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ON THE SETS WHERE A CONTINUOUS FUNCTION HAS INFINITE ONE-SIDED DERIVATIVES

Abstract

In the present paper I give a characterization by help of measure and Borel classes of the set of points at which the continuous function possesses an infinite one-sided derivative. The main theorem is as follows. Let E_1 and E_2 be disjoint subsets of $\mathbb{R}$. There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $E_1 = \{x : f'_+(x) = +\infty\}$ and $E_2 = \{x : f'_+(x) = -\infty\}$ if and only if (i) E_1 and E_2 are of type $F_{\sigma\delta}$ and measure zero and (ii) there exist disjoint sets F_1 and F_2 of type F_σ such that $E_1 \subset F_1$ and $E_2 \subset F_2$.

1 Introduction

In [4] and [2], the following theorems are proved:

Theorem I. *(Theorem 2 of [4]). Let E_1 and E_2 be disjoint subsets of $\mathbb{R}$. There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$E_1 = \{x : f'_-(x) = +\infty\} \quad \textit{and} \quad E_2 = \{x : f'_-(x) = -\infty\}$$

if and only if $m(E_1) = m(E_2) = 0$, where f'_- is the left-hand derivative and m denotes the Lebesgue measure.
Analogously for the right-hand derivative.

Theorem II. *(The main theorem of [2]). Let E_1 and E_2 be disjoint subsets of $\mathbb{R}$. There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$E_1 = \{x : f'(x) = +\infty\} \quad \textit{and} \quad E_2 = \{x : f'(x) = -\infty\}$$

if and only if

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(i) E_1 *and* E_2 *are of type* $F_{\sigma\delta}$ *and of measure zero, and*

(ii) there exist disjoint sets F_1 *and* F_2 *of type* F_σ *such that* $E_1 \subset F_1$ *and* $E_2 \subset F_2$*.*

In the present paper we prove analogous theorems for the one-sided derivatives of continuous functions. To prove these results we use the construction from [2].

We shall apply the following notations:

$D^- f(x)$, $D_- f(x)$, $D^+ f(x)$, $D_+ f(x)$ – the upper left-hand, lower left-hand, upper right-hand, lower right-hand Dini derivatives of a function f at a point x;

$\int_A f(x)dx$ – the Lebesgue integral of f on the set A;

C_f – the set of all continuity points of the function f;

D_f – the set of all discontinuity points of the function f;

2 Main results

Theorem 1 (Main Theorem)**.**

1) Let $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function and let*

$$E_1 = \{x \,:\, f_+^{'}(x) = +\infty\} \quad \text{and} \quad E_2 = \{x \,:\, f_+^{'}(x) = -\infty\}.$$

Then we have

(i) E_1 *and* E_2 *are of type* $F_{\sigma\delta}$ *and of measure zero;*

(ii) There exist disjoint F_σ*-type sets* F_1 *and* F_2 *such that* $E_1 \subset F_1$ *and* $E_2 \subset F_2$*.*

2) Let E_1 *and* E_2 *be subsets of* $\mathbb{R}$ *that satisfy the following conditions:*

(i) E_1 *and* E_2 *are of type* $F_{\sigma\delta}$ *and of measure zero;*

(ii) There exist disjoint F_σ*-type sets* F_1 *and* F_2 *such that* $E_1 \subset F_1$ *and* $E_2 \subset F_2$*.*

Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ *such that*

$$E_1 = \{x \,:\, f_+^{'}(x) = +\infty\} \quad \text{and} \quad E_2 = \{x \,:\, f_+^{'}(x) = -\infty\}.$$

Moreover, if $x \notin E_1 \cup E_2$ *then*

$$D_+ f(x) < +\infty\,; \quad D^+ f(x) > -\infty\,; \quad D_- f(x) < +\infty\,; \quad D^- f(x) > -\infty\,.$$

To prove this theorem we shall need the following five lemmas.

Lemma 1. *Let* $f : \mathbb{R} \to \mathbb{R}$. *Then*

$$\{x : f_+^{'}(x) = +\infty\} \subset F_1 \quad \textit{and} \quad \{x : f_+^{'}(x) = -\infty\} \subset F_2\,,$$

where $F_1 \cap F_2 = \emptyset$, $F_i = A_i \setminus B_i$, $A_i \in F_\sigma$ *and* $B_i \subset D_f$ *for* $i = 1, 2$.

Proof. We have

$$\{x : f_+^{'}(x) = +\infty\} = \{x : D_+f(x) = +\infty\} = \cap_{n=1}^{\infty}\{x : D_+f(x) > n\}$$

and

$$\{x : f_+^{'}(x) = -\infty\} = \{x : D^+f(x) = -\infty\} = \cap_{n=1}^{\infty}\{x : D^+f(x) < -n\}\,.$$

Hence

$$\{x : f_+^{'}(x) = +\infty\} \subset \{x : D_+f(x) > n\}\,, \quad n \in \mathbb{N}$$

and

$$\{x : f_+^{'}(x) = -\infty\} \subset \{x : D^+f(x) < -n\}\,, \quad n \in \mathbb{N}.$$

Of course

$$\{x : D_+f(x) > n\} \cap \{x : D^+f(x) < -n\} = \emptyset\,.$$

By Lemma 8 of [5],

$$\{x : D_+f(x) \le n\} \cap C_f \in G_\delta \quad \text{and} \quad \{x : D^+f(x) \ge -n\} \cap C_f \in G_\delta\,.$$

Hence

$$\{x : D_+f(x) > n\} \cup D_f \in F_\sigma \quad \text{and} \quad \{x : D^+f(x) < -n\} \cup D_f \in F_\sigma\,.$$

We have

$$\{x : D_+f(x) > n\} = [\{x : D_+f(x) > n\} \cup D_f] \setminus [D_f \setminus \{x : D_+f(x) > n\}]$$

and

$$\{x : D^+f(x) < -n\} = \left[\{x : D^+f(x) < -n\} \cup D_f\right] \setminus \left[D_f \setminus \{x : D^+f(x) < -n\}\right].$$

Denoting

- $F_1 = \{x : D_+f(x) > n\}$,
- $F_2 = \{x : D^+f(x) < -n\}$,

- $A_1 = \{x : D_+ f(x) > n\} \cup D_f$,
- $A_2 = \{x : D^+ f(x) < -n\} \cup D_f$,

we obtain our assertion. □

Lemma 2. *Let $u : \mathbb{R} \to \mathbb{R}$ be a nonnegative integrable function, let M be a set of measure zero and $L^* = \cup_{j=1}^{\infty}(a_j, b_j)$ such that $M \subset L^*$. For every interval (a_j, b_j) we denote by*

$$\Delta_{j,n} = \left(a_j + \frac{b_j - a_j}{2^{n+1}}, a_j + \frac{b_j - a_j}{2^n}\right)$$

$$\Delta'_{j,n} = \left(b_j - \frac{b_j - a_j}{2^n}, b_j - \frac{b_j - a_j}{2^{n+1}}\right)$$

Then for each positive integer k, there exists an open set $L_k \subset L^*$ such that

1) $M \subset L_k$;

2) $\int_{L_k \cap \Delta_{j,n}} u(x)dx < \frac{1}{2^k} \cdot \frac{m(\Delta_{j,n})}{2}$ and $\int_{L_k \cap \Delta'_{j,n}} u(x)dx < \frac{1}{2^k} \cdot \frac{m(\Delta'_{j,n})}{2}$.

3) $\int_{L_k \cap (a_j, b_j)} u(x)dx < \frac{1}{2^k}(b_j - a_j)$;

4) If $x_o \notin L^*$ then for every h, we have

$$\int_{L_k^h} u(x)dx < \frac{1}{2^k}|h|, \quad \text{where}$$

$$L_k^h = [x_o, x_o + h] \cap L_k \quad \text{for} \quad h > 0 \quad \text{and} \quad L_k^h = [x_o + h, x_o] \cap L_k \quad \text{for} \quad h < 0\,.$$

5) For every $x_o \in L^*$ there exists $h > 0$ (respectively $h < 0$) such that

$$\int_{L_k^h} u(x)dx < \frac{1}{2^{k-1}}|h| \quad \text{and}$$

$$(x_o, x_o + h) \subset L^* \ \text{(respectively} \ (x_o + h, x_o) \subset L^*).$$

Proof. 1)–4) See Codyks' paper [2] (pp. 435-436).

5) Since $x_o \in L^*$, there exists j such that $x_o \in (a_j, b_j)$. Let $\Delta_{j,n}$ and $\Delta'_{j,n}$ be the intervals defined in our hypothesis. Suppose that

$$\Delta_{j,n} = (p_n^j, q_n^j) \quad \text{and} \quad \Delta'_{j,n} = (r_n^j, t_n^j)\,.$$

Of course $p_n^j = q_{n+1}^j$ and $t_n^j = r_{n+1}^j$, and also

$$m(\Delta_{j,n+1}) = \frac{1}{2} m(\Delta_{j,n}), \quad m(\Delta'_{j,n+1}) = \frac{1}{2} m(\Delta'_{j,n}). \tag{1}$$

If $x_o \in (a_j, \frac{a_j+b_j}{2}]$ then there exists $h \geq \frac{1}{2}(b_j - a_j)$ such that $[x_o, x_o + h] \subset (a_j, b_j)$. From condition 3) we obtain

$$\int_{L_k^h} u(x)dx \leq \int_{L_k \cap (a_j, b_j)} u(x)dx < \frac{1}{2^k}(b_j - a_j),$$

hence

$$\int_{L_k^h} u(x)dx < \frac{1}{2^{k-1}} h.$$

If $x_o \in (\frac{a_j+b_j}{2}, b_j)$ then there exists $\Delta'_{j,n}$ such that $x_o \in \overline{\Delta}'_{j,n}$. Suppose that

$$(t_n^j - x_o) \geq \frac{1}{2} m(\Delta'_{j,n}).$$

Then there exists $h \geq \frac{1}{2} m(\Delta'_{j,n})$ such that $x_o + h \in \overline{\Delta}'_{j,n}$. From condition 2) we obtain

$$\int_{L_k^h} u(x)dx \leq \int_{L_k \cap \Delta'_{j,n}} u(x)dx < \frac{1}{2^k} \cdot \frac{m(\Delta'_{j,n})}{2}.$$

It follows that

$$\int_{L_k^h} u(x)dx < \frac{1}{2^k} h \quad \text{so} \quad \int_{L_k^h} u(x)dx < \frac{1}{2^{k-1}} h.$$

Suppose that

$$(t_n^j - x_o) < \frac{1}{2} m(\Delta'_{j,n}).$$

Then there exists $h > 0$ such that $x_o + h \in \overline{\Delta}'_{j,n+1}$ and $h \geq \frac{1}{2} m(\Delta'_{j,n})$.
By (1) we have

$$h > \frac{1}{2} m(\Delta'_{j,n+1}).$$

It follows that

$$\int_{L_k^h} u(x)dx = \int_{[x_o, t_n^j] \cap L_k} u(x)dx + \int_{[t_n^j, x_o + h] \cap L_k} u(x)dx.$$

From condition 2) we obtain

$$\int_{[x_o,t_n^j]\cap L_k} u(x)dx \leq \int_{L_k\cap\Delta'_{j,n}} u(x)dx < \frac{1}{2^k}\cdot\frac{m(\Delta'_{j,n})}{2}\,.$$

and

$$\int_{[t_n^j,x_o+h]} u(x)dx \leq \int_{L_k\cap\Delta'_{j,n+1}} u(x)dx < \frac{1}{2^k}\cdot\frac{m(\Delta'_{j,n+1})}{2}\,.$$

Hence

$$\int_{L_k^h} u(x)dx < \frac{1}{2^k}\left(\frac{m(\Delta'_{j,n})}{2}+\frac{m(\Delta'_{j,n+1})}{2}\right),$$

in consequence

$$\int_{L_k^h} u(x)dx < \frac{2}{2^k}h = \frac{1}{2^{k-1}}h\,.$$

Analogously it follows that there exists $h<0$ that satisfies condition 5). □

Lemma 3. *Let E^1, E^2, H_1, H_2 be sets such that $E^i\in F_{\sigma\delta}$, $H_i\in F_\sigma$, $m(E^i)=0$, $E^i\subset H_i$ $(i=1,2)$ and $H_1\cap H_2=\emptyset$. Let*

$$E^i=\cap_{n=1}^{\infty}E_n^i\,,\quad E_n^i=\cup_{k=1}^{\infty}E_{n,k}^i\,,\quad H_i=\cup_{k=1}^{\infty}F_k^i\,,$$

where $E_{n,k}^i$ and F_k^i are closed sets, $i=1,2$. Let G_n be an open set such that $E^1\cup E^2\subset G_n$ and $m(G_n)<\frac{1}{2^n}$, where $n=1,2,\ldots$. We may additionally suppose that

$$G_{n+1}\subset G_n\,,\quad E_n^i\subset G_n\,,\quad F_k^i\subset F_{k+1}^i\,,\quad E_{n,k}^i\subset E_{n,k+1}^i\,,\quad E_{n+1,k}^i\subset E_{n_k}^i\subset F_k^i\,.$$

Let us denote by C_k^1 and C_k^2 some disjoint open sets such that $F_k^i\subset C_k^i$ $(i=1,2)$. Let $u:\mathbb{R}\to\mathbb{R}$ be defined as follows:

$$u(x)=\sum_{n=1}^{\infty}u_n(x)\,,\quad \text{where}\quad u_n(x)=\begin{cases}1 & \text{if } x\in G_n\\ 0 & \text{if } x\notin G_n\end{cases}$$

(clearly u is an integrable function). Then there exist some sets $e_n^1\subset E_n^1$ and $e_n^2\subset E_n^2$ where

$$e_n^i=\cup_{k=1}^{\infty}e_{n,k}^i\,,\quad e_{n,k}^i\in F_\sigma\cap G_\delta\quad (i=1,2)\,,$$

such that the sets $e_{n,k}^i$ satisfy the following conditions:

1) $e^i_{n,1} = E^i_{n,1} \subset C^i_1$, $e^i_{n,k} \subset e^i_{n,k+1}$, $E^i \cap E^i_{n,k} \subset e^i_{n,k} \subset E^i_{n,k} \subset C^i_k$.

2) *For every positive integer* $k \geq 2$ *there exist some open sets* $g^*_{n,k}$, $g_{n,k}$, *and for* $k = 1$ *there exists the open set* $g_{n,1}$, *such that:*

a) $G_n = g_{n,1} \supset e^i_{n,1}$*;*

b) $g_{n,k-1} \setminus \left(e^1_{n,k-1} \cup e^2_{n,k-1}\right) \supset g^*_{n,k} \supset g_{n,k} \supset \left(e^i_{n,k} \setminus e^i_{n,k-1}\right)$ *;*

c) *the points of the set* $e^i_{n,1}$ *are the points of density of the set* $g_{n,1} \setminus g^*_{n,2}$, *while for* $k \geq 2$, *the points of the set* $e^i_{n,k} \setminus e^i_{n,k-1}$ *are the points of density of the set* $g_{n,k} \setminus g^*_{n,k+1}$ *;*

d) *for every component* I *of the set* $g^*_{n,k}$ *we have*

$$\int_{I \cap g_{n,k}} u(x)dx < \frac{1}{2^k} m(I)\,;$$

e) *for* $x_o \in g^*_{n,k}$ *there exists* $h > 0$ *such that* $[x_o, x_o + h] \subset g^*_{n,k}$ *and there exists* $h < 0$ *such that* $[x_o + h, x_o] \subset g^*_{n,k}$, *and*

$$\int_{g^h_{n,k}} u(x)dx < \frac{1}{2^{k-1}} |h|\,,$$

where $g^h_{n,k} = g_{n,k} \cap [x_o, x_o + h]$ *for* $h > 0$ *and* $g^h_{n,k} = g_{n,k} \cap [x_o + h, x_o]$ *for* $h < 0$*;*

f) *if* $c_o \notin g^*_{n,k}$ *then for every* h,

$$\int_{g^h_{n,k}} u(x)dx < \frac{1}{2^k} |h|\,.$$

PROOF. For a), b), c), d), f) see [2] (pp. 436-440). Condition e) follows from Lemma 2, 5). □

Corollary 1. *([2], p.440).*

1) *The sets* e^1_n *and* e^2_n *are of type* F_σ*;*

2) $E^i \subset e^i_n \subset E^i_n \quad (i = 1, 2)$*;*

3) *The points of the set* e^1_n *are the density points of the set*

$$\Omega^1_n = \cup_{k=1}^\infty [(g_{n,k} \setminus g^*_{n,k+1}) \cap C^1_k]\,;$$

The points of the set e^2_n *are the density points of the set*

$$\Omega^2_n = \cup_{k=1}^\infty [(g_{n,k} \setminus g^*_{n,k+1}) \cap C^2_k]\,;$$

4) $\Omega_n^1 \cap \Omega_n^2 = \emptyset$.

Lemma 4. *(Lemma 6 of [2], p. 440). Let* $K \in G_\delta$ *and* $K \subset G$*, where* G *is an open set. Then* $K = \cap_{j=1}^{\infty} K^{(j)}$*, where* $K^{(j)}$ *is an open set such that*

$$K^{(j+1)} \subset K^{(j)} \subset G\,, \quad \textit{and} \quad \overline{K}^{(j+1)} \subset K^{(j)} \cup \overline{K}\,.$$

Corollary 2. *The sets* $e_{n,1}^i$ *and* $e_{n,k}^i \setminus e_{n,k-1}^i$ *(*$n = 1, 2, \ldots,\ k = 2, 3, \ldots$*) from Lemma 3 are of the form:*

1) $e_{n,1}^i = \cap_{j=1}^{\infty} G_{n,1}^{(j)_i}$*, where* $G_{n,1}^{(j)_i}$ *are open sets satisfying*

$$\overline{G}_{n,1}^{(j+1)_i} \subset G_{n,1}^{(j)_i} \subset g_{n,1}\,;$$

2) $e_{n,k}^i \setminus e_{n,k-1}^i = \cap_{j=1}^{\infty} G_{n,k}^{(j)_i}$*, where* $G_{n,k}^{(j)_i}$ *are open sets satisfying*

$$G_{n,k}^{(j+1)_i} \subset G_{n,k}^{(j)_i} \subset g_{n,k} \quad \textit{and} \quad \overline{G}_{n,k}^{(j+1)_i} \subset G_{n,k}^{(j)_i} \cup E_{n,k}^i\,.$$

PROOF. See Lemma 4. □

Lemma 5. *Let the functions* $v_n^{(1)}(x)$ *and* $v_n^{(2)}(x)$ *be defined by*

$$v_n^{(1)}(x) = \begin{cases} 1 & \textit{if } x \in G^{(j)_1} \setminus G^{(j+1)_1} \\ +\infty & \textit{if } x \in \cap_{j=1}^{\infty} G_n^{(j)_1} \\ 0 & \textit{elsewhere} \end{cases}$$

and

$$v_n^{(2)}(x) = \begin{cases} -1 & \textit{if } x \in G^{(j)_2} \setminus G^{(j+1)_2} \\ -\infty & \textit{if } x \in \cap_{j=1}^{\infty} G_n^{(j)_2} \\ 0 & \textit{elsewhere} \end{cases}$$

where $G_n^{(j)_i} = \cup_{k=1}^{\infty} G_{n,k}^{(j)_i}$ *and* $G_{n,k}^{(j)_i}$ *are the sets from Corollary 2. We have*

$$e_n^i \subset G_n^{(j)_i} \subset G_n\,.$$

Let

$$w_n^{(1)}(x) = \begin{cases} \min[v_n^{(1)}(x), u(x)] & \textit{if } x \in \Omega_n^1 \\ 0 & \textit{elsewhere} \end{cases}$$

and

$$w_n^{(2)}(x) = \begin{cases} \max[v_n^{(2)}(x), -u(x)] & \text{if } x \in \Omega_n^2 \\ 0 & \text{elsewhere} \end{cases}$$

where u, Ω_n^1 and Ω_n^2 are defined in Lemma 3. Then the functions

$$W_n^{(1)}(x) = \int_0^x w_n^{(1)}(t)dt \quad \text{and} \quad W_n^{(2)}(x) = \int_0^x w_n^{(2)}(t)dt$$

have the following properties:

1) $0 \le D_+ W_n^{(1)}(x) < +\infty$ *and* $0 \le D_- W_n^{(1)}(x) < +\infty$ *for* $x \notin E_n^1$,

2) $0 \le D^+ W_n^{(1)}(x) < +\infty$ *and* $0 \le D^- W_n^{(1)}(x) < +\infty$ *for* $x \in E^2$,

3) $-\infty < D^+ W_n^{(2)}(x) \le 0$ *and* $-\infty < D^- W_n^{(2)}(x) \le 0$ *for* $x \notin E_n^2$,

4) $-\infty < D_+ W_n^{(2)}(x) \le 0$ *and* $-\infty < D_- W_n^{(2)}(x) \le 0$ *for* $x \in E_n^1$.

The sets E^i and E_n^i $(i = 1, 2)$ are defined in Lemma 3 and Corollary 1.

PROOF. Consider first the case where $x \in g_{n,k}^*$ for every $k \ge 2$. Let $k_1 \ge 2$. Then there exists a component I_1 of the set g_{n,k_1}^* such that $x \in I_1$. By the definition of $W_n^{(1)}$, for any $h > 0$ with $[x, x+h] \subset I_1$, and for any $h < 0$ with $[x+h, x] \subset I_1$ we have

$$\int_x^{x+h} w_n^{(1)}(t)dt = \int_{g_{n,k_1}^h} w_n^{(1)}(t)dt \le \int_{g_{n,k_1}^h} u(t)dt\,,$$

where

$$g_{n,k_1}^h = [x, x+h] \cap g_{n,k_1} \quad \text{for} \quad h > 0\,,$$

and

$$g_{n,k_1}^h = [x+h, x] \cap g_{n,k_1} \quad \text{for} \quad h < 0\,.$$

By Lemma 3, e), there exist $h_1 > 0$ with $[x, x+h_1] \subset I_1$, and $h_1^{'} < 0$ with $[x+h_1^{'}, x] \subset I_1$ such that

$$\int_x^{x+h_1} w_n^{(1)}(t)dt \le \int_{g_{n,k_1}^{h_1}} u(t)dt < \frac{1}{2^{k_1-1}}|h_1|\,,$$

and

$$\int_x^{x+h_1^{'}} w_n^{(1)}(t)dt \le \int_{g_{n,k_1}^{h_1^{'}}} u(t)dt < \frac{1}{2^{k_1-1}}|h_1^{'}|\,,$$

hence

$$-\frac{1}{2^{k_1-1}}|h_1^{'}| < \int_x^{x+h_1^{'}} w_n^{(1)}(t)dt\,.$$

By this inequality we have

$$0 \leq \frac{W_n^{(1)}(x+h_1) - W_n^{(1)}(x)}{h_1} = \frac{\int_x^{x+h_1} w_n^{(1)}(t)dt}{h_1} < \frac{1}{2^{k_1-1}}\,,$$

and

$$0 \leq \frac{W_n^{(1)}(x+h_1^{'}) - W_n^{(1)}(x)}{h_1^{'}} = \frac{\int_x^{x+h_1^{'}} w_n^{(1)}(t)dt}{h_1^{'}} < \frac{1}{2^{k_1-1}}\,.$$

Let $k_2 > k_1$. Then there exists a component I_2 of the set g^*_{n,k_2} such that $x \in I_2$. Similarly as above, we obtain that there exist $h_2 > 0$ and $h_2^{'} < 0$ such that

$$0 \leq \frac{W_n^{(1)}(x+h_2) - W_n^{(1)}(x)}{h_2} < \frac{1}{2^{k_2-1}}\,,$$

and

$$0 \leq \frac{W_n^{(1)}(x+h_2^{'}) - W_n^{(1)}(x)}{h_2^{'}} < \frac{1}{2^{k_2-1}}\,,$$

and so on.

By Lemma 3 it follows that $m(I_n) \to 0$ if $k_n \to \infty$. Then $h_n \to 0$ and $h_n^{'} \to 0$. Finally if $x \in g^*_{n,k}$ for every k then $D_+W_n^{(1)}(x) = 0$ and $D_-W_n^{(1)}(x) = 0$.

Consider the case where $x \notin E_n^1$ and $x \notin g^*_{n,k}$ for $k \geq k_o$, where k_o is a positive integer. From Lemma 3, $E_n^1 = \cup_{k=1}^{\infty} E_{n,k}^1$ and $E_n^1 = e_{n,1}^1$. Since $x \notin E_n^1$ it follows that $x \notin E_{n,k}^1$ for every k. Hence $x \notin e_{n,1}^1$. By Corollary 2, $e_{n,1}^1 = \cap_{j=1}^{\infty} G_{n,1}^{(j)_1}$. It follows that there exists some j_1 such that $x \notin G_{n,1}^{(j_1)_1}$. Since $x \notin E_n^1 \supset e_n^1 \supset e_{n,2}^1 \setminus e_{n,1}^1$ (see Lemma 3), we obtain that $x \notin e_{n,2}^1 \setminus e_{n,1}^1$. By Corollary 2, $e_{n,2}^1 \setminus e_{n,1}^1 = \cap_{j=1}^{\infty} G_{n,2}^{(j)_1}$. Then there exists some j_2 such that $x \notin G_{n,2}^{(j_2)_1}$.

Continuing the above procedure we have

$$x \notin e_{n,k_o-1}^1 \setminus e_{n,k_o-2}^1 \subset E_n^1\,,$$

and there exists j_{k_o-1} such that $x \notin G_{n,k_o-1}^{(j_{k_o-1})_1}$. Let

$$j_o = \max(j_1, j_2, \ldots, j_{k_o-1}\}\,.$$

From the properties of the sets $G_{n,k}^{(j)_i}$ (see Corollary 2), we get

$$x \notin \cup_{k<k_o} G_{n,k}^{(j_o)_1} \cup E_n^1 .$$

This implies the fact that: the distance of the point x to the set $\cup_{k<k_o} \overline{G}_{n,k}^{(j_o+1)_1}$ is positive, therefore to $\cup_{k<k_o} G_{n,k}^{(j_o+1)_1}$ the distance is positive too. Then there exists an open interval Δ such that $x \in \Delta$ and

$$\Delta \cap \left(\cup_{k<k_o} G_{n,k}^{(j_o+1)_1} \right) = \emptyset .$$

There exist $h > 0$ with $[x, x+h] \subset \Delta$, and $h < 0$ with $[x+h, x] \subset \Delta]$ such that

$$0 \le \frac{W_n^{(1)}(x+h) - W_n^{(1)}(x)}{h} = \frac{\int_x^{x+h} w_n^{(1)}(t)dt}{h} =$$
$$= \frac{\int_{g_{n,k_o}^h} w_n^{(1)}(t)dt}{|h|} + \frac{\int_{\sim g_{n,k_o}^h} w_n^{(1)}(t)dt}{|h|} ,$$

where $\sim g_{n,k_o}^h = [x, x+h] \setminus g_{n,k_o}^h$. Since $x \notin g_{n,k}^*$, by Lemma 3, f), we have

$$\frac{\int_{g_{n,k_o}^h} w_n^{(1)}(t)dt}{|h|} \le \frac{\int_{g_{n,k_o}^h} u(t)dt}{|h|} < \frac{1}{2^{k_o}} < 1 .$$

From Lemma 3 we obtain the following fact: if $t \notin g_{n,k_o}$ then $t \notin g_{n,k}$ for $k \ge k_o$, therefore $t \notin G_{n,k}^{(j)_1}$ for $k \ge k_o$ and $j \ge 1$ (see Corollary 2). If $t \in [x, x+h]$ then $t \notin \cup_{k<k_o} G_{n,k}^{(j_o+1)_1}$. It follows that if $t \in \sim g_{n,k_o}^h$ then

$$t \notin G_n^{(j_o+1)_1} = \cup_{k=1}^{\infty} G_{n,k}^{(j_o+1)_1} \quad \text{and} \quad 0 \le w_n^{(1)}(t) \le j_o .$$

This implies that

$$\frac{\int_{g_{n,k_o}^h} w_n^{(1)}(t)dt}{|h|} \le j_o .$$

We have

$$0 \le D^+ W_n^{(1)}(x) < j_o + 1 ,$$
$$0 \le D_+ W_n^{(1)}(x) < j_o + 1 ,$$
$$0 \le D^- W_n^{(1)}(x) < j_o + 1 ,$$
$$0 \le D_- W_n^{(1)}(x) < j_o + 1 .$$

Hence, for $x \notin E_n^1$ we have

$$0 \le D_+ W_n^{(1)}(x) < +\infty \quad \text{and} \quad 0 \le D_- W_n^{(1)}(x) < +\infty \,.$$

Consider the case where $x \in E^2$. By Lemma 3, there exists some k such that $x \in E_{n,k}^2$. By Lemma 3, 1), we have $E^2 \cap E_{n,k}^2 \subset e_{n,k}^2$, hence $x \in e_n^2$. Then there exists some k_o such that $x \in e_{n,k_o-1}^2$. Since

$$g_{n,k_o}^* \subset g_{n,k_o-1} \setminus \left(e_{n,k_o-1}^1 \cup e_{n,k_o-1}^2\right) ,$$

it follows that $x \notin g_{n,k_o}^*$ and $x \notin E_n^1$. From this fact we have

$$0 \le D^+ W_n^{(1)}(x) < +\infty \quad \text{and} \quad 0 \le D^- W_n^{(1)}(x) < +\infty \,.$$

Analogously we can prove 3) and 4). □

PROOF. [Proof of Theorem 1]

1) This follows from Theorem 1 of [3] and Lemma 1, where $B_1 = B_2 = \emptyset$.

2) The proof is as that of Theorem 2 of [2] (see pp. 445-449), using properties 1)–4) from Lemma 5. □

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that every point $x \in D_f$ is isolated in D_f. Then there exist $F_1, F_2 \in F_\sigma$ with $F_1 \cap F_2 = \emptyset$ such that*

$$\{x : f_+^{'}(x) = +\infty\} \subset F_1 \quad \textit{and} \quad \{x : f_+^{'}(x) = -\infty\} \subset F_2 \,.$$

PROOF. This theorem follows from Lemma 1. Since $B_1 \subset D_f$ and $B_2 \subset D_f$ it follows that every point of B_1 or B_2 is isolated in B_1, B_2. The sets B_1 and B_2 are at most countable. From the definition of an isolated point we have that:

$$\mathbb{R} \setminus B_1 = \cup_{n=1}^{\infty} I_n^1 \cup P_1 \quad \text{and} \quad \mathbb{R} \setminus B_2 = \cup_{n=1}^{\infty} I_n^2 \cup P_2 \,,$$

where I_n^1, I_n^2 are intervals such that $I_k^i \cap I_1^i = \emptyset$ for $k \ne 1$ $(i = 1, 2)$ and $I_k^i = (a_k^i, b_k^i)$ with $a_k^i, b_k^i \in B_i$, P_i is the set of all accumulation points of B_i. Of course $\cup_{n=1}^{\infty} I_n^i$ is an open set and P_i are closed sets. Hence $\mathbb{R} \setminus B_1 \in F_\sigma$ and $\mathbb{R} \setminus B_2 \in F_\sigma$. It follows that $F_i \in A_i \setminus B_i \in F_\sigma$ $(i = 1, 2)$. This proves our theorem. □

Remark 1. From Theorem 1 of [3] and our Theorem 2, it follows that in Theorem 1 we may replace condition "$f : \mathbb{R} \to \mathbb{R}$, f a continuous function" with "$f : \mathbb{R} \to \mathbb{R}$ such that every point $x \in D_f$ is an isolated point of D_f".

3 Infinite one-sided derivatives and Baire functions

Theorem 3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a Baire function of α class ($f\in\mathcal{B}_\alpha$). Then there exist some sets F_i, $i=1,2,3,4$, such that*

$$\{x\,:\,f_+^{'}(x)=+\infty\}\subset F_1\,,\quad \{x\,:\,f_+^{'}(x)=-\infty\}\subset F_2\,,$$

$$\{x\,:\,f_-^{'}(x)=+\infty\}\subset F_3\,,\quad \{x\,:\,f_-^{'}(x)=-\infty\}\subset F_4\,,$$

where $F_2,F_4\in F_{\alpha+1}$, $F_1,F_3\in G_{\alpha+1}$, and $F_1\cap F_2=\emptyset$, $F_3\cap F_4=\emptyset$.

Proof. The case $\alpha=0$ follows from Theorem 1, 1).

Let $\alpha>0$. We have

$$\{x\,:\,f_+^{'}(x)=+\infty\}=\{x\,:\,D_+f(x)=+\infty\}=\cap_{p=1}^{\infty}\{x\,:\,D_+f(x)>p\}$$

and

$$\{x\,:\,f_+^{'}(x)=-\infty\}=\{x\,:\,D^+f(x)=-\infty\}=\cap_{p=1}^{\infty}\{x\,:\,D^+f(x)<-p\}\,.$$

Hence

$$\{x\,:\,f_+^{'}(x)=+\infty\}\subset\{x\,:\,D^+f(x)>p\}$$

and

$$\{x\,:\,f_+^{'}(x)=-\infty\}\subset\{x\,:\,D^+f(x)<-p\}\,,$$

where $p\in\mathbb{N}$. Let

$$F_{m,n}(x)=\sup_{\frac{1}{m}\le h\le\frac{1}{n}}\frac{f(x+h)-f(x)}{h}\,,$$

$$H_{m,n}(x)=\inf_{\frac{1}{m}\le h\le\frac{1}{n}}\frac{f(x+h)-f(x)}{h}\,,$$

$$F_n(x)=\sup_{0<h\le\frac{1}{n}}\frac{f(x+h)-f(x)}{h}\,,$$

$$H_n(x)=\inf_{0<h\le\frac{1}{n}}\frac{f(x+h)-f(x)}{h}\,.$$

We have

$$F_n(x)=\lim_{m\to\infty}F_{m,n}(x)\,,\quad H_n(x)=\lim_{m\to\infty}H_{m,n}\,,$$

$$F_n(x)\ge F_{n+1}(x)\,,\quad H_n(x)\le H_{n+1}(x)\,.$$

By [1], $F_{m,n} \in \mathcal{B}_\alpha$ and $H_{m,n} \in \mathcal{B}_\alpha$. Consequently $F_n, H_n \in \mathcal{B}_{\alpha+1}$. We prove that

$$D^+ f(x) = \lim_{n\to\infty} F_n(x) \quad \text{and} \quad D_+ f(x) = \lim_{n\to\infty} H_n(x)\,.$$

Let $y = \lim_{n\to\infty} F_n(x)$. Then there exist $y_n < F_n(x)$ such that $y_n \to y$ if $n \to \infty$. Since $y_n < F_n(x)$ and by the definition of $F_n(x)$, we know that there exists $h_n \in (0, \frac{1}{n}]$ such that

$$F_n(x) \geq \frac{f(x+h_n) - f(x)}{h_n} > y_n\,.$$

Hence

$$\lim_{n\to\infty} \frac{f(x+h_n) - f(x)}{h_n} = y \quad \text{and} \quad D^+ f(x) \geq y\,.$$

Let $z_n < D^+ f(x)$, $z_n \to D^+ f(x)$. For every n there exists $h_n \in (0, \frac{1}{n}]$ such that

$$\frac{f(x+h_n) - f(x)}{h_n} > z_n\,.$$

Hence $F_n(x) > z_n$ and $y = \lim_{n\to\infty} F_n(x) \geq D^+ f(x)$. Finally $D^+ f(x) = \lim_{n\to\infty} F_n(x)$. Analogously $D_+ f(x) = \lim_{n\to\infty} H_n(x)$.

From the definitions of the limits we have:

$$D^+ f(x) = \inf_n \{F_n(x)\} \quad \text{and} \quad D_+ f(x) = \sup_n \{H_n(x)\}\,.$$

Hence

$$\{x : D^+ f(x) < -p\} \subset \cup_{n=1}^{\infty} \{x : F_n(x) < -p\}\,,$$

$$\{x : D_+ f(x) > -p\} \subset \cup_{n=1}^{\infty} \{x : H_n(x) > p\}\,.$$

Because $F_n, H_n \in \mathcal{B}_{\alpha+1}$, it follows that $\{x : F_n(x) < -p\} \in F_{\alpha+1}$ if α is even, and $\{x : H_n(x) > p\} \in G_{\alpha+1}$ if α is odd. It follows that the sets

$$F_1 = \cup_{n=1}^{\infty} \{x : F_n(x) < -p\} \quad \text{and} \quad F_2 = \cup_{n=1}^{\infty} \{x : H_n(x) > p\}$$

are identical. From the properties of F_n and H_n we have $F_1 \cap F_2 = \emptyset$.

For F_3, F_4 and the left-hand derivatives the proof is similar. □

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