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# UNIFORM INTEGRABILITY AND MEAN CONVERGENCE FOR THE VECTOR-VALUED MCSHANE INTEGRAL

**Abstract**

We show that a pointwise convergent, uniformly integrable sequence of Banach space valued, McShane integrable functions converges in mean. We also show that uniform integrability holds in a vector-valued generalization of the Beppo Levi convergence theorem.

It has been observed in [3, 4, 5], [7] that uniform integrability for the Henstock-Kurzweil integral is a sufficient condition to "take the limit under the integral sign." In this note we point out that uniform integrability for the McShane integral is actually a sufficient condition for mean or $L^1$ convergence. Our methods extend easily to functions with values in a Banach space so we consider this case where the results give significant improvements to the scalar case. We also show that the conclusion of the vector-valued generalization of the Monotone Convergence (Beppo Levi) Theorem given in [10] can be improved to uniform integrability.

We fix the notation and terminology which will be used in the sequel. It should be noted that we will work in $\mathbb{R}$ whereas the results in [3, 4, 5] are for compact intervals in $\mathbb{R}$. Let $X$ be a (real) Banach space and let $\mathbb{R}^*$ be the extended real line with the points $\pm\infty$ added. If $f$ is any function $f : \mathbb{R} \to X$, we always assume that $f$ is extended to $\mathbb{R}^*$ by setting $f(\pm\infty) = 0$.

A gauge is a function $\gamma$ on $\mathbb{R}^*$ whose value at a point $t$ is a neighborhood $\gamma(t)$ of $t$, where $\gamma(t)$ is bounded whenever $t \in \mathbb{R}$. [A neighborhood of $\infty$ is an interval of the form $(a, \infty]$; similarly for $-\infty$.] A partition of $\mathbb{R}$ is a finite collection of left-closed intervals $\{I_i : i = 1, \dots, n\}$ such that $I_i \cap I_j = \phi$ for $i \neq j$ and $\mathbb{R} = \bigcup_{i=1}^{n} I_i$ (here we agree that $(-\infty, a)$ is left-closed). A tagged partition of $\mathbb{R}$ is a finite collection of pairs $\{(I_i, t_i) : i = 1, \dots, n\}$ such that

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$\{I_i : i = 1, \ldots, n\}$ is a partition of $\mathbb{R}$ and $t_i \in \mathbb{R}^*$; $t_i$ is called the tag associated with $I_i$. Note that it is not required that $t_i \in I_i$; this requirement is what distinguishes the McShane and the Henstock-Kurzweil integral ([1],[7],[8],[3]). If $\gamma$ is a gauge on $\mathbb{R}^*$, a tagged partition $\mathcal{D} = \{(I_i, t_i) : i = 1, \ldots, n\}$ is $\gamma$-fine if $\bar{I}_i \subset \gamma(t_i)$ for $i = 1, \ldots, n$. If $\mathcal{D} = \{(I_i, t_i) : i = 1, \ldots, n\}$ is a tagged partition and $f : \mathbb{R} \to X$, we write $S(f, \mathcal{D}) = \sum_{i=1}^{n} f(t_i)\ell(I_i)$ for the Riemann sum of $f$ with respect to $\mathcal{D}$ where $\ell$ is Lebesgue measure on $\mathbb{R}$ [here we make the usual agreement that $0 \cdot \infty = 0$].

**Definition 1.** *A function $f : \mathbb{R} \to X$ is (McShane) integrable over $\mathbb{R}$ if there exists $v \in X$ such that for every $\varepsilon > 0$ there exists a gauge $\gamma$ on $\mathbb{R}^*$ such that $\|S(f, \mathcal{D}) - v\| < \varepsilon$ whenever $\mathcal{D}$ is $\gamma$-fine.*

The vector $v$ is called the (McShane) integral of $f$ over $\mathbb{R}$ and is denoted by $\int_{\mathbb{R}} f$. We refer the reader to [8], [3] for basic properties of the McShane integral.

Let $M(\mathbb{R}, X)$ be the space of all $X$-valued integrable functions defined on $\mathbb{R}$; if $X = \mathbb{R}$, we abbreviate $M(\mathbb{R}, \mathbb{R}) = M(\mathbb{R})$. The space $M(\mathbb{R})$ is complete under the semi-norm $\|f\|_1 = \int_{\mathbb{R}} |f|$ ([8, VI.4.3]). We define a semi-norm on $M(\mathbb{R}, X)$ by $\|f\|_1 = \sup\{\int_{\mathbb{R}} |x'f| : x' \in X', \|x'\| \leq 1\}$; in general, $\| \|_1$ is not complete ([10]; see [4], [2] for properties of the vector-valued McShane integral and its comparison with the Pettis and Bochner integrals). We describe another semi-norm which is equivalent to $\| \|_1$ and is useful in estimation. Let $\mathcal{A}$ be the algebra of subsets of $\mathbb{R}$ generated by the left-closed subintervals of $\mathbb{R}$; thus, every element of $\mathcal{A}$ is a finite, pairwise disjoint union of left-closed intervals ([9, 2.1.11]). If $f \in M(\mathbb{R}, X)$, $A \in \mathcal{A}$ and $C_A$ denotes the characteristic function of $A$, then $C_A f$ is also integrable and $\|f\|_1' = \sup\{\left\|\int_{\mathbb{R}} C_A f\right\| = \left\|\int_A f\right\| : A \in \mathcal{A}\}$ defines a semi-norm on $M(\mathbb{R}, X)$ which is equivalent to $\| \|_1$ ([10]).

We next define uniform (McShane) integrability and show that uniform integrability implies convergence in $\| \|_1$. A family $\mathcal{F}$ of $X$-valued functions defined on $\mathbb{R}$ is uniformly integrable if for every $\varepsilon > 0$ there exists a gauge $\gamma$ on $\mathbb{R}^*$ such that $\left\|S(f, \mathcal{D}) - \int_{\mathbb{R}} f\right\| < \varepsilon$ for every $f \in \mathcal{F}$ and $\mathcal{D}$ $\gamma$-fine; that is, the gauge is independent of the functions in $\mathcal{F}$.

For our next theorems we require an important result called the Henstock Lemma. If $\mathcal{D} = \{(I_i, t_i) : i = 1, \ldots, n\}$ is any collection of pairwise disjoint, left-closed subintervals $\{I_i\}$ and $t_i \in \mathbb{R}^*$, then $\mathcal{D}$ is called a partial tagged partition of $\mathbb{R}$ (it is not required that $\bigcup_{i=1}^{n} I_i = \mathbb{R}$); if $\gamma$ is a gauge on $\mathbb{R}^*$, $\mathcal{D}$ is $\gamma$-fine if $\overline{I}_i \subset \gamma(t_i)$ for $i = 1, \ldots, n$. We again write $S(f, \mathcal{D}) = \sum_{i=1}^{n} f(t_i)\,\ell(I_i)$.

**Lemma 2.** *(Henstock) Let $f \in M(\mathbb{R}, X)$ and $\varepsilon > 0$. Suppose the gauge $\gamma$ on $\mathbb{R}^*$ is such that $\left\| S(f, \mathcal{D}) - \int_{\mathbb{R}} f \right\| < \varepsilon$ for every $\gamma$-fine tagged partition $\mathcal{D}$ of $\mathbb{R}$. If $\mathcal{D}$ is any $\gamma$-fine partial tagged partition and $I = \bigcup_{i=1}^{n} I_i$, then $\left\| S(f, \mathcal{D}) - \int_I f \right\| \leq \varepsilon$.*

See [4] for Lemma 2.

We first establish an interesting preliminary result.

**Theorem 3.** *Let $f_k \in M(\mathbb{R}, X)$* for every $k \in \mathbb{N}$. *If $\{f_k\}$ is uniformly integrable over $\mathbb{R}$, then*

$$\lim_{b \to \infty} \left\| C_{[b,\infty)} f_k \right\|_1' = 0$$

*uniformly for $k \in \mathbb{N}$.*

PROOF. Let $\varepsilon > 0$. There exists a gauge $\gamma$ with $\gamma(z)$ bounded for every $z \in \mathbb{R}$ such that $\left\| \int_{\mathbb{R}} f_k - S(f_k, \mathcal{D}) \right\| < \varepsilon$ when $\mathcal{D}$ is $\gamma$-fine. Fix such a $\mathcal{D} = \{(I_i, t_i) : 1 \leq i \leq m\}$ and assume that $I_1 = [b, \infty)$, $t_1 = \infty$. If $a \geq b$, let $A$ be an element of $\mathcal{A}$ with $A \subset [a, \infty)$ and $A = \bigcup_{i=1}^{n} J_i$, $J_i$ a left-closed interval and $\{J_i\}$ pairwise disjoint. Then $\mathcal{J} = \{(J_i, \infty) : 1 \leq i \leq n\}$ is a $\gamma$-fine partial tagged partition so Henstock's Lemma implies that

$$\left\| \int_A f_k - S(f_k, \mathcal{J}) \right\| = \left\| \int_A f_k \right\| \leq \varepsilon.$$

Since $A \in \mathcal{A}$ is arbitrary, $\left\| C_{[a,\infty)} f_k \right\|_1' \leq \varepsilon$ for $a \geq b$.

We next establish our mean convergence result.

**Theorem 4.** *Let $f_k : \mathbb{R} \to X$ be integrable for every $k \in \mathbb{N}$ and suppose $\{f_k\}$ converges pointwise to $f$. If $\{f_k\}$ is uniformly integrable, then $f$ is integrable and $\|f_k - f\|_1 \to 0$.*

PROOF. Let $\varepsilon > 0$. There exists a gauge $\gamma$ with $\gamma(z)$ bounded for every $z \in \mathbb{R}$ such that $\left\| \int_{\mathbb{R}} f_k - S(f_k, \mathcal{D}) \right\| < \varepsilon$ whenever $\mathcal{D}$ is $\gamma$-fine.

Fix a tagged partition $\mathcal{D} = \{(I_i, t_i) : 1 \leq i \leq m\}$ which is $\gamma$-fine. Since $\{f_k\}$ is pointwise convergent and $f_k(\pm\infty) = 0$, there exists $N$ such that $k$, $j \geq N$ implies $\|S(f_k, \mathcal{D}) - S(f_j, \mathcal{D})\| = \left\| \sum_{i=1}^{m} (f_k(t_i) - f_j(t_i)) \ell(I_i) \right\| < \varepsilon$. Therefore, if $k$, $j \geq N$, we have

$$\left\|\int_{\mathbb{R}} f_k - \int_{\mathbb{R}} f_j\right\| \leq \left\|\int_{\mathbb{R}} f_k - S\left(f_k, \mathcal{D}\right)\right\| + \left\|S\left(f_k, \mathcal{D}\right) - S\left(f_j, \mathcal{D}\right)\right\| + \left\|S\left(f_j, \mathcal{D}\right) - \int_{\mathbb{R}} f_j\right\| < 3\varepsilon.$$

Hence, $\lim_k \int_{\mathbb{R}} f_k = L$ exists in $X$.

We claim that $\int_{\mathbb{R}} f = L$. Let $\mathcal{E}$ be $\gamma$-fine. Pick $n_0$ such that $k \geq n_0$ implies $\left\|\int_{\mathbb{R}} f_k - L\right\| < \varepsilon$. As above there exists $n_1 \geq n_0$ such that whenever $k \geq n_1$, it follows that $\left\|S\left(f, \mathcal{E}\right) - S\left(f_k, \mathcal{E}\right)\right\| < \varepsilon$. Then

$$\left\|L - S\left(f, \mathcal{E}\right)\right\| \leq \left\|L - \int_{\mathbb{R}} f_{n_1}\right\| + \left\|\int_{\mathbb{R}} f_{n_1} - S\left(f_{n_1}, \mathcal{E}\right)\right\| + \left\|S\left(f_{n_1}, \mathcal{E}\right) - S\left(f, \mathcal{E}\right)\right\| < 3\varepsilon$$

and the claim is established.

For the last statement we may assume that $f = 0$ since $f_k - f \to 0$ pointwise and $\{f_k - f\}$ is uniformly integrable. With $\mathcal{D}$ fixed as above, assume that $I_1 = [b, \infty)$ and $I_2 = (-\infty, a)$ and set $I = \mathbb{R} \setminus I_1 \cup I_2$. Let $A$ be an arbitrary element of $\mathcal{A}$ with $A = \bigcup_{j=1}^{n} J_j$, $J_j$ a left-closed interval, $\{J_j\}$ pairwise disjoint. Then $\mathcal{E} = \{(I_i \cap J_j, t_i) : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is $\gamma$-fine so Henstock's Lemma implies $\left\|\sum_{i=1}^{m}\sum_{j=1}^{n}\left\{\int_{I_i \cap J_j} f_k - f_k\left(t_i\right)\ell\left(I_i \cap J_j\right)\right\}\right\| \leq \varepsilon$. Hence,

$$\begin{aligned}\left\|\int_A f_k\right\| &\leq \varepsilon + \left\|\sum_{i=1}^{m}\sum_{j=1}^{n} f_k\left(t_i\right)\ell\left(I_i \cap J_j\right)\right\| = \varepsilon + \left\|\sum_{i=3}^{m} f_k\left(t_i\right)\ell\left(I_i \cap A\right)\right\| \\ &\leq \varepsilon + \sup_{3 \leq i \leq m}\left\|f_k\left(t_i\right)\right\|\ell\left(I\right),\end{aligned}$$

and $k$ can be taken large enough so the last term is less than $\varepsilon$. Since $A$ is arbitrary, it follows that $\left\|f_k\right\|_1' \leq 2\varepsilon$ for large $k$.

It has been previously observed that $f$ is integrable and uniform integrability of a pointwise convergent sequence $\{f_k\}$ implies that $\lim \int_{\mathbb{R}} f_k = \int_{\mathbb{R}} f$ ([7], [3, 4, 5]). Since $\|\ \|_1$ and $\|\ \|_1'$ are equivalent, the conclusion of Theorem 4 implies that $\lim \int_A f_k = \int_A f$ uniformly for $A \in \mathcal{A}$ giving a significant improvement particularly in the vector-valued result for the McShane integral.

In [10] we established a convergence theorem for vector-valued McShane integrable functions which easily implies the Monotone Convergence (Beppo Levi) Theorem for scalar-valued functions. We now show that the conclusion of this generalized Monotone Convergence Theorem can be improved from $\| \|_1$-convergence to uniform integrability.

Since we are working over $\mathbb{R}$ instead of a bounded interval we need a preliminary lemma.

**Lemma 5.** *There exists a positive McShane integrable function $\varphi : \mathbb{R} \to (0, \infty)$ and a gauge $\gamma\,(= \gamma_\varphi)$ such that $0 \leq S\,(\varphi, \mathcal{D}) \leq 1$ whenever $\mathcal{D}$ is a $\gamma$-fine partial tagged partition.*

PROOF. Let $\varphi$ be positive with $\int_{\mathbb{R}} \varphi = \frac{1}{2}$. Let $\gamma$ be a gauge with $\left|\frac{1}{2} - S\,(\varphi, \mathcal{D})\right| < \frac{1}{2}$ whenever $\mathcal{D}$ is a $\gamma$-fine tagged partition of $\mathbb{R}$. Since $\varphi$ is positive, the result follows immediately from Henstock's Lemma.

**Theorem 6.** *Let $f_k \subset M\,(\mathbb{R}, X)$ and suppose $\sum\limits_{k=1}^{\infty} f_k = f$ pointwise with $\sum\limits_{k=1}^{\infty} \|f_k\|_1 < \infty$. If $F_n = \sum\limits_{k=1}^{n} f_k$, then (i) $\{F_n\}$ is uniformly integrable, (ii) $f$ is integrable and (iii) $\|F_n - f\|_1 \to 0$.*

PROOF. Let $\varepsilon > 0$. For each $n$ pick a gauge $\gamma_n$ with $\gamma_n(z)$ bounded for every $z \in \mathbb{R}$ such that $\left\|\int\limits_{\mathbb{R}} F_n - S\,(F_n, \mathcal{D})\right\| < \varepsilon/2^n$ whenever $\mathcal{D}$ is $\gamma_n$-fine. Pick $n_0$ such that $\sum\limits_{k=n_0}^{\infty} \|f_k\| < \varepsilon$, and for every $t \in \mathbb{R}$ pick $n\,(t) \geq n_0$ such that $k \geq j \geq n\,(t)$ implies $\left\|\sum\limits_{i=j}^{k} f_i\,(t)\right\| < \varepsilon\varphi\,(t)$, where $\varphi$ and $\gamma_\varphi$ are as in Lemma 5.

Define a gauge $\gamma$ on $\mathbb{R}$ by $\gamma\,(t) = \left(\bigcap\limits_{j=1}^{n(t)} \gamma_j\,(t)\right) \cap \gamma_\varphi\,(t)$ for $t \in \mathbb{R}$ and $\gamma\,(\pm\,\infty) = \left(\bigcap\limits_{j=1}^{n_0} \gamma_j\,(\pm\,\infty)\right) \cap \gamma_\varphi\,(\pm\,\infty)$ and set $n\,(\pm\,\infty) = n_0$. Suppose $\mathcal{D} = \{(I_i, t_i) : 1 \leq i \leq m\}$ is $\gamma$-fine. To establish (i), first note that $\mathcal{D}$ is $\gamma_i$-fine for $i = 1, \ldots, n_0$ implies that $\left\|\int\limits_{\mathbb{R}} F_i - S\,(F_i, \mathcal{D})\right\| < \varepsilon/2^i < \varepsilon$.

So, now fix $n > n_0$. Set $d_1 = \{i : 1 \leq i \leq m, n\,(t_i) \geq n\}$ and $d_2 = \{i : 1 \leq i \leq m, n\,(t_i) < n\}$, and note $\mathcal{D}_1 = \{(I_i, t_i) : i \in d_1\}$ is $\gamma_n$-fine by the definition of $\gamma$. Set $I = \cup \{I_i : i \in d_1\}$. We have, using Henstock's Lemma,

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} F_n - S(F_n, \mathcal{D}) \right\| \le \\
&\quad \le \left\| \int_I F_n - S(F_n, \mathcal{D}_1) \right\| + \left\| \sum_{i \in d_2} \sum_{j=1}^{n} \left\{ \int_{I_i} f_j - f_j(t_i)\, \ell(I_i) \right\} \right\| \qquad (1) \\
&\quad \le \varepsilon/2^n + \left\| \sum_{i \in d_2} \sum_{j=1}^{n(t_i)} \left\{ \int_{I_i} f_j - f_j(t_i)\, \ell(I_i) \right\} \right\| + \\
&\qquad + \left\| \sum_{i \in d_2} \sum_{j=n(t_i)+1}^{n} \int_{I_i} f_j \right\| + \left\| \sum_{i \in d_2} \sum_{j=n(t_i)+1}^{n} f_j(t_i)\, \ell(I_i) \right\| \\
&\quad < \varepsilon + T_1 + T_2 + T_3,
\end{aligned}$$

with obvious notation for the $T_i$.

First, we estimate $T_3$:

$$T_3 \le \sum_{i \in d_2} \left\| \sum_{j=n(t_i)+1}^{n} f_j(t_i) \right\| \ell(I_i) \le \sum_{i \in d_2} \varepsilon \varphi(t_i)\, \ell(I_i) = \varepsilon S(\varphi, \mathcal{D}_2) \le \varepsilon,$$

where $\mathcal{D}_2 = \{(I_i, t_i) : i \in d_2\}$.

Next,

$$\begin{aligned}
T_2 &= \sup \left\{ \left| < x', \sum_{i \in d_2} \sum_{j=n(t_i)+1}^{n} \int_{I_i} f_j > \right| : \|x'\| \le 1 \right\} \\
&\le \sup \left\{ \sum_{i \in d_2} \sum_{j=n(t_i)+1}^{n} \int_{I_i} |x' f_j| : \|x'\| \le 1 \right\} \\
&\le \sup \left\{ \sum_{i \in d_2} \sum_{j=n_0}^{n} \int_{I_i} |x' f_j| : \|x'\| \le 1 \right\} \\
&\le \sup \left\{ \sum_{j=n_0}^{n} \int_{\mathbb{R}} |x' f_j| : \|x'\| \le 1 \right\} \le \sum_{j=n_0}^{n} \|f_k\|_1 < \varepsilon.
\end{aligned}$$

For $T_1$, let $s = \max\{n(t_i) : i \in d_2\}$. Then

$$\begin{aligned}
T_1 &= \left\| \sum_{i \in d_2} \left\{ \int\limits_{I_i} F_{n(t_i)} - F_{n(t_i)}(t_i)\ell\left(I_i\right) \right\} \right\| \\
&= \left\| \sum_{k=1}^{s} \sum_{\substack{i \\ n(t_i)=k}} \left\{ \int\limits_{I_i} F_{n(t_i)} - F_{n(t_i)}(t_i)\ell\left(I_i\right) \right\} \right\| \\
&\leq \sum_{k=1}^{s} \left\| \sum_{\substack{i \\ n(t_i)=k}} \left\{ \int_{I_i} F_{n(t_i)} - F_{n(t_i)}(t_i)\ell\left(I_i\right) \right\} \right\| \leq \sum_{k=1}^{s} \varepsilon/2^k < \varepsilon,
\end{aligned}$$

by Henstock's Lemma since $\{(I_i, t_i) : n\left(t_i\right) = k\}$ is $\gamma_k$-fine.

From (1), it follows that $\left\| \int\limits_{\mathbb{R}} F_n - S\left(F_n, \mathcal{D}\right) \right\| < 4\varepsilon$ as required, and (i) holds.

Conditions (ii) and (iii) now follow from Theorem 4.

For the McShane integral we have from Theorem 6 a version of the Monotone Convergence Theorem (MCT) for the McShane integral. The conclusion in part (i) strengthens the "usual" conclusions in the MCT (see for example [3, 10.10]).

**Corollary 7.** *(MCT). Let $f_k : \mathbb{R} \to \mathbb{R}$ be integrable for every $k \in \mathbb{N}$ and suppose that $f_k\left(t\right) \uparrow f\left(t\right)$ for every $t$. If $\sup \left\{ \int\limits_{\mathbb{R}} f_k : k \in \mathbb{N} \right\} < \infty$, then (i) $\{f_k\}$ is uniformly $M$-integrable, (ii) $f$ is integrable and (iii) $\int\limits_{\mathbb{R}} f_k \uparrow \int\limits_{\mathbb{R}} f$.*

Proof. Set $g_0 = 0$ and $g_k = f_k - f_{k-1}$ for $k \geq 1$. Then $\sum\limits_{k=1}^{n} g_k = f_n \to f$ pointwise and

$$\sum_{k=1}^{\infty} \int\limits_{\mathbb{R}} |g_k| = \lim_n \sum_{k=1}^{n} \int\limits_{\mathbb{R}} \left(f_k - f_{k-1}\right) = \lim_n \int\limits_{\mathbb{R}} f_k = \sup_k \int\limits_{\mathbb{R}} f_k < \infty.$$

Hence, Theorem 6 gives the result.

Similarly, it was noted by McLeod that the same improvement can be obtained for the Monotone Convergence Theorem for the Henstock-Kurzweil integral ([7, p. 98] ; a similar result is obtained by Gordon ([3, 13.18] ), but his proof uses Lebesgue integration.

We can use the MCT of Corollary 7 to obtain a similar generalization of the Dominated Convergence Theorem (DCT) for the McShane integral. A sequence $f_k : \mathbb{R} \to X$ is said to be uniformly McShane or $M$-Cauchy if for every $\varepsilon > 0$ there exists a gauge $\gamma$ and $N$ such that $\|S(f_i, \mathcal{D}) - S(f_j, \mathcal{D})\| < \varepsilon$ for $i, j \geq N$ and $\mathcal{D}$ $\gamma$-fine. As in Theorem 4 of [5], we have

**Proposition 8.** *Let $f_k : \mathbb{R} \to X$ be integrable for every $k \in \mathbb{N}$. Then $\{f_k\}$ is uniformly $M$-Cauchy if and only if $\{f_k\}$ is uniformly integrable and $\lim \int_{\mathbb{R}} f_k$ exists.*

**Corollary 9.** *(DCT) Let $f_k : \mathbb{R} \to \mathbb{R}$ be integrable for every $k \in \mathbb{N}$ and suppose $f_k \to f$ pointwise. Assume there exists $g : \mathbb{R} \to \mathbb{R}$, integrable and such that $|f_i - f_j| \leq g$ for all $i$, $j$. Then (i) $\{f_k\}$ is uniformly integrable, (ii) $f$ is integrable and (iii) $\int_{\mathbb{R}} |f_k - f| \to 0$.*

PROOF. Set $t_{jk} = \vee \{|f_m - f_n| : j \leq m \leq n \leq k\}$. For each $j$ $\{t_{jk}\}_k$ is increasing and converges to the function $t_j = \vee \{|f_m - f_n| : j \leq m \leq n\}$ with $0 \leq t_j \leq g$. Corollary 7 implies that $t_j$ is integrable and $\int_{\mathbb{R}} t_j \leq \int_{\mathbb{R}} g$. Now $0 \leq t_{j+1} \leq t_j$ and $t_j \to 0$ pointwise so Corollary 7 implies that $\int_{\mathbb{R}} t_j \downarrow 0$.

Let $\varepsilon > 0$. There exists $N$ such that $\int_{\mathbb{R}} t_N < \varepsilon$, and there exists a gauge $\gamma$ such that $\left| \int_{\mathbb{R}} t_N - S(t_N, \mathcal{D}) \right| < \varepsilon$ when $\mathcal{D}$ is $\gamma$-fine. If $i$, $j \geq N$, we have $|S(f_i, \mathcal{D}) - S(f_j, \mathcal{D})| \leq S(|f_i - f_j|, \mathcal{D}) \leq S(t_N, \mathcal{D}) < \int_{\mathbb{R}} t_N + \varepsilon < 2\varepsilon$ when $\mathcal{D}$ is $\gamma$-fine. Hence, $\{f_i\}$ is uniformly $M$-Cauchy. It follows from Proposition 8 that $\{f_i\}$ is uniformly integrable and (i) holds.

Conditions (ii) and (iii) follow from Theorem 6.

McLeod obtained a similar improvement in the DCT for the Henstock-Kurzweil integral ([7, p. 98] ); Gordon also obtained this result but employed the Lebesgue integral ([3, 13.17]).

Finally, in conclusion it should be noted that the results above are also valid with $\mathbb{R}$ being replaced by $\mathbb{R}^n$; only the notation becomes more cumbersome.

If $I = [a, b)$ is a bounded interval, it is straightforward to generalize the Henstock-Kurzweil integral to functions $f : I \to X$. If $HK(I, X)$ is the space of all $X$-valued Henstock-Kurzweil integrable functions defined on $I$, then $HK(I, X)$ has a natural semi-norm defined by $\|f\| = \sup \left\{ \left\| \int_a^t f \right\| : a \leq t \leq b \right\}$ ([6, 11.1]).

Problem: Are there analogues of Theorems 4 and 6 for the Henstock-Kurzweil integral?

The proofs of these results above are not valid for the Henstock-Kurzweil integral. In Theorem 4 McShane tagged partitions were used and the proof of Theorem 6 used the absolute integrability of the scalar-valued McShane integral in estimating $T_2$ so different techniques would be required.

The referee has observed that the sequence $f_k(t) = (\sin t)/t$ for $1 \leq t \leq 2k\pi$ and $f_k(t) = 0$ for $t > 2k\pi$ gives a counter-example to Theorem 3 for the Henstock-Kurzweil integral.

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