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SELF-AFFINE CURVES AND SEQUENTIAL MACHINES

Introduction

The nowhere differentiable function proposed by K. Weierstraß (see [59], [29]) inspired many mathematicians to look for simple examples of such functions and to investigate their properties. A similar source of impulses was given by the examples of space-filling curves constructed by G. Peano (see [37], [38]) and D. Hilbert (see [25]). A unified description of the construction methods leading to the previous examples was given by B. Mandelbrot, when he introduced the initiator-generator construction (see [34]). This general concept has been formalized and further developed in several directions: In the direction of recurrent sets initiated by M. F. Dekking (see [13], [14], [15]), and in the direction of iterated function systems (IFS) introduced by J. E. Hutchinson (see [26]). Iterated function systems have later been generalized by T. Bedford (see [4]), M. Barnsley (see [2]), C. Bandt (see [1]), D. Mauldin and S. Williams (see [35]) as well as many other authors.

The concept of M. F. Dekking probably is the most general and best known method to generate fractal curves. It involves combinatorial and geometrical aspects. The combinatorial part consists of a substitution rule, which assigns to each element of a finite set of symbols, called an alphabet, a word, i. e. a

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concatenation of finitely many symbols from the same alphabet. While the symbols correspond to the initiators in the sense of B. Mandelbrot, the words can be associated with the generators. The geometrical part of Dekking's method includes the geometrical interpretation of the symbols and of the words by non-empty compact subsets of the n-dimensional Euclidian space as well as a rescaling procedure. The set or the fractal curve, which is produced, is the limit of the rescaled geometrical representations corresponding to the words generated by the substitution starting from a given symbol or word.

S. Eilenberg introduced a different way to construct fractal curves by using sequential functions (see [19]). A sequential function, which is obtained by a finite state machine closely related to a finite automaton, assigns to each word with letters from an input alphabet a word with letters from an output alphabet. Assuming that a consistency condition is satisfied, an interpretation of the words as the digital representations of numbers in two different number systems gives rise to a continuous function from the real numbers into an Euclidian space. S. Eilenberg found sequential machines producing the square-filling curves of Peano and Hilbert with their standard parametrizations.

T. Kamae defined the general notion of a (m, α)-self-affine function and proved that it is generated by a special kind of finite automaton (see [27]). Later on C. Bandt observed that the graphs of (m, α)-self-affine functions can be constructed by sequential machines (see [1]).

Our paper was motivated by the desire to understand to what extend the notion of sequential functions is capable to construct self-affine functions and curves. We shall prove that each self-affine curve is generated by a sequential machine. This includes not only the curves obtained by the method of M. F. Dekking but also curves produced by iterated function systems. In order to expose the relations between the construction using sequential functions and the previous established methods we introduce several types of self-affine curves. Many classical examples will be discussed to illustrate the different types of self-affinity. In the general case, the realization of the curves by means of sequential functions is based on the classical Cantor representation of the real numbers in the unit interval. The combinatorial part of Dekking's construction, the substitution, will appear as an ingredient of the sequential machine. The geometrical part can be recovered in the consistency condition for sequential machines, which assures that a sequential machine leads to a continuous function.

For detailed information on the examples considered in this paper we refer to [17], [18], [22], [34] and [39]. Moreover these books provide an introduction to the concepts of fractal geometry, which are not explicated here. Further books with a specific treatment of curves are [49] and [56].

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1 Preliminaries

Notations

An alphabet $A = \{a_1, \ldots, a_n\}$ is a set containing a finite number of symbols. For an alphabet A we denote with $A^* = \{a_1 \ldots a_k : k \in \mathbb{N}, a_j \in A, 1 \leq j \leq k\}$ the set of all words with letters in A. The empty word is denoted with ε. With respect to the concatenation of words, $(v, w) \mapsto vw$ for all words v and w, A^* is provided with the structure of a free semi-group generated by the letters of A, whose unit element is ε.

As a particular case we denote by $[k]$ the alphabet whose elements are the numbers $\{0, 1, \ldots, k-1\}$ for $k \in \mathbb{N}$.

Let Q be any alphabet. A map $\theta : Q^* \to Q^*$ with $\theta(q) \neq \varepsilon$ for all $q \in Q$ is called a substitution, provided it is compatible with the semi-group structure of Q^*, i. e. for all $v, w \in Q^*$ we have $\theta(vw) = \theta(v)\theta(w)$. In the case that $\theta(q)$ has the same number m of letters for all $q \in Q$, we say that θ is a substitution of constant length m. Moreover, if $\theta(q) = q_0 \ldots q_{m-1}$, we write $\theta_j(q) = q_j$ for all $j \in [m]$, (see [19]).

Self-Affine Functions

In this section we introduce a family of self-affine functions, which has been defined by T. Kamae in [27] generalizing a concept of N. Kôno [32].

Definition 1. *A continuous non-zero function $f : [0,1] \to \mathbb{R}$ is called self-affine of order $\alpha \in (0,1]$ and with base $m \in \mathbb{N}$, $m \geq 2$ (or simply (m,α)-self-affine), if the following conditions are satisfied.*

(a) There is a finite number of continuous functions $x_0, \ldots, x_{N-1} : [0,1] \to \mathbb{R}$ with $x_j(0) = 0$ for all $j \in [N]$, and $x_0 = f$.

(b) There is a substitution $\theta : [N] \to [N]^*$ of constant length m such that for all $(j,h) \in [N] \times [m]$ and for $t \in [0,1]$ we have

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = \frac{x_{\theta_h(j)}(t)}{m^\alpha},$$

Remarks 1. T. Kamae defines self-affine functions by means of continuous functions $y_0, \ldots, y_{N-1} : [0,1] \to [0,1]$, where $y_j(0)$ in general is not equal to zero and therefore property (a) is not satisfied. For technical reasons we work with the functions $x_0, \ldots, x_{N-1} : [0,1] \to \mathbb{R}$ satisfying the properties (a) and (b), which can be obtained from $y_0, \ldots, y_{N-1}$ by the definition $x_j(t) = y_j(t) - y_j(0)$.

2. Additionally to the conditions (a) and (b) T. Kamae requires that for all $i, j \in [N]$ there are numbers $k \in \mathbb{N}$ and $h \in [m^k]$ such that

$$x_i\left(\frac{h+t}{m^k}\right) - x_i\left(\frac{h}{m^k}\right) = \frac{x_j(t)}{m^{\alpha k}}.$$

We omit this condition since it is not of any importance during the course of this paper.

Example 1. The Coordinate Functions of the Peano Square-Filling Curve, *see [37], [29] pp. 117–122, [36].*

The Peano curve $p : [0,1] \to [0,1]^2$ *is defined as follows. Let* $k : [3] \to [3]$ *be the map* $a \mapsto 2 - a$ *for* $a \in [3]$. *If* $t \in [0,1]$ *is given in its* 3*-adic representation* $t = \sum_{j=1}^{\infty} t_j 3^{-j}$, *then* $p(t) = (x(t), y(t))$, *where*

$$\begin{aligned} x(t) &= \textstyle\sum_{j=1}^{\infty} u_j 3^{-j}, \\ y(t) &= \textstyle\sum_{j=1}^{\infty} v_j 3^{-j}, \end{aligned}$$

and

$$\begin{array}{rclcrcl} u_1 &=& t_1, & \qquad & v_1 &=& k^{t_1} t_2, \\ u_2 &=& k^{t_2} t_3, & & v_2 &=& k^{t_1+t_3} t_4, \\ & \vdots & & & & \vdots & \\ u_n &=& k^{t_2+\ldots+t_{2n-2}} t_{2n-1}, & & v_n &=& k^{t_1+\ldots+t_{2n-1}} t_{2n}. \end{array}$$

Let $\theta : \{0,1\}^* \to \{0,1\}^*$ *be the substitution*

$$\begin{aligned} \theta(0) &= 0\,1\,0\,0\,1\,0\,0\,1\,0, \\ \theta(1) &= 1\,0\,1\,1\,0\,1\,1\,0\,1. \end{aligned}$$

Then the functions $x_0(t) = x(t)$ *and* $x_1(t) = -x(t)$ *are* $(9, \frac{1}{2})$*-self-affine, since we have for all* $h \in [9]$, $j \in \{0,1\}$ *and* $t \in [0,1]$

$$x_j\left(\frac{h+t}{9}\right) - x_j\left(\frac{h}{9}\right) = \frac{1}{3} x_{\theta_{h(j)}}(t).$$

If $\theta : \{0,1\}^* \to \{0,1\}^*$ is the substitution

$$\theta(0) = 0\,0\,0\,1\,1\,1\,0\,0\,0,$$
$$\theta(1) = 1\,1\,1\,0\,0\,0\,1\,1\,1,$$

and $y_0(t) = y(t)$ and $y_1(t) = -y(t)$, we have for all $h \in [9]$, $j \in \{0,1\}$ and $t \in [0,1]$

$$y_j\left(\frac{h+t}{9}\right) - y_j\left(\frac{h}{9}\right) = \frac{1}{3} y_{\theta_h(j)}(t).$$

Hence, also the functions y_0 and y_1 are $(9, \frac{1}{2})$-self-affine.

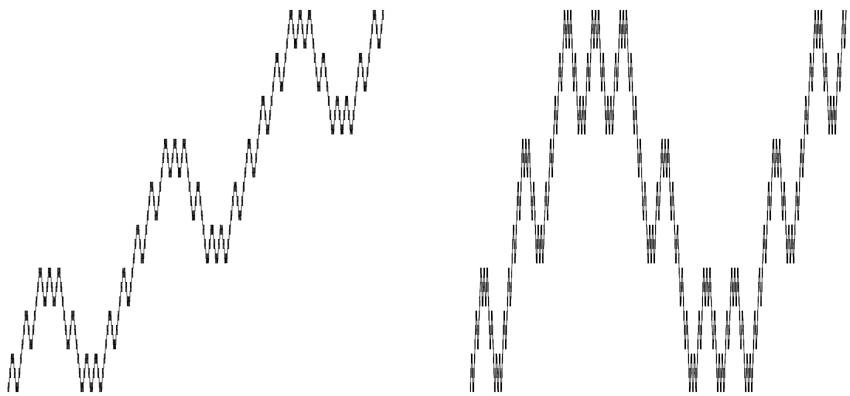

Figure 1: The fourth approximation to the first (left) and to the second (right) coordinate function of the Peano curve.

Example 2. *(***Kiesswetter's Nowhere Differentiable Function***, see [28]).*
The definition of K. Kiesswetters function f is based on the 4-adic representation of numbers in the unit interval. For $t \in [0,1]$, $t = \sum_{j=1}^{\infty} t_j 4^{-j}$, it is given by

$$f(t) = \sum_{\nu=1}^{\infty} (-1)^{N_\nu} \frac{d_\nu}{2^\nu},$$

with

$$d_\nu = \begin{cases} 0 & \textit{if } t_\nu = 0, \\ t_\nu - 2 & \textit{if } t_\nu > 0, \end{cases}$$

and $N_\nu = \#\{k \mid t_k = 0, k < \nu\}$. This function is $(4, \frac{1}{2})$-self-affine. Let $\theta : \{0,1\} \to \{0,1\}^*$ be defined by $\theta(0) = 1\,0\,0\,0$ and $\theta(1) = 0\,1\,1\,1$. Then we *obtain for* $f_0(t) = f(t)$ *and* $f_1(t) = -f(t)$ *that for all* $h \in [4]$, $q \in \{0,1\}$, *and* $t \in [0,1]$

$$f_q\left(\frac{h+t}{4}\right) - f_q\left(\frac{h}{4}\right) = \frac{1}{2} f_{\theta_h(q)}(t).$$

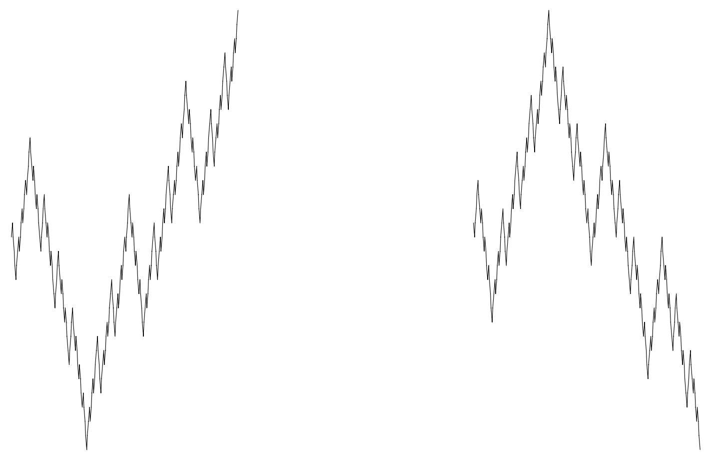

Figure 2: The fourth approximation to the Kiesswetter function f_0 and to $f_1 = -f_0$.

Example 3. The Devil's Staircase, *see [8].*

G. Cantor constructed a monotone function $c : [0,1] \to \mathbb{R}$, *which is known as the devil's staircase or as Cantor's singular function. It has a derivative except for points in the Cantor set, which has Lebesgue measure zero. Moreover, if the derivative exists for* $t \in [0,1]$ *then* $c'(t) = 0$. *On the Cantor set*

$$\mathcal{C} = \left\{ t = \sum_{j=1}^{\infty} t_j 3^{-j} \mid t_j \in \{0,2\} \quad \textit{for all } j \in \mathbb{N} \right\},$$

the function is given for $t = \sum_{j=1}^{\infty} t_j 3^{-j}$ *by*

$$c(t) = \sum_{j=1}^{\infty} \frac{x_j}{2^j} \quad \textit{with } x_j = \tfrac{t_j}{2}.$$

It takes the same values on the endpoints $a = 0.t_1 \dots t_n 0\overline{2}$ *and* $b = 0.t_1 \dots t_n 1$ *of the interval* $I_{t_1 \dots t_n 1} = [a,b]$, *and it is defined to be* $c(t) = c(a)$ *for all* $t \in [a,b]$.

Let $Q = \{0, 1\}$ and $\theta : Q \to Q^*$ be the substitution given by $\theta(0) = 0\,1\,0$ and $\theta(1) = 1\,1\,1$. Then the functions $c_0(t) = c(t)$ and $c_1(t) = 0$ for all $t \in [0, 1]$ are $(3, 1)$-self-affine, i. e. for all $t \in [0, 1]$, $i \in \{0, 1\}$, and $h \in [3]$ we have

$$c_i\left(\frac{h+t}{3}\right) - c_i\left(\frac{h}{3}\right) = \frac{1}{3}c_{\theta_h(i)}(t).$$

For this function the additional condition of T. Kamae is not satisfied (see the previous remark).

Sequential Functions

A sequential function is a function between the words over two different alphabets A and B. In this section we will give a brief introduction to sequential machines and sequential functions according to S. Eilenberg (see [19], chap. X). Moreover, we show how sequential functions can be applied to generate and describe (m, α)-self-affine functions $f : [0, 1] \to \mathbb{R}$.

Definition 2. *A sequential machine $\mathcal{M} = (Q, A, B, \sigma, \tau)$ consists of a finite set of states Q, an input alphabet A, an output alphabet B, a transition function $\sigma : Q \times A \to Q$, and an output function $\tau : Q \times A \to B$.*

The transition function and the output function can be uniquely extended to $\sigma : Q \times A^* \to Q$ and $\tau : Q \times A^* \to B^*$, such that $\sigma(q, \varepsilon) = q$ for all $q \in Q$, and for all words $v, w \in A^*$ and all $q \in Q$

$$\begin{aligned} \sigma(q, vw) &= \sigma(\sigma(q, v), w), \\ \tau(q, vw) &= \tau(q, v)\tau(\sigma(q, v), w). \end{aligned}$$

For each $q \in Q$ the sequential machine $\mathcal{M} = (Q, A, B, \sigma, \tau)$ defines a function $f_q^* : A^* \to B^*$, which is called the sequential function corresponding to $\mathcal{M}$ and q. It is defined by

$$f_q^*(a_1 \dots a_n) = \tau(q, a_1 \dots a_n).$$

Usually a sequential machine $\mathcal{M} = (Q, A, B, \sigma, \tau)$ is represented by a directed graph $\Gamma_{\mathcal{M}}$, whose vertices are the states $q \in Q$. An arrow with label a/b leads from a vertex q_1 to a vertex q_2 if and only if $\sigma(q_1, a) = q_2$ and $\tau(q_1, a) = b$.

Remark A sequential function f_q^* has the following properties.

(a) If $f_q^*(a_1 \dots a_n) = b_1 \dots b_m$, then $n = m$.

(b) $f_q^*(a_1 \dots a_{n+s}) = b_1 \dots b_{n+s}$ implies that $f_q^*(a_1 \dots a_n) = b_1 \dots b_n$ for $s, n \in \mathbb{N}$.

All sequential machines $\mathcal{M} = (Q, A, B, \sigma, \tau)$ considered in this section will be used to generate continuous functions $f : [0,1] \to \mathbb{R}$. The input alphabet is $A = [m]$ for some number $m \in \mathbb{N}$, and the output alphabet B is a subset of $\mathbb{R}$ with $0 \in B$. We will assume that $\tau(q, 0) = 0$ for all $q \in Q$ during the course of this section.

Let $r \in \mathbb{R}$ be a number with $r > 1$. We say that the machine $\mathcal{M} = (Q, [m], B, \sigma, \tau)$ is *r-consistent*, if for all $q \in Q$ and for all $t \in [0,1]$ with two different m-adic representations

$$t = \sum_{j=1}^{\infty} a_j m^{-j} = \sum_{j=1}^{\infty} \tilde{a}_j m^{-j},$$

we have

$$\sum_{j=1}^{\infty} b_j r^{-j} = \sum_{j=1}^{\infty} \tilde{b}_j r^{-j},$$

where for all $n \in \mathbb{N}$

$$\begin{array}{rcl} f_q^*(a_1 \dots a_n) & = & b_1 \dots b_n, \\ f_q^*(\tilde{a}_1 \dots \tilde{a}_n) & = & \tilde{b}_1 \dots \tilde{b}_n. \end{array}$$

Definition 3. *Let $\mathcal{M} = (Q, [m], B, \sigma, \tau)$ be a r-consistent sequential machine. Then for each $q \in Q$ the sequential function f_q^* gives rise to a continuous function $f_q : [0,1] \to \mathbb{R}$. For $t \in [0,1]$ with $t = \sum_{j=1}^{\infty} t_j m^{-j}$ and $f_q^*(t_1 \dots t_n) = d_1 \dots d_n$ for all $n \in \mathbb{N}$ we define*

$$f_q(t) = \sum_{j=1}^{\infty} d_j r^{-j}.$$

We say that f_q is generated by the sequential function f_q^*.

Proposition 1. *Let $\mathcal{M} = (Q, [m], B, \sigma, \tau)$ be a r-consistent sequential machine such that $B \subset \mathbb{R}$, $0 \in B$, and $\tau(q, 0) = 0$ for all $q \in Q$. Then for each $q \in Q$ the function $f_q : [0,1] \to \mathbb{R}$ generated by f_q^* is (m, α)-self-affine, where α is determined by $m^\alpha = r$.*

PROOF. We set $N = \#Q$, and for x_j we take the function f_{q_j} which is generated by $f_{q_j}^*$, where $Q = \{q_0, \dots, q_{N-1}\}$. The condition $\tau(q, 0) = 0$ for all $q \in Q$ ensures that $x_j(0) = 0$ for all $j \in [N]$.
Now conversely the following proposition will show that an arbitrary (m, α)-self-affine function $f : [0,1] \to \mathbb{R}$ can be generated by a sequential function. This relation has been observed also by C. Bandt in [1] without using the concept of sequential functions. $\square$

Proposition 2. *Let $f : [0,1] \to \mathbb{R}$ be a (m,α)-self-affine function. Then there exists an alphabet $B \subset \mathbb{R}$ and an m^α-consistent sequential machine $\mathcal{M} = (Q,[m],B,\sigma,\tau)$, such that f is generated by $f_q^* : [m]^* \to B^*$ for some $q \in Q$.*

PROOF. Let $x_0,\ldots,x_{N-1} : [0,1] \to \mathbb{R}$ be the collection of continuous functions, such that $f = x_0$ and for all $(j,h) \in [N] \times [m]$ and $t \in [0,1]$

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = \frac{x_{\theta_h(j)}(t)}{m^\alpha}.$$

We will define a sequential machine $(Q,[m],B,\sigma,\tau)$. Then an easy calculation shows that the corresponding sequential function $f_{x_0}^* : [m]^* \to B$ generates f.

- the set of states is $Q = \{x_0,\ldots,x_{N-1}\}$,
- the input alphabet is $A = [m]$,
- the output alphabet is

$$B = \left\{ m^\alpha x_j\left(\frac{h}{m}\right) \mid j \in [N], h \in [m] \right\},$$

- the transition function is given by $\sigma(x_j,h) = x_{\theta_h(j)}$,
- the output function is $\tau(x_j,h) = m^\alpha x_j\left(\frac{h}{m}\right)$.

The m^α-consistency of this machine can be concluded from the continuity of the functions $x_0,\ldots,x_{N-1}$. □

Remark Self-affine functions $f : [0,1] \to \mathbb{R}$ with $m^\alpha \in \mathbb{N}$ and $x_j(0), x_j(1) \in \{0,1\}$ for all $j \in [N]$ have been studied by S. Takahashi in [55]. If we do not assume that $x_j(0) = 0$, we obtain $r = m^\alpha$ and $B = [r]$ in this case.

2 (m,L)-Self-Affine Curves

In this section we extend the definition of (m,α)-self-affine functions $f : [0,1] \to \mathbb{R}$ to curves $\varphi : [0,1] \to \mathbb{R}^2$. We will show that self-affine curves can be realized by sequential functions. Moreover it turns out that the family of (m,L)-self-affine curves coincides with a class of recurrent curves considered by F. M. Dekking in [14]. In this section we only consider curves in $\mathbb{R}^2$. However, the results can be extended directly to curves in $\mathbb{R}^n$.

Recall that a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ is called an expanding map, if the absolute values of all eigenvalues of L are larger than 1.

Definition 4. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear expanding map and $m \in \mathbb{N}$. A curve $\varphi : [0,1] \to \mathbb{R}^2$ is called an (m,L)-self-affine curve, if the following conditions are satisfied.*

(a) There is a finite number of curves $x_0, \ldots, x_{N-1} : [0,1] \to \mathbb{R}^2$ *with* $x_j(0) = (0,0)$ *for all* $j \in [N]$, *and* $x_0 = \varphi$.

(b) There is a substitution $\theta : [N] \to [N]^*$ *of constant length* m, *such that for all* $(j,h) \in [N] \times [m]$ *and for* $t \in [0,1]$ *we have*

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = L^{-1}(x_{\theta_h(j)}(t)).$$

Example 4. Hilbert's Square-Filling Curve, *see [25], see also [14].*

Set $Q = \{0,1,2,3\}$ *and let* $\theta : Q \to Q^*$ *be the substitution*

$$\begin{array}{rcl} \theta(0) &=& 1\,0\,0\,3, \\ \theta(1) &=& 0\,1\,1\,2, \\ \theta(2) &=& 3\,2\,2\,1, \\ \theta(3) &=& 2\,3\,3\,0. \end{array}$$

Let $h : [0,1] \to [0,1]^2$ *be the Hilbert curve and define*

$$\begin{array}{rclcrcl} h_0(t) &=& h(t) & \qquad & h_2(t) &=& -h(t), \\[2ex] h_1(t) &=& \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} h(t), & \qquad & h_3(t) &=& \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} h(t). \end{array}$$

Then $\{h_0, h_1, h_2, h_3\}$ *is a family of* $(4, L)$*-self-affine curves, where* $L(x,y) = (2x, 2y)$.

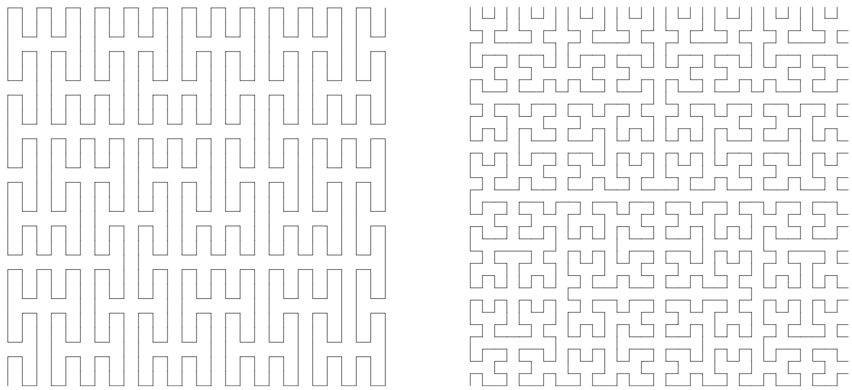

Figure 3: The third approximation to the Peano curve (left) and the fifth approximation to the Hilbert curve (right).

Example 5. Peano's Square-Filling Curve.

Let $p : [0,1] \to [0,1]^2$ *be the Peano curve,* $p(t) = (x(t), y(t))$ *for all* $t \in [0,1]$*. We take the set of states* $Q = \{0,1,2,3\}$ *together with the substitution* $\theta : Q \to Q^*$*, which is given by*

$$\begin{aligned}\theta(0) &= 0\,1\,0\,3\,2\,3\,0\,1\,0,\\ \theta(1) &= 1\,0\,1\,2\,3\,2\,1\,0\,1,\\ \theta(2) &= 2\,3\,2\,1\,0\,1\,2\,3\,2,\\ \theta(3) &= 3\,2\,3\,0\,1\,0\,3\,2\,3.\end{aligned}$$

The following family of curves $\{p_0, p_1, p_2, p_3\}$ *is* $(9, L)$*-self-affine for the expanding map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $L(x, y) = (3x, 3y)$*. For all* $t \in [0,1]$

$$\begin{aligned}&p_0(t) = p(t),\\ &p_1(t) = (-x(t), y(t)),\\ &p_2(t) = (-x(t), -y(t)),\\ &p_3(t) = (x(t), -y(t)).\end{aligned}$$

Example 6. Heighway's Dragon, *see [12], [16]*

Recall that the approximation to Heighway's dragon can be obtained by folding a sheet of paper a certain number of times in the middle. Unfolding the paper with angles of 90^0 *at any crease establishes the curve. Let* $Q = [8]$

Figure 4: The 12th approximation to Heighway's dragon with an unfolding angle of 90^0.

be a set with eight states and define $\theta : [8] \to [8]^*$ *to be the substitution*

$$\begin{array}{lcllcl} \theta(0) & = & 1\,6\,3\,6, & \theta(4) & = & 2\,7\,2\,5, \\ \theta(1) & = & 2\,7\,0\,7, & \theta(5) & = & 3\,4\,3\,6, \\ \theta(2) & = & 3\,4\,1\,4, & \theta(6) & = & 0\,5\,0\,7, \\ \theta(3) & = & 0\,5\,2\,5, & \theta(7) & = & 1\,6\,1\,4. \end{array}$$

Let $d_0 : [0,1] \to \mathbb{R}^2$ *denote the Heighway dragon curve. We now introduce the following family of curves* $\{d_0, d_1, \ldots, d_7\}$*, which is* $(4, L)$*-self-affine with* $L(x, y) = (2x, 2y)$*.*

$$d_1(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d_0(t), \qquad d_2(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} d_0(t),$$

$$d_3(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d_0(t),$$

and $d_j(t) = d_{j-4}(-t) - d_{j-4}(1)$ *for* $j = 4, 5, 6, 7$*.*

Example 7. Heighway's Dragon with Unfolding Angle 120^0**.**

Now we consider the dragon curve $\varphi : [0,1] \to \mathbb{R}^2$*, whose approximations are obtained by using angles of* 120^0 *instead of* 90^0*, as in the previous example.*

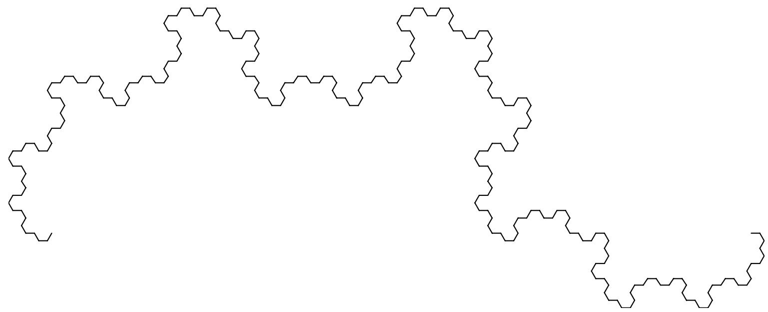

Figure 5: The 7th approximation to Heighway's dragon with an unfolding angle of 120^0.

In this case let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the expanding map $L(x, y) = (3x, 3y)$, and let $\theta : [12] \to [12]^*$ be given by

$$\begin{array}{rcl@{\qquad}rcl}
\theta(0) & = & 1\ 9\ 5\ 9, & \theta(6) & = & 6\ 2\ 6\ 4, \\
\theta(1) & = & 2\ 10\ 0\ 10, & \theta(7) & = & 7\ 3\ 7\ 5, \\
\theta(2) & = & 3\ 11\ 1\ 11, & \theta(8) & = & 8\ 4\ 8\ 0, \\
\theta(3) & = & 4\ 6\ 2\ 6, & \theta(9) & = & 9\ 5\ 9\ 1, \\
\theta(4) & = & 5\ 7\ 3\ 7, & \theta(10) & = & 10\ 0\ 10\ 2, \\
\theta(5) & = & 6\ 8\ 4\ 8, & \theta(11) & = & 11\ 1\ 11\ 3.
\end{array}$$

Set $\varphi_0(t) = \varphi(t)$, and for $j = 1, \ldots, 5$ define

$$\varphi_j(t) = \begin{pmatrix} \cos(\frac{2\pi j}{3}) & -\sin(\frac{2\pi j}{3}) \\ \sin(\frac{2\pi j}{3}) & \cos(\frac{2\pi j}{3}) \end{pmatrix} \varphi(t).$$

Finally, for $j = 6, \ldots, 11$ set $\varphi_j(t) = \varphi_{6-j}(1-t) - \varphi_{6-j}(1)$. Then the family of curves $\{\varphi_j : [0,1] \to \mathbb{R}^2 \mid j \in [12]\}$ is $(4, L)$-self-affine.

Remark The Heighway dragon curve is $(4, L)$-self-affine for any choice of a rational unfolding angle $\alpha \in [\frac{\pi}{3}, \pi]$. The set of states Q then has exactly $2k$ elements, if $k \in \mathbb{N}$ is the smallest number such that $k\beta \equiv 0 \bmod 2\pi$, where $\beta = \frac{\pi-\alpha}{2}$. If α is an irrational number, the Heighway dragon curve is not $(4, L)$-self-affine for any L, but we will see later that it will be self-affine in a more general sense.

Example 8. Lévy's Dragon, *see [33].*

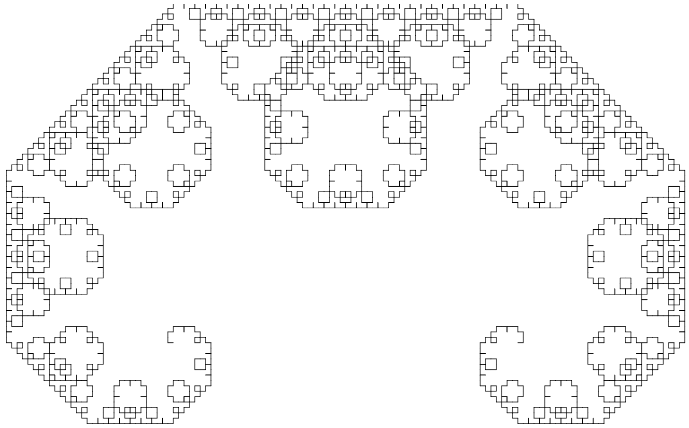

Figure 6: The 6th approximation to Levy's dragon.

Let $l : [0,1] \to \mathbb{R}^2$ be the Lévy dragon and $\theta : \{0,1,2,3\} \to \{0,1,2,3\}^*$ the substitution

$$\begin{aligned}\theta(0) &= 3\,0\,0\,1,\\ \theta(1) &= 0\,1\,1\,2,\\ \theta(2) &= 1\,2\,2\,3,\\ \theta(3) &= 2\,3\,3\,0.\end{aligned}$$

The following family of curves $\{l_0, l_1, l_2, l_3\}$ is $(4, L)$-self-affine with $L(x,y) = (2x, 2y)$.

$$l_0(t) = l(t), \qquad l_2(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} l(t),$$

$$l_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} l(t), \qquad l_3(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} l(t).$$

Example 9. Sierpinski's Gasket Curve, *see [52].*

The gasket curve $s : [0,1] \to \mathbb{R}^2$ has been defined by W. Sierpinski as an example for a curve, where each point is a point of ramification. We set $Q = [6]$ and define the substitution $\theta : [6] \to [6]^*$ to be

$$\begin{aligned}\theta(0) &= 5\,0\,4, & \theta(3) &= 1\,3\,2,\\ \theta(1) &= 3\,1\,5, & \theta(4) &= 2\,4\,0,\\ \theta(2) &= 4\,2\,3, & \theta(5) &= 0\,5\,1.\end{aligned}$$

Moreover we set $s_0(t) = s(t)$ *and*

$$s_1(t) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} s_0(t), \qquad s_2(t) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} s_0(t).$$

For $j = 3, 4, 5$ *we set* $s_j(t) = s_{j-3}(1-t) - s_{j-3}(1)$. *Then the family* $\{s_0, \ldots, s_5\}$ *is* $(3, L)$*-self-affine, where* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *is the expanding map* $L(x, y) = (2x, 2y)$.

Example 10. The Graph of the x-coordinate Function of the Peano Curve.

Let $x : [0, 1] \to \mathbb{R}$ *be the* x*-coordinate function of the Peano curve and* $\varphi_0(t) = (t, x(t))$. *If* $\varphi_1(t) = (t, -x(t))$ *and* $L(x, y) = (9x, 3y)$, *the curves* φ_0 *and* φ_1 *are* $(9, L)$*-self-affine. For all* $j \in [2]$, $h \in [9]$ *and* $t \in [0, 1]$ *we have*

$$\varphi_j\left(\frac{h+t}{9}\right) - \varphi_j\left(\frac{h}{9}\right) = L^{-1}(\varphi_{\theta_h(j)}(t)),$$

where $\theta : [4] \to [4]^*$ *is the substitution of example 5.*

Example 11. The Graph of Kiesswetter's Function.

Let $f : [0, 1] \to \mathbb{R}$ *be the Kiesswetter function and* $\varphi_0 : [0, 1] \to \mathbb{R}^2$ *its graph,* $\varphi_0(t) = (t, f(t))$ *for all* $t \in [0, 1]$. *If* $\varphi_1(t) = (t, -\varphi_0(t))$, *the curves* $\{\varphi_0, \varphi_1\}$ *are* $(4, L)$*-self-affine. For a verification take again the substitution* $\theta : \{0, 1\} \to \{0, 1\}^*$ *which is given by* $\theta(0) = 1\,0\,0\,0$ *and* $\theta(1) = 0\,1\,1\,1$, *and* $L(x, y) = (4x, 2y)$. *Then we obtain for all* $q \in \{0, 1\}$,$h \in [4]$, *and* $t \in [0, 1]$

$$\varphi_q\left(\frac{h+t}{4}\right) - \varphi_q\left(\frac{h}{4}\right) = L^{-1}(\varphi_{\theta_h(q)}(t)).$$

Remark In general, if $f : [0, 1] \to \mathbb{R}$ is a (m, α)-self-affine function, the curve $\gamma_f(t) = (t, f(t))$ for all $t \in [0, 1]$ is a (m, L)-self-affine curve for the same substitution θ and $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$L(x, y) = \begin{pmatrix} m & 0 \\ 0 & m^\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Similar as before we assume that $\mathcal{M} = (Q, [m], B, \sigma, \tau)$ is a sequential machine with $B \subset \mathbb{R}^2$ such that $\tau(q, 0) = (0, 0)$ for all $q \in Q$. We say that $\mathcal{M}$ is L-*consistent*, if for all $q \in Q$ and for all $t \in [0, 1]$ with two different m-adic representations

$$t = \sum_{j=1}^{\infty} a_j m^{-j} = \sum_{j=1}^{\infty} \tilde{a}_j m^{-j}$$

and

$$\begin{aligned} f_q^*(a_1 \dots a_n) &= b_1 \dots b_n \\ f_q^*(\tilde{a}_1 \dots \tilde{a}_n) &= \tilde{b}_1 \dots \tilde{b}_n \end{aligned}$$

for all $n \in \mathbb{N}$, it follows that

$$\sum_{j=1}^{\infty} L^{-j}(b_j) = \sum_{j=1}^{\infty} L^{-j}(\tilde{b}_j).$$

Proposition 3. *Let $\mathcal{M} = (Q, [m], B, \sigma, \tau)$ be an L-consistent sequential machine such that $B \subset \mathbb{R}^2$, $(0,0) \in B$, and $\tau(q,0) = 0$ for all $q \in Q$. Then each $q \in Q$ gives rise to a sequential function f_q^*. The sequential function generates a continuous function f_q that is (m, L)-self-affine.*

Proposition 4. *Let $\varphi : [0,1] \to \mathbb{R}^2$ be an (m, L)-self-affine curve. Then there exists an alphabet $B \subset \mathbb{R}^2$ and an L-consistent sequential machine $\mathcal{M} = (Q, [m], B, \sigma, \tau)$, such that φ is generated by $f_q^* : [m]^* \to B^*$ for some $q \in Q$.*

Proof. Let $x_0, \dots, x_{N-1} : [0,1] \to \mathbb{R}^2$ be the curves, such that $\varphi = x_0$, and let $\theta : [N] \to [N]^*$ be the substitution such that for all $(j, h) \in [N] \times [m]$ and $t \in [0,1]$

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = L^{-1}(x_{\theta_h(j)}(t)).$$

In the same way as in Proposition 1 we find the sequential machine $\mathcal{M} = (Q, A, B, \sigma, \tau)$ and the sequential function $f_{x_0}^*$ generating $\varphi : [0,1] \to \mathbb{R}^2$ by the following definition.

- the set of states is $Q = \{x_0, \dots, x_{N-1}\}$,
- the input alphabet is $A = [m]$,
- the output alphabet is

$$B = \left\{ L \circ x_j\left(\frac{h}{m}\right) \mid j \in [N], h \in [m] \right\},$$

- the transition function is given by $\sigma(x_j, h) = x_{\theta_h(j)}$,
- the output function is $\tau(x_j, h) = L \circ x_j\left(\frac{h}{m}\right)$.

Again, the L-consistency of $\mathcal{M} = (Q, [m], B, \sigma, \tau)$ can be concluded from the continuity of the curves $x_0, \dots, x_{N-1}$, and the condition $x_j(0) = 0$ for all $j \in [N]$ implies that $\tau(q, 0) = (0,0)$ holds for all $q \in Q$. □

Now we consider in which way (m, L)-self-affine curves are related to the recurrent curves studied in [14]. Let $\theta : Q \to Q^*$ be a substitution and $\mu : Q^* \to \mathbb{R}^2$ a homomorphism. Suppose that we can find a linear expanding map $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, such that $L_\theta \circ \mu = \mu \circ \theta$. According to [14] we will call a system (Q, θ, μ, L) with $L \circ \mu = \mu \circ \theta$ an L-system.

Let $\mathcal{H}(\mathbb{R}^2)$ denote the metric space of all non-empty compact subsets of $\mathbb{R}^2$ together with the Hausdorff metric (see [22]). The polygon map $K : Q \to \mathcal{H}(\mathbb{R}^2)$ is defined by

$$K(q) = \{t \cdot \mu(q) : t \in [0,1]\}.$$

It can be extended to $K : Q^* \to \mathcal{H}(\mathbb{R}^2)$ by

$$K(q_1 \dots q_n) = K(q_1) \cup \{K(q_2) + \mu(q_1)\} \cup \dots \cup \{K(q_n) + \mu(q_1 \dots q_{n-1})\}.$$

It is proved in [14] that the sequence of polygons $L_\theta^{-n}(K(\theta^n(q_0)))$ converges with respect to the Hausdorff metric to a set $K_\theta(q_0)$, which is a curve.

We set $\theta^n(q_0) = w_{n1} \dots w_{nr(n)}$. In order to give a parametrization for $K_\theta(q_0)$, we define $\varphi_n : [0,1] \to \mathbb{R}^2$ in such a way, that $\varphi_n(t)$ is parametrized uniformly on the interval $[jr(n)^{-1}, (j+1)r(n)^{-1}]$ for all $j \in [r(n)]$. It is shown in [14] that the polygonal curves $\varphi_n(t)$ converge uniformly to a curve $\varphi : [0,1] \to \mathbb{R}^2$, which is a parametrization of $K_\theta(q_0)$.

Proposition 5. *Let $\varphi : [0,1] \to \mathbb{R}^2$ be the parametrization of a recurrent curve $K_\theta(q_0)$, where $\theta : Q \to Q^*$ is a substitution of constant length m. Then the curve φ is (m, L_θ)-self-affine.*

PROOF. We consider the parametrizations $\varphi_{q,n} : [0,1] \to \mathbb{R}^2$ of $L^{-n}(K(\theta^n(q)))$ and the parametrizations $\varphi_q(t)$ of $K_\theta(q)$ for all $q \in Q$, which is explicitly

$$\varphi_q(t) = \lim_{n \to \infty} \varphi_{q,n}(t).$$

Then for all $h \in [m]$, $n \in \mathbb{N}$ and $t \in [0,1]$ it follows that

$$\varphi_{q,n}\left(\frac{h+t}{m}\right) - \varphi_{q,n}\left(\frac{h}{m}\right) = L_\theta^{-1}(\varphi_{\theta_h(q),n-1}(t)).$$

In the limit $n \to \infty$ we therefore obtain for all $q \in Q$, $h \in [m]$ and $t \in [0,1]$

$$\varphi_q\left(\frac{h+t}{m}\right) - \varphi_q\left(\frac{h}{m}\right) = L_\theta^{-1}(\varphi_{\theta_h(q)}(t)).$$

$\square$

Finally we will see that each (m, L)-self-affine curve $\varphi : [0,1] \to \mathbb{R}^2$ is a recurrent curve corresponding to a substitution $\theta : Q \to Q^*$ of constant length.

Proposition 6. *Let $\varphi : [0,1] \to \mathbb{R}^2$ be an (m, L)-self-affine curve. Then there is a homomorphism $\mu : Q^* \to \mathbb{R}^2$, such that $L \circ \mu = \mu \circ \theta$, and a state $q_0 \in Q$ such that φ is the parametrization of $K_\theta(q_0)$.*

Proof. Let $Q = \{x_0, \ldots, x_{N-1}\}$ be the family of curves, such that $x_0 = \varphi$, and let $\theta : Q \to Q^*$ be the substitution such that for all $j \in [N]$, $h \in [m]$, and $t \in [0,1]$

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = L^{-1}(x_{\theta_h(j)}(t)),$$

where $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear expanding map. The substitution $\theta : Q \to Q^*$ is given by $\theta(x_j) = x_{\theta_0(j)} \ldots x_{\theta_{m-1}(j)}$ for all $x_j \in Q$. The homomorphism $\mu : Q^* \to \mathbb{R}^2$ is given by $\mu(x_j) = x_j(1)$. Then for $q = x_j \in Q$

$$\begin{aligned}
\mu(\theta(q)) &= \mu(x_{\theta_0(j)} \ldots x_{\theta_{m-1}(j)}) \\
&= x_{\theta_0(j)}(1) + \ldots + x_{\theta_{m-1}(j)}(1) \\
&= \sum_{h=0}^{m-1} L\left(x_j\left(\frac{h+1}{m}\right) - x_j\left(\frac{h}{m}\right)\right) \\
&= L(x_j(1)) \\
&= L(\theta(q)),
\end{aligned}$$

which shows that $\mu(\theta(q)) = L(\mu(q))$ for all $q \in Q$. Therefore the conditions from [14] are satisfied for $\theta : Q^* \to Q^*$, $\mu : Q^* \to \mathbb{R}^2$, and $L : \mathbb{R}^2 \to \mathbb{R}^2$. If g_q denotes the curve constructed in [14] corresponding to $q \in Q$, the curves g_q and φ_q coincide. Since they are continuous maps, it is sufficient to verify that for all $h \in [m^k]$

$$\varphi_q\left(\frac{h}{m^k}\right) = g_q\left(\frac{h}{m^k}\right).$$

This follows directly from the definition by induction with respect to k. □

3 Cantor Representations of Reals in the Unit Interval Corresponding to a Substitution

The idea of representing a real number $z \in [0,1]$ in its m-adic expansion is motivated by the geometric fact that each $z \in [0,1]$ is the intersection of at least one sequence of subintervals of length m^{-n}, where the $(n+1)$st interval is obtained from the nth interval by a decomposition into m smaller intervals all of which having the same length.

In the same way each $z \in [0,1]$ can be described as the intersection of a sequence of nested intervals, where again the $(n+1)$st interval is obtained from the nth interval by an equidistant decomposition into subintervals, but the number of subintervals depends on the nth interval itself. This process leads to the Cantor representation of real numbers in the unit interval $[0,1] = I$ (see [7]).

Let Q be an alphabet and $\theta : Q \to Q^*$ a substitution, such that $r(q) \geq 2$ is the number of symbols in the word $\theta(q)$ for all $q \in Q$. We assign a finite directed graph Γ_θ with labeled arrows to the substitution θ. The vertices of Γ_θ are the letters of the alphabet Q, and an arrow has the initial vertex q_1, the terminal vertex q_2, and the label $h \in [r(q_1)]$, when $\theta_h(q_1) = q_2$.

Example 12. *The graph* Γ_θ *for* $\theta : \{a,b\} \to \{a,b\}^*$, *where* $a \mapsto a\,b\,a$ *and* $b \mapsto b\,a$ *is shown in Figure 7.*

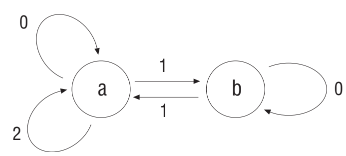

Figure 7: The graph for the substitution θ of Example 12.

We call a (finite or infinite) sequence of arrows $(e_j)_{j\geq 1}$ a directed path in Γ_θ, if the terminal vertex of e_j is the initial vertex of e_{j+1} for all j. The initial vertex of e_1 is the initial vertex of the path. We denote by Σ_q the set of all infinite paths in Γ_θ with initial vertex q. To each directed path $\underline{e} = (e_j)_{j\geq 1}$ in Σ_q corresponds a number

$$\pi_q(\underline{e}) = \sum_{j=1}^{\infty} \frac{h_j}{r(q_0)\cdot \ldots \cdot r(q_{j-1})},$$

where q_j are the initial vertices of e_j and $h_j \in [r(q_{j-1})]$ are the labels of e_j for all $j \in \mathbb{N}$. Here we set $q_0 = q$. Note that $\pi_q(\underline{e}) \in [0,1]$ for all $\underline{e}$. Therefore π_q is a map from Σ_q to the unit interval.

We will show that $\pi_q : \Sigma_q \to [0,1]$ is onto, i.e. for each $t \in [0,1]$ there is a directed path $\underline{e} \in \Sigma_q$, such that $\pi_q(\underline{e}) = t$. Let the tree T_q with root $q \in Q$ be the set of all finite paths in Γ_θ starting in $q \in Q$. We define q to be the vertex of the 0th generation in the tree T_q. Assume that the vertices of the kth generation in T_q are already defined for $k \in \mathbb{N}$. Then $\overline{v}$ is a vertex of the $(k+1)$st generation with label $\overline{q}$ if there is a vertex $\tilde{v}$ of the kth generation with label $\tilde{q}$ and a $h \in [r(\tilde{q})]$, such that $\theta_h(\tilde{q}) = \overline{q}$.

Example 13. *The first two generations of the trees* T_a *and* T_b *corresponding to the substitution* $\theta : \{a,b\} \to \{a,b\}^*$ *given by* $a \mapsto a\,b\,a$ *and* $b \mapsto b\,a$ *are illustrated in the Figures 8 and 9.*

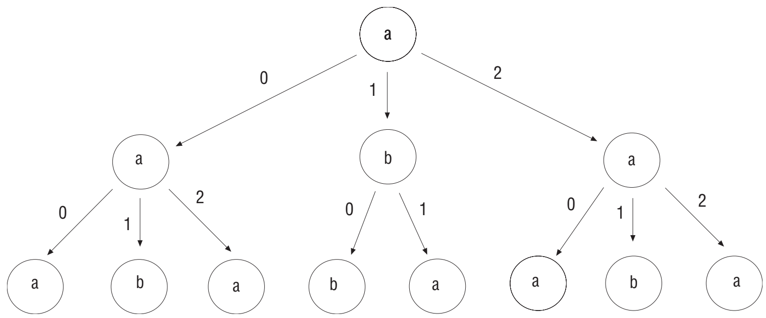

Figure 8: The tree T_a corresponding to the substitution θ of Example 13.

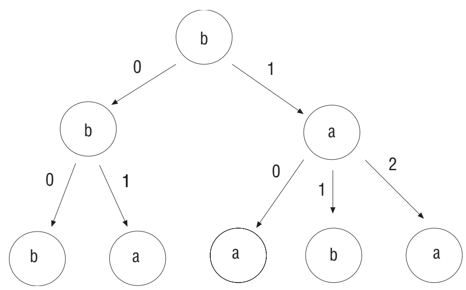

Figure 9: The tree T_b corresponding to the substitution θ of Example 13.

To each vertex v of the tree T_q corresponds a subinterval of $[0,1]$, which can be defined as follows. Let $\underline{e} = e_1 \dots e_k$ be the unique path in T_q connecting the root $q = q_0$ with v. If the label of e_j is $h_j \in [r(q_{j-1})]$ for $j = 1, \dots, k$, the interval $I_{e_1 \dots e_k}$ corresponding to $\underline{e}$ is

$$I_{e_1 \dots e_k} = \psi_{h_1, r(q_0)} \circ \dots \circ \psi_{h_k, r(q_{k-1})}(I),$$

where $\psi_{h,r} : [0,1] \to [0,1]$ is defined for $t \in [0,1]$ by

$$\psi_{h,r}(t) = \frac{h+t}{r}.$$

Then $I_{e_1 \dots e_k} \subset I_{e_1 \dots e_{k-1}}$, and $I_{e_1 \dots e_k} = \cup I_{e_1 \dots e_k e_{k+1}}$. These properties imply that there is at least one path $\underline{e} = (e_k)_{k \geq 1} \in \Sigma_q$ such that $t \in I_{e_1 \dots e_n}$ for all $n \in \mathbb{N}$. From the definition of π_q we have $\pi_q(\underline{e}) = \cap_{n=1}^{\infty} I_{e_1 \dots e_n}$, and therefore

$t = \pi_q(\underline{e})$. In particular,

$$t = \sum_{n=1}^{\infty} \frac{h_n}{r(q_0) \cdot \ldots \cdot r(q_{n-1})}.$$

We call this representation of t the (θ, q)-representation.

Remarks 1. If $\theta : Q \to Q^*$ is a substitution of constant length, i.e. $r(q) = r$ for all $q \in Q$, then the (θ, q)-representation is the r-adic representation.

2. The (θ, q)-representation of a number $t \in [0, 1]$ is not unique in general.

3. For each $t \in [0, 1]$ and a path $\underline{e}$ in the tree T_q such that $\pi(\underline{e}) = t$, the (θ, q)-representation of t corresponding to $\underline{e}$ is the Cantor representation with respect to the sequence $(r(q_n))_{n \geq 0}$ (see [7], [6], pp. 215–221).

4 (θ, L)-Self-Affine Curves

We have seen in Section 2 that recurrent curves $K_\theta(q)$ in the sense of Dekking corresponding to a substitution $\theta : Q \to Q^*$ of constant length m, a homomorphism $\mu : Q^* \to \mathbb{R}^2$, and a linear expanding map $L : \mathbb{R}^2 \to \mathbb{R}^2$ can be described by an (m, L)-self-affine curve $\varphi : [0, 1] \to \mathbb{R}^2$. In particular, φ can be generated by a sequential function.

In this section we will show how sequential machines can be generalized to produce recurrent curves corresponding to arbitrary substitutions. The functions generated by these generalized sequential machines will be called (θ, L)-self-affine curves. During the course of this section let $\theta : Q \to Q^*$ be a substitution such that $r(q) \geq 2$ is the number of symbols of $\theta(q)$ for all $q \in Q$. In particular, (m, L)-self-affine curves are (θ, L)-self-affine curves, where θ is a substitution of constant length.

Definition 5. *Let $\theta : Q \to Q^*$ be a substitution and $L : \mathbb{R}^2 \to \mathbb{R}^2$ a linear expanding map. A curve $\varphi : [0, 1] \to \mathbb{R}^2$ is called a (θ, L)-self-affine curve, if the following conditions a. and b. are satisfied.*

(a) There is a family of curves $\{x_q : [0, 1] \to \mathbb{R}^2 \mid q \in Q\}$ with $x_q(0) = (0, 0)$ for all $q \in Q$ and $x_0 = \varphi$.

(b) For all $t \in [0, 1]$, for $q \in Q$ and $\tilde{q} = \theta_h(q)$ with $h \in [r(q)]$ one has

$$x_q\left(\frac{h + t}{r(q)}\right) - x_q\left(\frac{h}{r(q)}\right) = L^{-1}(x_{\tilde{q}}(t)).$$

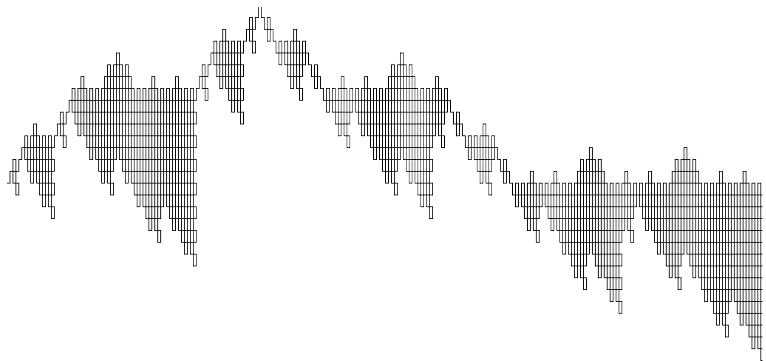

Figure 10: Third approximation to the curve f_a of M. F. Dekking.

Example 14. Dekking's Square-Filling Curve, *see [14], ex. 4.7.*
Let $Q = \{0, 1, 2, 3\}$ be the set of states and $\theta : Q \to Q^*$ the substitution

$$\begin{aligned} \theta(0) &= 0\,1\,0\,3\,0\,3\,0\,1, \\ \theta(1) &= 2\,1\,2\,1\,0\,3\,0\,1, \\ \theta(2) &= 2\,1\,2\,1\,2\,3\,0\,3\,2\,1\,2\,3, \\ \theta(3) &= 0\,3\,2\,3. \end{aligned}$$

Let $\mu : Q \to \mathbb{R}^2$ be defined by $\mu(0) = (1, 0)$, $\mu(1) = (0, 2)$, $\mu(2) = (-1, 0)$, and $\mu(3) = (0, -2)$. It is easy to check that for $L(x, y) = (4x, 2y)$ we have $L \circ \mu = \mu \circ \theta$. We consider the curves $\{f_0, f_1, f_2, f_3\}$, where $f_q(t) = K_\theta(q)$ for $q \in Q$. Then we obtain

$$\begin{aligned} f_q\left(\tfrac{h+t}{8}\right) - f_q\left(\tfrac{h}{8}\right) &= L^{-1} f_{\theta_h(q)}(t) && \text{for } q \in \{0,1\} \text{ and } h \in [8], \\ f_2\left(\tfrac{h+t}{12}\right) - f_2\left(\tfrac{h}{12}\right) &= L^{-1} f_{\theta_h(2)}(t) && \text{for } h \in [12], \\ f_3\left(\tfrac{h+t}{4}\right) - f_3\left(\tfrac{h}{4}\right) &= L^{-1} f_{\theta_h(3)}(t) && \text{for } h \in [4]. \end{aligned}$$

Hence, the family of curves $\{f_0, f_1, f_2, f_3\}$ is (θ, L)-self-affine.

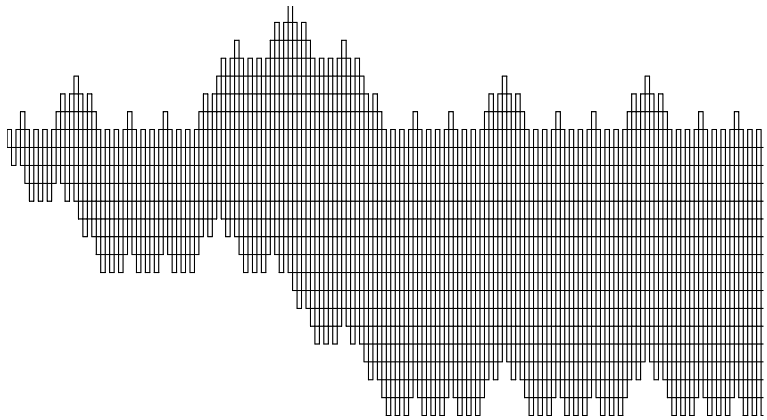

Figure 11: Third approximation to the curve f_b of M. F. Dekking.

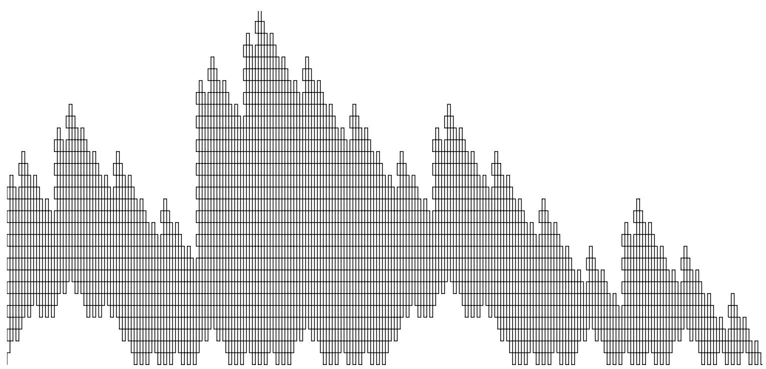

Figure 12: Third approximation to the curve f_c of M. F. Dekking.

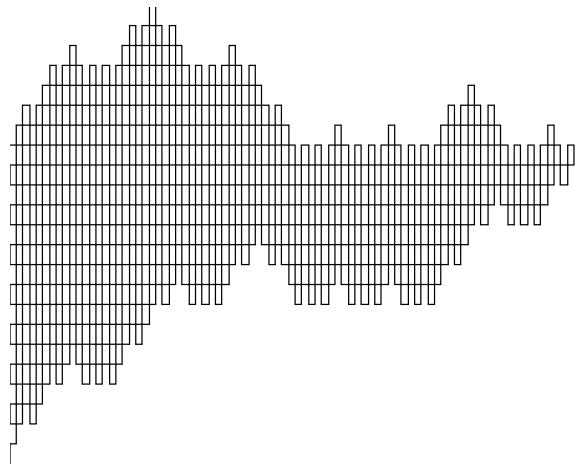

Figure 13: Third approximation to the curve f_d of M. F. Dekking.

Definition 6. *Let $\theta : Q \to Q^*$ be a substitution, and $r_{max} = \max\{r(q) : q \in Q\}$. Then the sequential machine $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ corresponding to θ consists of the set of states Q, the initial alphabet $A = [r_{max}]$, and an output alphabet $B \subset \mathbb{R}^2$ containing $(0,0)$. Transition function σ and output function τ are defined on the set*

$$\Omega = \{(q,h) : q \in Q, h \in [r(q)]\} \subset Q \times [r_{max}].$$

The transition function is given by $\sigma(q,h) = \theta_h(q)$. The output function satisfies $\tau(q,0) = (0,0)$ for all $q \in Q$.

The transition and output function can be extended to all $(q, h_1 \dots h_n) \in Q \times [r_{max}]^*$, where $\underline{e} = e_1 \dots e_n$ is a path in the tree T_q and $h_j \in [r(j-1)]$ are the labels of e_j for all $j \in \mathbb{N}$, $q_0 = q$.

Example 15. The Sequential Machine $\mathcal{M}_\theta$ *for the substitution $\theta : \{a,b\} \to \{a,b\}^*$ given by $a \mapsto a\,b\,a$ and $b \mapsto b\,a$ is shown in Figure 14. Here $B = \{d_0, d_1, d_2, \delta_0, \delta_1\} \subset \mathbb{R}^2$.*

Now let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear expanding map, and let $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ be a sequential machine with $B \subset \mathbb{R}^2$ such that $\tau(q,0) = (0,0)$ for all $q \in Q$. We say that $\mathcal{M}_\theta$ is *L-consistent*, if for all $q \in Q$ and all $t \in [0,1]$ with two different (θ, q)-representations

$$t = \sum_{n=1}^{\infty} \frac{h_n}{r(q_0) \cdot \ldots \cdot r(q_{n-1})} = \sum_{n=1}^{\infty} \frac{\tilde{h}_n}{r(\tilde{q}_0) \cdot \ldots \cdot r(\tilde{q}_{n-1})},$$

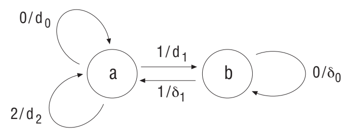

Figure 14: The sequential machine $\mathcal{M}_\theta$ for the substitution θ of Example 15.

where

$$\begin{array}{rcl} f_q^*(h_1 \dots h_n) & = & d_1 \dots d_n \\ f_q^*(\tilde{h}_1 \dots \tilde{h}_n) & = & \tilde{d}_1 \dots \tilde{d}_n \end{array}$$

holds for all $n \in \mathbb{N}$ we obtain

$$\sum_{n=1}^{\infty} L^{-n}(d_n) = \sum_{n=1}^{\infty} L^{-n}(\tilde{d}_n).$$

If $\mathcal{M}_\theta$ is L-consistent, we find for each $q \in Q$ a curve $f_q : [0,1] \to \mathbb{R}^2$. If $0.h_1\, h_2 \dots$ is the (θ, q)-representation of t corresponding to the path $\underline{e} = (e_n)_{n \geq 1}$ with initial vertex q, and if for all $n \in \mathbb{N}$

$$f_q^*(h_1 \dots h_n) = d_1 \dots d_n,$$

then f_q is defined by

$$f_q(t) = \sum_{n=1}^{\infty} L^{-n}(d_n).$$

The condition $\tau(q,0) = (0,0)$ implies that $f_q(0) = (0,0)$ for all $q \in Q$.

Proposition 7. *The curves $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ generated by an L-consistent sequential machine $\mathcal{M}_\theta$, which corresponds to a substitution θ are (θ, L)-self-affine.*

PROOF. Suppose that $\sigma(q_0, h_1) = \tilde{q}_0$ for $h_1 \in [r(q_0)]$ and

$$t = \sum_{n=1}^{\infty} \frac{h_n}{r(q_0) \cdot \ldots \cdot r(q_{n-1})} = \sum_{n=1}^{\infty} \frac{\tilde{h}_n}{r(\tilde{q}_0) \cdot \ldots \cdot r(\tilde{q}_{n-1})}$$

are the (θ, q_0)-representation of $t \in [0,1]$ and the $(\theta, \tilde{q}_0)$-representation respectively. Note that $q_1 = \tilde{q}_0$. Then the (θ, q_0)-representation of $\frac{t+h_1}{r(q_0)}$ is

$$\frac{t + h_1}{r(q_0)} = \frac{h_1}{r(q_0)} + \sum_{n=1}^{\infty} \frac{\tilde{h}_n}{r(q_0) \cdot r(\tilde{q}_0) \cdot \ldots \cdot r(\tilde{q}_{n-1})}$$

From the definition of f_{q_0} it follows that for $d = \tau(q_0, h_1)$

$$f_{q_0}\left(\frac{t+h_1}{r(q_0)}\right) = L^{-1}(d) + L^{-1}(f_{\tilde{q}_0}(t)).$$

The condition $\tau(q_0, 0) = (0, 0)$ implies that

$$f_{q_0}\left(\frac{h_1}{r(q_0)}\right) = L^{-1}(d),$$

and therefore,

$$f_{q_0}\left(\frac{t+h_1}{r(q_0)}\right) - f_{q_0}\left(\frac{h_1}{r(q_0)}\right) = L^{-1}(f_{\tilde{q}_0}(t)).$$

□

Proposition 8 shows that the opposite assertion is also true.

Proposition 8. *Let $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ be a family of (θ, L)-self-affine curves. Then there is an L-consistent sequential machine $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ corresponding to a substitution θ, such that $\mathcal{M}_\theta$ generates the curves f_q.*

PROOF. Let $\mathcal{M}_\theta$ be the sequential machine corresponding to the substitution $\theta : Q \to Q^*$. The output alphabet $B \subset \mathbb{R}^2$ is the set

$$B = \{(0,0)\} \bigcup \{f_{\theta(q,0)}(1) + \ldots + f_{\theta(q,h)}(1) \mid q \in Q, h \in [r(q)]\},$$

and the output function is given by $\tau(q, 0) = (0, 0)$ for all $q \in Q$ and

$$\tau(q,h) = f_{\theta(q,0)}(1) + \ldots + f_{\theta(q,h-1)}(1)$$

for all (q, h) with $h \in [r(q)]$. Since the curves f_q are continuous, $\mathcal{M}_\theta$ is L-consistent, the curve generated by the sequential function f_q^* is f_q. □

Finally we want to discuss how the (θ, L)-self-affine curves are related to the L-systems (Q, θ, μ, L) in the sense of F. M. Dekking, where $\theta : Q \to Q^*$ in general is a substitution of nonconstant length, L is a linear expanding map, and $\mu : Q \to \mathbb{R}^2$ is a homomorphism, such that $L \circ \mu = \mu \circ \theta$. Then (Q, θ, μ, L) generates a curve $\varphi_q : [0,1] \to \mathbb{R}^2$ given by

$$\varphi_q(t) = \lim_{n\to\infty} \varphi_{q,n}(t),$$

where $\varphi_{q,n}(t)$ is the parametrization of the polygon $K_{q,n} = L^{-n}(K(\theta^n(q)))$. Here we denote with $K : Q^* \to \mathbb{R}^2$ the polygon map.

Proposition 9. *The curves $\varphi_q : [0,1] \to \mathbb{R}^2$ are (θ, L)-self-affine.*

PROOF. From the definition of Dekking follows the recursive relation

$$\varphi_{q,n+1}\left(\frac{h+t}{r(q)}\right) - \varphi_{q,n+1}\left(\frac{h}{r(q)}\right) = L^{-1}\varphi_{\theta_h(q),n}(t)$$

for all $q \in Q$, $t \in [0,1]$ and $n \in \mathbb{N}$. If n tends to infinity we obtain the assertion. □

Proposition 10. *Let $\mathcal{M}_\theta$ be a L-consistent sequential machine generating the curves $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$. There is a L-system with substitution $\theta : Q \to Q^*$ such that f_q is the parametrization of the curve*

$$K_q = \lim_{n\to\infty} L^{-n}(K(\theta^n(q))).$$

PROOF. The homomorphism $\mu : Q \to \mathbb{R}^2$ can be defined by $\mu(q) = f_q(1)$. Then (Q, θ, μ, L) is an L-system, i.e. $L \circ \mu = \mu \circ \theta$, such that the curves f_q are generated by (Q, θ, μ, L). □

5 (θ, γ)-Self-Affine Curves

We have seen in the previous section how sequential functions generate curves in $\mathbb{R}^2$ which are (θ, L)-self-affine with respect to a linear expanding map $L : \mathbb{R}^2 \to \mathbb{R}^2$. Now we consider more generally sequential functions whose interpretation leads to curves in $\mathbb{R}^2$ which are self-affine with respect to a finite number of linear expanding maps. Just as in the previous section $\theta : Q \to Q^*$ is a substitution such that $r(q) \geq 2$ is the length of the word $\theta(q)$ for all $q \in Q$. Let

$$\gamma : \bigcup_{q\in Q} \{q\} \times [r(q)] \longrightarrow GL(\mathbb{R}^2)$$

be a map into the group $GL(\mathbb{R}^2)$ of all invertible linear maps of $\mathbb{R}^2$, such that $\gamma(q,h)$ are expanding maps for all $q \in Q$ and $h \in [r(q)]$.

Definition 7. *A curve $\varphi : [0,1] \to \mathbb{R}^2$ is called a (θ, γ)-self-affine curve, if the following conditions are satisfied.*

(a) There is a family of curves $\{x_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ with $x_q(0) = (0,0)$ for all $q \in Q$ and $x_0 = \varphi$.

(b) For all $t \in [0,1]$, for $q \in Q$ and $\tilde{q} = \theta_h(q)$ with $h \in [r(q)]$ holds

$$x_q\left(\frac{h+t}{r(q)}\right) - x_q\left(\frac{h}{r(q)}\right) = \gamma(q,h)^{-1}(x_{\tilde{q}}(t)).$$

If in particular θ is a substitution of constant length m, we also say that φ is a (m,γ)-self-affine curve.

Let $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ be a sequential machine with $B \subset \mathbb{R}^2$ and $\tau(q,0) = (0,0)$ for all $q \in Q$. If $\gamma : \bigcup_{q\in Q}\{q\} \times [r(q)] \to GL(\mathbb{R}^2)$ is a map such that $\gamma(q,h)$ is a linear expanding map for all $q \in Q$ and $h \in r(q)$, we say that $\mathcal{M}_\theta$ is *γ-consistent*, if for all $q \in Q$ and all $t \in [0,1]$ with two different (θ, q)-representations

$$t = \sum_{n=1}^{\infty} \frac{h_n}{r(q_0)\cdot \ldots \cdot r(q_{n-1})} = \sum_{n=1}^{\infty} \frac{\tilde{h}_n}{r(\tilde{q}_0)\cdot \ldots \cdot r(\tilde{q}_{n-1})},$$

where

$$\begin{array}{rcl} f_q^*(h_1 \ldots h_n) & = & d_1 \ldots d_n \\ f_q^*(\tilde{h}_1 \ldots \tilde{h}_n) & = & \tilde{d}_1 \ldots \tilde{d}_n \end{array}$$

holds for all $n \in \mathbb{N}$, we obtain that

$$\sum_{n=1}^{\infty} \alpha_n(d_n) = \sum_{n=1}^{\infty} \tilde{\alpha}_n(\tilde{d}_n),$$

with

$$\alpha_n = \prod_{j=1}^{n} \gamma(\sigma(q, h_1 \ldots h_{j-1}), h_j)^{-1},$$

$$\tilde{\alpha}_n = \prod_{j=1}^{n} \gamma(\sigma(q, \tilde{h}_1 \ldots \tilde{h}_{j-1}), \tilde{h}_j)^{-1}.$$

If $\mathcal{M}_\theta$ is a γ-consistent sequential machine, we find for each $q \in Q$ a curve $f_q : [0,1] \to \mathbb{R}^2$. If $0.h_1\,h_2\ldots$ is the (q,θ)-representation of t corresponding to the path $\underline{e} = (e_n)_{n\geq 1}$ with initial vertex q, and if $f_q^*(h_1 \ldots h_n) = d_1 \ldots d_n$ for all $n \in \mathbb{N}$, then f_q is defined by

$$f_q(t) = \sum_{n=1}^{\infty} \alpha_n(d_n).$$

Proposition 11. *The curves $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ generated by a γ-consistent sequential machine $\mathcal{M}_\theta$, which corresponds to a substitution θ are (θ,γ)-self-affine.*

Proposition 12. *Let $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ be a family of (θ,γ)-self-affine curves. Then there is a γ-consistent sequential machine $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ corresponding to a substitution θ, such that $\mathcal{M}_\theta$ generates the curves f_q.*

These propositions can be proved completely in the same way as the Propositions 6 and 7 in the previous section. Note that (θ, L)-self-affine curves are also (θ, γ)-self-affine.

Example 16. Hilbert's Square-Filling Curve.
Let $h : [0,1] \to [0,1]^2$ be the Hilbert curve and $Q = \{0\}$ and $\theta(0) = 0\,0\,0\,0$. The map $\gamma : [4] \to GL(\mathbb{R}^2)$ let be given by

$$\gamma(0) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \gamma(1) = \gamma(2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \gamma(3) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

Then it can be varified that the Hilbert curve is (θ, γ)-self-affine.

Example 17. Peano's Square-Filling Curve.
If $Q = \{0\}$ is the one point set, and

$$\theta(0) = \underbrace{0 \ldots 0}_{9 \ times} .$$

The Peano curve is (θ, γ)-self-affine, where $\gamma : [9] \to GL(\mathbb{R}^2)$ is defined by

$$\gamma(h) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \text{ for } h = 0,2,6,8, \quad \gamma(h) = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \text{ for } h = 1,7,$$

$$\gamma(h) = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \text{ for } h = 3,5, \quad \gamma(h) = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \text{ for } h = 4.$$

Example 18. Sierpinski's Gasket Gurve.
Let $Q = \{0\}$ be the one point set and $\theta(0) = 0\,0\,0$. Define $\gamma : [3] \to GL(\mathbb{R}^2)$, such that $\gamma(0)$ is a reflection along the direction of $\frac{\pi}{6}$ together with a scaling by a factor of 2, $\gamma(1)$ is just a scaling by the factor 2, and $\gamma(2)$ is a reflection along the direction of $-\frac{\pi}{6}$ together with a scaling by the factor 2. Hence,

$$\gamma(0) = \begin{pmatrix} 2 & 2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} \quad \gamma(1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \gamma(2) = \begin{pmatrix} 2 & -2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}.$$

Then the Sierpinski gasket curve is (θ, γ)-self-affine.

Example 19. Lévy's dragon.
Define $Q = \{0\}$ to be the one point set and $\theta(0) = 0\,0$, and let $\gamma : [2] \to GL(\mathbb{R}^2)$ be defined by

$$\gamma(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \gamma(1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then the Levy dragon $l : [0,1] \to \mathbb{R}^2$ is a (θ, γ)-self-affine curve.

Example 20. Heighway's Dragon Curve with Unfolding Angle $\alpha \in (\frac{\pi}{3}, \pi]$.
Let $f_0 : [0,1] \to \mathbb{R}^2$ be the Heighway dragon curve and let $f_1 : [0,1] \to \mathbb{R}^2$ be defined by $f_1(t) = f_0(1-t) - (1,0)$. Set $\beta = \frac{\pi - \alpha}{2}$. Define $\theta : \{0,1\} \to \{0,1\}^*$ by $\theta(0) = \theta(1) = 0\,1$ and $\gamma : \{0,1\} \times [2] \to GL(\mathbb{R}^2)$ by

$$\begin{aligned}
\gamma(0,0) &= 2\cos\beta \begin{pmatrix} \cos(-\beta) & -\sin(-\beta) \\ \sin(-\beta) & \cos(-\beta) \end{pmatrix}, \\
\gamma(0,1) &= 2\cos\beta \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}, \\
\gamma(1,0) &= 2\cos\beta \begin{pmatrix} \cos(-(\alpha+\beta)) & -\sin(-(\alpha+\beta)) \\ \sin(-(\alpha+\beta)) & \cos(-(\alpha+\beta)) \end{pmatrix}, \\
\gamma(1,1) &= 2\cos\beta \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.
\end{aligned}$$

Then for all $(q,h) \in \{0,1\} \times [2]$ holds

$$f_q\left(\frac{t+h}{2}\right) - f_q\left(\frac{h}{2}\right) = \gamma(q,h)^{-1} f_{\theta_h(q)}(t).$$

In contrast to the case where f_0 is described as a $(4,L)$-self-affine curve, the condition that the angle α is rational is not necessary (see the remark on page 458).

Example 21. Steinitz's Nowhere Differentiable Function, *see [54].*
E. Steinitz constructed a nowhere differentiable continuous function $f : [0,1] \to \mathbb{R}$. We consider the graph of f as a curve $\varphi_0 : [0,1] \to \mathbb{R}^2$, $\varphi(t) = (t, f(t))$ for all $t \in [0,1]$. Take the set of states $Q = \{0,1\}$ and the substitution $\theta(0) = \theta(1) = 0\,0\,1\,1\,0\,0$. Then θ is a substitution of constant length 6. Let $\gamma : Q \times [6] \to GL(\mathbb{R}^2)$ be the map defined by

$$\gamma(0,h) = \begin{pmatrix} 6 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \text{for } h = 0,1,4,5, \quad \gamma(0,h) = \begin{pmatrix} 6 & 0 \\ 0 & \frac{6}{5} \end{pmatrix} \text{for } h = 2,3,$$

$$\gamma(1,h) = \begin{pmatrix} 6 & 0 \\ 0 & -\frac{3}{2} \end{pmatrix} \text{for } h = 0,1,4,5, \quad \gamma(1,h) = \begin{pmatrix} 6 & 0 \\ 0 & -\frac{6}{5} \end{pmatrix} \text{for } h = 2,3.$$

Then we obtain the (θ,γ)-self-affine curves φ_0 and φ_1.

Example 22. Tagaki's Nowhere Differentiable Function, *see [57], [30], [47].*

Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined for all $t \in \mathbb{R}$ by

$$g(t) = |t - \lfloor t + \frac{1}{2} \rfloor|,$$

where $\lfloor a \rfloor = \max\{n \in \mathbb{Z} : n \leq a\}$. Then the function of T. Tagaki is for $t \in \mathbb{R}$

$$f(t) = \sum_{k=0}^{\infty} 2^{-n} g(2^n t).$$

It is continuous and nowhere differentiable. K. Knopp considered the functions

$$f(t) = \sum_{n=1}^{\infty} \alpha^n g(b^n t),$$

with $b \in 2\mathbb{N}$, $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}$ (see [30], p. 18). He proved that it is nowhere differentiable for $\alpha b > 4$. The function studied by L. B. van der Waerden is the special case of K. Knopp's functions, which is obtained for $\alpha = \frac{1}{10}$ and $b = 10$. It is also continuous and nowhere differentiable (see also [23], [58]).

Here we shall consider the function of T. Tagaki. Similar arguments can be applied for the other examples. For $t \in [0,1]$ set $\varphi(t) = (t, f(t))$ and $Q = \{0\}$ a one point set. Then $\theta(0) = 0\,0$ is a substitution of length 2. Let $\gamma : Q \times [2] \to GL(\mathbb{R}^2)$ be defined by

$$\gamma(0) = \begin{pmatrix} 2 & 0 \\ -2 & 2 \end{pmatrix}, \qquad \gamma(1) = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}.$$

Then φ is a (θ,γ)-self-affine curve.

Example 23. Salem's Singular Sunction

R. Salem constructed a continuous function $s : [0,1] \to \mathbb{R}$, which is strictly increasing and $s'(x) = 0$ for almost all $x \in [0,1]$. With $Q = \{0\}$, $\theta(0) = 0\,0$ and $\gamma : [2] \to GL(\mathbb{R}^2)$ defined by

$$\gamma(0) = \begin{pmatrix} 2 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad \gamma(1) = \begin{pmatrix} 2 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$$

Salem's function is $(2,\gamma)$-self-affine, where $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0\lambda_1 > 0$, and $\lambda_0 + \lambda_1 = 1$, and moreover $\lambda_0 \neq \frac{1}{2}$ (see [50]).

Remark Tagaki's function f is not (m, α)-self-affine for any $\alpha > 0$. Otherwise it would be α-Hölder continuous, which contradicts the known relation

$$f(t+h) - f(t) = O\left(|h| \log(|h|^{-1})\right)$$

for all $t \in [0,1]$ and $h \to 0$ (see [24]).

Example 24. The Self-Affine Curves of G. de Rham, *see [43], [44], [45], [46].*

G. de Rham considered a curve $f : [0,1] \to \mathbb{R}^2$ which is (θ, γ)-self-affine with $\theta : \{0\} \to \{0\}^*$, $\theta(0) = 0\,0$ and $\gamma : [2] \to GL(\mathbb{R}^2)$, having the property that $\gamma(0)^{-1}(x_1, y_1) = \gamma(1)^{-1}(x_0, y_0)$, where (x_j, y_j) is the fixed point of the contraction map $\gamma(j)^{-1}$, $j \in [2]$. The curve is the unique non-zero solution of the equation

$$f\left(\frac{h+t}{2}\right) - f\left(\frac{h}{2}\right) = \gamma(h)^{-1} f(h),$$

for $h \in [2]$. Without loss of generality we assume that $f(0) = (0,0)$. Particular cases are the curves for which $\gamma(0)^{-1}(z) = a\overline{z}$ and $\gamma(1)^{-1}(z) = a + (1-a)\overline{z}$ for $z \in \mathbb{C}$ and $a \in \mathbb{C}$. G. de Rham observed that for $a = \frac{1}{2} + \frac{i\sqrt{3}}{6}$ the corresponding (θ, γ)-self-affine curve is von Koch's curve (see [31]). For $|a - \frac{1}{2}| = \frac{1}{2}$ with $a \in \mathbb{C}$ we obtain the area filling curves of E. Cesàro (see [9]), W. Sierpinski (see [51]), and G. Polya (see [41]). For further examples see [44], [45], [46].

Example 25. The Scaling Function ${}_2\Phi$ of I. Daubechies, *see [10], [11], [40].*

The Daubechies scaling function ${}_2\Phi = \Phi$, which is shown in Figure 15, is a continuous function with support $[0,3]$ that satisfies the scaling equation

$$\Phi(x) = a\Phi(2x) + \overline{1-a}\Phi(2x-1) + (1-a)\Phi(2x-2) + \overline{a}\Phi(2x-3),$$

where $a = \frac{1+\sqrt{3}}{4}$ and $\overline{\alpha + \beta\sqrt{3}} = \alpha - \beta\sqrt{3}$ for $\alpha, \beta \in \mathbb{Q}$.

It is possible to generate the graph $\lambda(t) = (t, \Phi(t))$ for $t \in [0,3]$ by a sequential machine using the dyadic representations of the real numbers in $[0,3]$ as the input. However, we did not consider such machines, and therefore we will here present another method to generate $\lambda(t)$ for $t \in [0,3]$. For $s = 0,1,2$ let us consider the self-affine curves $\lambda_s : [0,1] \to \mathbb{R}^2$, where $\lambda_s(t) = (t, \varphi_s(t))$ is defined by the relations

$$\lambda_s\left(\frac{t+u}{2}\right) - c_{s,u} = L_{s,u}\lambda_s(t).$$

Here $u \in \{0,1\}$, $c_{0,0} = c_{1,0} = c_{2,0} = (0,0)$, and

$$c_{0,1} = \left(\tfrac{1}{2}, 2a^2\right), \quad c_{1,1} = \left(\tfrac{1}{2}, -2a\right), \quad c_{2,1} = \left(\tfrac{1}{2}, \tfrac{1}{4}\right),$$

and moreover

$$\begin{array}{llll} L_{0,0} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & a \end{pmatrix}, & & L_{0,1} = \begin{pmatrix} \frac{1}{2} & 0 \\ a & \overline{a} \end{pmatrix}, \\ L_{1,0} = \begin{pmatrix} \frac{1}{2} & 0 \\ a & \overline{a} \end{pmatrix}, & & L_{1,1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -a & \overline{a} \end{pmatrix}, \\ L_{2,0} = \begin{pmatrix} \frac{1}{2} & 0 \\ -a & \overline{a} \end{pmatrix}, & & L_{2,1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \overline{a} \end{pmatrix}. \end{array}$$

The sequential machines $\mathcal{M}_s$, which generate the curves $\lambda_s(t)$ for $s = 0,1,2$ satisfy the consistency condition. Therefore λ_s is a continuous (θ, γ_s)-self-affine curve, where $Q = \{0\}$, $\theta(0) = 00$, and $\gamma_s : [2] \to GL(\mathbb{R}^2)$ is given by $\gamma_s(u) = L_{s,u}^{-1}$ for $u \in [2]$.

The scaling function Φ is related to φ_0, φ_1, and φ_2 by

$$_2\Phi(t) = \begin{cases} \varphi_0(t) & \text{if } t \in [0,1], \\ \frac{1+\sqrt{3}}{2} + \varphi_1(t-1) & \text{if } t \in [1,2], \\ \frac{1-\sqrt{3}}{2} + \varphi_2(t-2) & \text{if } t \in [2,3]. \end{cases} \tag{1}$$

Remark T. Bedford [4] generalized Dekking's construction of curves with L-systems using different expanding maps for different letters of the substitution. These curves are (θ, γ)-self-affine.

6 Fractal Interpolation and (θ, β, γ)-Self-Affine Curves

The following interpolation problem has been examined by M. Barnsley (see [2], [3]). Let $N \in \mathbb{N}$ be a positive integer and

$$\{(x_n, F_n), n \in [N+1]\}$$

a data set with $x_n,\ F_n \in \mathbb{R}$ and $x_0 < x_1 < \ldots < x_N$. The problem is to find a continuous function $f : [x_0, x_N] \to \mathbb{R}$ with $f(x_j) = F_j$ for all $j \in [N+1]$, such that the graph Γ_f has a given box dimension. M. Barnsley constructed such a function f using iterated function systems. We will consider this problem for

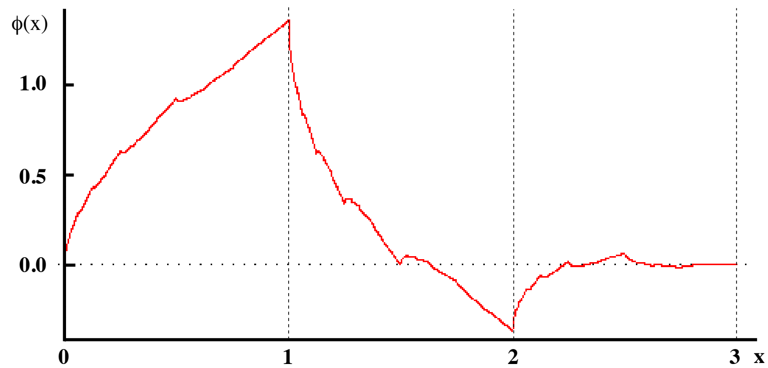

Figure 15: The Daubechies scaling function Φ. The decomposition of Φ into the functions φ_0, φ_1, and φ_2 given by equation 1 is indicated by the vertical dotted lines.

$x_0 = 0$, $x_N = 1$ and $F_0 = 0$, and we will show that the graph Γ_f corresponding to an interpolation function is generated by a sequential machine, whose inputs are the appropriate Cantor representations of the numbers $t \in [0, 1]$. These interpolation functions are self-affine in a more general sense, and in this section we will give the suitable definitions. Let $\theta : Q \to Q^*$ be a substitution, such that $r(q)$ is the length of the word $\theta(q)$ for all $q \in Q$, and

$$\gamma : \bigcup_{q \in Q} \{q\} \times [r(q)] \to GL(\mathbb{R}^2)$$

such that $\gamma(q, h)$ is a linear expanding map for all $q \in Q$ and $h \in [r(q)]$. Moreover, let

$$\beta : \bigcup_{q \in Q} \{q\} \times [r(q)] \to \mathbb{R}$$

be a map such that $\beta(q, h) > 0$ and for all $q \in Q$

$$\sum_{h=0}^{r(q)-1} \beta(q, h)^{-1} = 1.$$

For all $q \in Q$ and $h \in [r(q)]$ we set

$$t_h(q) = \beta(q, h) \sum_{k=0}^{h-1} \beta(q, k)^{-1}.$$

Definition 8. *A curve $\varphi : [0,1] \to \mathbb{R}^2$ is called (θ, β, γ)-self-affine, if the following conditions are satisfied.*

(a) There is a family of curves $\{x_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ with $x_q(0) = (0,0)$ for all $q \in Q$ and $x_0 = \varphi$.

(b) For all $t \in [0,1]$, for $q \in Q$ and $h \in [r(q)]$ one has that

$$x_q\left(\frac{t_h(q)+t}{\beta(q,h)}\right) - x_q\left(\frac{t_h(q)}{\beta(q,h)}\right) = \gamma(q,h)^{-1}(x_{\theta_h(q)}(t)).$$

Recall that the Cantor representation of a number $t \in [0,1]$ has been generated by constructing a sequence of nested intervals $I_{e_1 \dots e_n}$ according to a substitution $\theta : Q \to Q^*$, such that

$$t = \bigcap_{n \in \mathbb{N}} I_{e_1 \dots e_n}.$$

Thereby the $(n+1)$st interval has been obtained by a decomposition of the nth interval into a certain number of subintervals of the same length. In the following we represent the numbers $t \in [0,1]$ more generally by sequences of nested intervals, where the nth interval again is decomposed in a certain number of subintervals according to a substitution θ, but the subintervals do not necessarily have the same length. Take a substitution $\theta : Q \to Q^*$ and β as above. Corresponding to θ and β we will find for each $t \in [0,1]$ a Cantor (θ, β, q)-representation. Consider the tree T_q with root $q \in Q$ corresponding to θ. Now there is an arrow e with label (t,b) pointing from q_1 to q_2 if and only if there is a number $h \in [r(q_1)]$, such that $\theta_h(q_1) = q_2$, $b = \beta(q_1, h)$, and $t = t_h(q_1)$. We say that h is associated to the label (t,b). Let $\underline{e} = e_1 \dots e_n$ be a path in T_q such that the label of e_j is (t_j, b_j) for $j = 1, \dots, n$. To $\underline{e}$ corresponds a subinterval $I_{\underline{e}}$ of $I = [0,1]$ defined by

$$I_{\underline{e}} = I_{e_1 \dots e_n} = \psi_{t_1, b_1} \circ \dots \circ \psi_{t_n, b_n}(I),$$

where as before $\psi_{s,r} : [0,1] \to [0,1]$ is given for $t \in [0,1]$ by

$$\psi_{s,r}(t) = \frac{s+t}{r}.$$

For each $t \in [0,1]$ there is at least one path $\underline{e} = (e_n)_{n \geq 1}$ in the set Σ_q of all infinite paths with root q, such that

$$t = \bigcap_{n=1}^{\infty} I_{e_1 \dots e_n}$$

and consequently

$$t = \sum_{n=1}^{\infty} \frac{t_n}{b_1 \dots b_n},$$

where (t_n, b_n) is the label of e_n for all $n \in \mathbb{N}$. We call this representation of t the Cantor (θ, β, q)-representation corresponding to the directed path $\underline{e}$.

Remark In the case of $\beta(q,h) = r(q)$ for all $q \in Q$ and $h \in [r(q)]$ the (θ, β, q)-representation coincides with the (θ, q)-representation.

Let $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ be the sequential machine corresponding to the substitution θ, where $B \subset \mathbb{R}^2$ and $\tau(q, 0) = (0, 0)$ for all $q \in Q$. The transition function of $\mathcal{M}_\theta$ is

$$\sigma : \bigcup_{q \in Q} \{q\} \times [r(q)] \longrightarrow Q$$

and the output function

$$\tau : \bigcup_{q \in Q} \{q\} \times [r(q)] \longrightarrow B.$$

The machine is called (β, γ)-*consistent* if for all $q \in Q$ and the following condition is satisfied. Suppose that $t \in [0,1]$ has two different (θ, β, q)-representations corresponding to paths $\underline{e}$ and $\underline{\tilde{e}}$ with labels (t_j, b_j) and $(\tilde{t}_j, \tilde{b}_j)$ for all $j \in \mathbb{N}$, such that

$$t = \sum_{n=1}^{\infty} \frac{t_n}{b_1 \cdot \ldots \cdot b_n} = \sum_{n=1}^{\infty} \frac{\tilde{t}_n}{\tilde{b}_1 \cdot \ldots \cdot \tilde{b}_n}.$$

Let h_j and $\tilde{h}_j$ be associated to the labels (t_j, b_j) and $(\tilde{t}_j, \tilde{b}_j)$ for all $j \in \mathbb{N}$. If $f_q^*(h_1 \dots h_n) = d_1 \dots d_n$ and $f_q^*(\tilde{h}_1 \dots \tilde{h}_n) = \tilde{d}_1 \dots \tilde{d}_n$ holds for all $n \in \mathbb{N}$, then the machine $\mathcal{M}_\theta$ is (β, γ)-*consistent*, if

$$\sum_{n=1}^{\infty} \alpha_n(d_n) = \sum_{n=1}^{\infty} \tilde{\alpha}_n(\tilde{d}_n)$$

with

$$\alpha_n = \prod_{j=1}^{n} \gamma(\sigma(q, h_1 \dots h_{j-1}), h_j)^{-1},$$

$$\tilde{\alpha}_n = \prod_{j=1}^{n} \gamma(\sigma(q, \tilde{h}_1 \dots \tilde{h}_{j-1}), \tilde{h}_j)^{-1}.$$

If $\mathcal{M}_\theta$ is (β, γ)-consistent, we find for each $q \in Q$ a curve $f_q : [0,1] \to \mathbb{R}^2$. If $0.t_1\, t_2 \ldots$ is the (q, β, θ)-representation of t corresponding to the path $\underline{e} = (e_n)_{n\geq 1}$ with initial vertex q and labels (t_n, b_n), and if $f_q^*(h_1 \ldots h_n) = d_1 \ldots d_n$ for all $n \in \mathbb{N}$, then f_q is defined by

$$f_q(t) = \sum_{n=1}^{\infty} \alpha_n(d_n).$$

Proposition 13. *The curves $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ generated by a (β, γ)-consistent sequential machine $\mathcal{M}_\theta$, which corresponds to a substitution θ, are (θ, β, γ)-self-affine.*

Proposition 14. *Let $\{f_q : [0,1] \to \mathbb{R}^2 \mid q \in Q\}$ be a family of (θ, β, γ)-self-affine curves. Then there is a (β, γ)-consistent sequential machine $\mathcal{M}_\theta = (Q, A, B, \sigma, \tau)$ corresponding to a substitution θ, such that $\mathcal{M}_\theta$ generates the curves f_q.*

The proofs of these propositions are essentially the same as in the previous chapters. To find the output function τ of the (β, γ)-consistent sequential machine corresponding to a given (θ, β, γ)-self-affine curves, take the output alphabet

$$B = \{\gamma(q,h) f_q\left(\frac{t_h(q)}{\beta(q,h)}\right) : q \in Q, h \in [r(q)]\},$$

and let

$$\tau : \bigcup_{q \in Q} \{q\} \times [r(q)] \longrightarrow B$$

be defined by $\tau(q, 0) = (0,0)$ and for $h \in [N] - \{0\}$ by

$$\tau(q,h) = \gamma(q,h) f_q\left(\frac{t_h(q)}{\beta(q,h)}\right).$$

Now consider an interpolation function f constructed by M. Barnsley corresponding to the interpolation data $\{(x_j, F_j) : j = 0, \ldots, N\}$ with $x_0 = F_0 = 0$, $x_N = 1$ and $0 = x_0 < x_1 < \ldots < x_N = 1$. For $t \in [0,1]$ let $\varphi(t) = (t, f(t))$ be the graph of f considered as a curve. Set $Q = \{0\}$ and $\theta(0) = 0$. Then β and γ do not depend on Q. Let $\beta : [N] \to \mathbb{R}$ be defined by $\beta(h) = (x_{h+1} - x_h)^{-1}$ for $h \in [N]$ and set $t_h = \beta(h) \sum_{j=1}^{h-1} \beta(j)^{-1}$. Let $\gamma : [N] \to GL(\mathbb{R}^2)$ be given for all $h \in [N]$ by the invers of the shear transformation, i. e.

$$\gamma(h)^{-1} = \begin{pmatrix} \beta(h)^{-1} & 0 \\ c(h) & d(h) \end{pmatrix},$$

where $c(h) = F_{h+1} - F_h - d_h F_h$, and $d(h)$ are the free parameters discussed by M. Barnsley with $|d_h| < 1$. Then φ is a (θ, β, γ)-self-affine curve, i.e. for $t \in [0, 1]$ and $h \in [N]$

$$\varphi\left(\frac{t_h + t}{\beta(h)}\right) - \varphi\left(\frac{t_h}{\beta(h)}\right) = \gamma(h)^{-1}(\varphi(t)).$$

Remark Let $\mathcal{H}(\mathbb{R}^2)$ denote the space of all non-empty compact subsets of $\mathbb{R}^2$. The set $(f_q(I))_{q\in Q} \in \mathcal{H}(\mathbb{R}^2)^{|Q|}$ for a given family of (θ, β, γ)-self-affine curves $(f_q)_{q\in Q}$ is the attractor of an hierarchical iterated function system (see [4] and [1] for the case of $\beta(q, h) = r(q)$). More precisely, hierarchical iterated function systems are generalizations of iterated function systems (see [17], [22], [26], [39]). They are defined as follows. Let $\theta : Q \to Q^*$ be a substitution and let $\{X_q \mid q \in Q\}$ be a set of complete metric spaces. Suppose that for $q_1, q_2 \in Q$ and $h \in [r(q_1)]$ with $\theta_h(q_1) = q_2$ we have contractions

$$\varphi_{q_1,q_2;h} : X_{q_2} \to X_{q_1}.$$

If for each $q_1 \in Q$ there is at least one contraction map $\varphi_{q_1,q_2;h}$, the system

$$\{X_q, \varphi_{q_1,q_2;h} \mid q, q_1, q_2 \in Q, h \in [r(q_1)]\}$$

is called a hierarchical iterated function system or simply HIFS. The attractor of a HIFS is a vector of compact sets $\mathcal{A} = (A_q)_{q\in Q}$, such that $A_q \subset X_q$ for all $q \in Q$, and

$$A_q = \bigcup_{\tilde{q}\in Q, \tilde{q}=\theta_h(q)} \varphi_{q,\tilde{q};h}(X_{\tilde{q}}).$$

If $\{f_q \mid q \in Q\}$ is a family of (θ, β, γ)-self-affine functions, the vector of compact sets $(f_q(I))_{q\in Q}$ can be described by an HIFS with metric spaces $X_q = \mathbb{R}^2$ for all $q \in Q$. If $\theta_h(q_1) = q_2$ for $h \in [r(q_1)]$ we have a contraction map $\varphi_{q_1,q_2;h} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi_{q_1,q_2;h}(x, y) = f_{q_1}\left(\frac{t_h}{\beta(q_1, h)}\right) + \gamma(q_1, h)^{-1}(x, y),$$

where as before $t_h = \beta(h) \sum_{j=0}^{h-1} \beta(j)^{-1}$.

Remark Examples for interpolation functions are the examples 16, 17, 21, 22, 23, and 25.

7 Final Remarks

The self-affinity of the logistic equation. For $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ consider the parabola $l(t) = \lambda t(1-t)$ for all $t \in [0,1]$. The graph $\varphi(t) = (t, l(t))$ is a $(2,\gamma)$-self-affine curve, if $\theta : \{0\} \to \{0\}^*$ is the substitution $\theta(0) = 0\,0$ and $\gamma : [2] \to GL(\mathbb{R}^2)$ is given by

$$\gamma(0) = \begin{pmatrix} 2 & 0 \\ -2\lambda & 4 \end{pmatrix}, \quad \gamma(1) = \begin{pmatrix} 2 & 0 \\ 2\lambda & 4 \end{pmatrix}.$$

More generally, if $\theta : \{0\} \to \{0\}^*$ is the substitution of arbitrary length m, then $\phi(t) = (t, l(t))$ is (m,γ)-self-affine. M. Hata observed that for $\lambda = 2$ the function $l(t)$ is generated by a similar procedure as Tagaki's function (see [24]). We have

$$2t(1-t) = \sum_{n=0}^{\infty} 4^{-n} g(2^n t),$$

where g is the same function as in Example 22. To prove this equality it is sufficient to show that the graph of the infinite sum on the right side is a $(2,\gamma)$-self-affine curve with the same θ and γ as for the function $l(t)$ for $\lambda = 2$. For all $h \in [2^k]$ and $t = \frac{h}{2^k}$ the values $l(t)$ and $\sum_{n=0}^{\infty} 4^{-n} g(2^n t)$ coincide.

The self-affinity of the standard parabola If $f(t) = t^2$ is the standard parabola and $g(t) = (t, f(t))$ is its graph, then g is (m,γ)-self-affine for all $m \in \mathbb{N}$. Again let $\theta : \{0\} \to \{0\}^*$ be the substitution of length m. For all $h \in [m]$ we define

$$\gamma(h) = \begin{pmatrix} m & 0 \\ -2hm & m^2 \end{pmatrix}.$$

Then for all $h \in [m]$

$$g\left(\frac{t+h}{m}\right) - g\left(\frac{h}{m}\right) = \gamma(h)^{-1} g(t).$$

T. Kamae proved that if a function is both (m,α)-self-affine and $(\tilde{m},\tilde{\alpha})$ self-affine, then m and $\tilde{m}$ must be multiplicatively dependent, i. e. there are numbers k and l such that $m^k = \tilde{m}^l$ (see [27]). The fact that g is (m,γ)-self-affine for arbitrary $m \in \mathbb{N}$ shows that the result of T. Kamae cannot be generalized to (m,γ)-self-affine curves.

If $f : [0,1] \to \mathbb{R}$ is a polynomial with degree strictly larger than 2, the curve $g(t) = (t, f(t))$ is not (m,γ) self-affine for any $m \in \mathbb{N}$. However, in the same way as for the graph of the standard parabola it can be shown that $g(t) = (t, t^2, t^3)$ is (m, γ)-self-affine in $\mathbb{R}^3$ for arbitrary m. Again we define $\theta : \{0\} \to \{0\}^*$ to be the substitution of length m, and for all $h \in [m]$

$$\gamma(h) = \begin{pmatrix} m & 0 & 0 \\ -2hm & m^2 & 0 \\ 3h^2m & -3hm^2 & m^3 \end{pmatrix}.$$

The existence of (θ, β, γ)-self-affine curves. Consider the substitution $\theta : Q \to Q^*$ where $r(q)$ is the length of the word $\theta(q)$ for all $q \in Q$, together with a map

$$\gamma : \bigcup_{q \in Q} \{q\} \times [r(q)] \longrightarrow GL(\mathbb{R}^2)$$

such that $\gamma(q, h)$ are expanding maps for all $q \in Q$ and $h \in [r(q)]$. Moreover let

$$\beta : \bigcup_{q \in Q} \{q\} \times [r(q)] \longrightarrow \mathbb{R}$$

be a map satisfying $\beta(q, h) > 0$ for all $q \in Q$ and $h \in [r(q)]$ and

$$\sum_{h=0}^{r(q)-1} \beta(q, h)^{-1} = 1.$$

Let $a(q, h) \in \mathbb{R}^2$ for all $q \in Q$ and $h \in [r(q)]$ and $a(q, 0) = (0, 0)$ for all $q \in Q$. The question we want to discuss here is the existence of a family $\{f_q : [0, 1] \to \mathbb{R} \mid q \in Q\}$ of (θ, β, γ)-self-affine curves such that

$$f_q \left(\frac{t_h(q)}{\beta(q, h)} \right) = a(q, h),$$

i. e. the family $\{f_q : q \in Q\}$ is a non-zero solution of the system of equations

$$f_q \left(\frac{t_h(q) + t}{\beta(q, h)} \right) - a(q, h) = \gamma(q, h)^{-1} f_{\theta_h(q)}(t)$$

for all $t \in [0, 1]$. In general such a non-zero continuous solution does not exist for arbitrary θ, γ, β, and $a(q, h)$. In all examples we consider some additional condition, which is necessary and sufficient for the existence of a solution. The condition we impose is the (β, γ)-consistency of the machine $\mathcal{M}_\theta$ with output alphabet $B = \{\gamma(q,h)a(q,h)\}$. This condition is equivalent to conditions imposed in the work of other authors (see for example [14], [2], [26], [43], [44], [45], [46], [20]). It should be mentioned that a different treatment of self-affine functions is given in [53].

Functions and curves, which are not self-affine 1. Although it may appear that many of the examples of nowhere differentiable functions and area-filling curves are self-affine, there are also many examples which are not self-affine, i.e. the function of G. Faber which is

$$f(t) = \sum_{n=1}^{\infty} 2^{-n} g\left(2^{n!} t\right),$$

where g is the same function as in Example 22 (see [20]). Faber proved that this function is not α-Hölder continuous for any $\alpha > 0$. Since (m,α)-self-affine functions are α-Hölder continuous for some $\alpha > 0$, f can not be (m,α)-self-affine (see [32], [27], [21]).

2. P. Wingren proved in [60] that the graph of the function

$$w(t) = \sum_{n=0}^{\infty} 2^{-n} g\left(2^{2^n}\right)$$

for $t \in [0,1]$ has Hausdorff dimension 2 (here g is the function from Example 22). The function w is not (m,α)-self-affine for any $m \geq 2$ and $\alpha \in (0,1]$. Otherwise w would be α-Hölder continuous and therefore the box counting dimension of the graph of w would be equal to $2-\alpha$ (see [22], p. 147, Cor. 11.2). Since the box counting dimension is larger or at least equal to the Hausdorff dimension (see [22], chap. 3), this implies the contradiction $\alpha = 0$.

3. In general the nowhere differentiable curves constructed by K. Knopp are not self-affine, since in each step of the construction he is allowed to use different polygonal curves (see [30]). Examples of space-filling curves which are not self-affine are the curves examined by Prusinkiewicz et al. in [42] except for the uniform curves, which can be described by sequential functions (see [48]).

4. Faber's and Wingren's functions are not (θ,β,γ)-self-affine for $Q = \{0\}$, the substitution $\theta(0) = 0\,0$ and the inverse of the shear transformation

$$\gamma(h)^{-1} = \begin{pmatrix} \beta(h)^{-1} & 0 \\ c(h) & d(h) \end{pmatrix},$$

for $h \in [2]$, if $d(h) > \beta(h)^{-1}$. Otherwise it could be concluded from [5] that Faber's and Wingren's functions are α-Hölder continuous for some $\alpha > 0$.

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