c})$	$\mathrm{cf}(\mathfrak{c})$	1	1	1	$\mathfrak{c}$
$\mathrm{c}_{\mathrm{in}}(\mathcal{F})$	1	1	1	1	1	1	$(2^{\mathfrak{c}})^+$	$(2^{\mathfrak{c}})^+$
$\mathrm{c}_{\mathrm{r}}(\mathcal{F})$	1	1	1	1	1	1	$(2^{\mathfrak{c}})^+$	$(2^{\mathfrak{c}})^+$
$\mathrm{c}_{\mathrm{l}}(\mathcal{F})$	1	1	1	1	1	1	$(2^{\mathfrak{c}})^+$	$(2^{\mathfrak{c}})^+$

Table 1.

where $\kappa = \mathrm{a}(\mathrm{ACS}) = \mathrm{a}(\mathrm{Conn}) = \mathrm{a}(\mathrm{D})$ satisfies the inequalities $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$ and $\mathrm{cf}(\kappa) > \mathfrak{c}$. Moreover, it is consistent with ZFC that κ can be equal to any regular cardinal between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$ and that it can be equal to $2^{\mathfrak{c}}$ independently of the cofinality of $2^{\mathfrak{c}}$ [30]. Additionally, the equalities (i, j) are proved

in [32]	for $i = 1, 2$	and	$j = 1, 7, 8$;
in [94]	for $i = 2$	and	$j = 2, 3, 4$;
in [92]	for $i = 3, 4$	and	$j = 1, 2, 3, 4, 5, 7, 8$;
in [93]	for $i = 5, 6$	and	$j = 7, 8$.

Here (i, j) denotes the coordinates of given equality in our table; i denotes the number of the line and j the number of the column. The other equalities easily follow from the monotonicity of considered cardinal functions.

Table 1 can be easily complete for the other Darboux like classes.

Theorem 9.36. *The following equalities hold for the classes* WCIVP*,* PB *and* ACS $\cap$ PR*:*

$\mathcal{F}$	ACS $\cap$ PR	WCIVP	PB
$\mathrm{a}(\mathcal{F})$	$\mathfrak{c}^+$	$\mathfrak{c}^+$	$\mathfrak{c}^+$
$\mathrm{m}(\mathcal{F})$	2	2	2
$\mathrm{c_{out}}(\mathcal{F})$	1	1	1
$\mathrm{c_{in}}(\mathcal{F})$	1	$(2^{\mathfrak{c}})^+$	$(2^{\mathfrak{c}})^+$
$\mathrm{c_r}(\mathcal{F})$	1	$(2^{\mathfrak{c}})^+$	$(2^{\mathfrak{c}})^+$
$\mathrm{c_l}(\mathcal{F})$	1	$(2^{\mathfrak{c}})^+$	$(2^{\mathfrak{c}})^+$

Table 2.

From cardinal functions defined above, the function $\mathrm{a}(\mathcal{F})$ has been studied most extensively. In particular, K. Ciesielski and J. Wojciechowski proved recently the following results.

Theorem 9.37. (Ciesielski, Wojciechowski [33]) *In the class of real functions defined on R^n, with $n > 1$, the following equalities hold:*

$$\mathrm{a}(\mathrm{Ext}) = \mathrm{a}(\mathrm{PC}) = \mathrm{a}(\mathrm{Conn}) = \mathrm{a}(\mathrm{D}) = 2.$$

F. Jordan in [70] considered the function $\mathrm{a}(\mathcal{F})$ for complements of Darboux like classes. An analogous function has been also studied for bounded Darboux like functions in [29]. It is surprising that the result obtained for bounded functions are essentially different than that described above. In particular, K. Ciesielski and A. Maliszewski proved that

$$\mathrm{a}_b(\mathrm{ACS}) = \mathrm{a}_b(\mathrm{Conn}) = \mathrm{a}_b(\mathrm{D}) = \mathrm{cf}(\mathfrak{c}).$$

D. Banaszewski studied cardinal functions for additive Darboux like functions [2].

For other results concerning cardinal functions, see the recent survey by K. Ciesielski [27].

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