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# ON COUNTABLY GENERATED INVARIANT $\sigma$-ALGEBRAS WHICH DO NOT ADMIT MEASURE TYPE FUNCTIONALS

**Abstract**

Using the Banach-Kuratowski matrix, we give a generalization of the result from [6] concerning the nonexistence of nonzero $\sigma$-finite diffused measures on some countably generated invariant $\sigma$-algebras of sets.

The present note is devoted to certain countably generated invariant $\sigma$-algebras of sets, which do not admit nonzero $\sigma$-finite diffused (i.e. continuous) measures. The main tool of this note is the so-called Banach-Kuratowski matrix (see [1]) which shows, in particular, that, under the Continuum Hypothesis, there exists a countably generated $\sigma$-algebra $S$ of subsets of the real line $\mathbf{R}$, such that all one-element subsets of $\mathbf{R}$ belong to $S$ and there is no nonzero $\sigma$-finite diffused measure defined on $S$.

Here we construct (under the Continuum Hypothesis, of course) a certain Banach-Kuratowski matrix consisting of almost invariant sets with respect to the group of all isometric transformations of $\mathbf{R}$. We then apply that matrix and obtain a generalization of a result given in the paper of Pelc and Prikry[6]. Notice that the method used in our further considerations is taken from the papers [4] and [5].

First of all let us recall the construction of a Banach-Kuratowski matrix. Let $\omega$ denote, as usual, the least infinite ordinal number and let $F = \omega^\omega$ denote the family of all functions acting from $\omega$ into $\omega$. Let $f$ and $g$ be any two functions from $F$. We put $f \prec g$ if and only if there exists a natural

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number $n = n(f, g)$ such that $f(m) \leq g(m)$, for all natural numbers $m \geq n$. Evidently, the relation $\prec$ is a preordering on $F$. As usual, we denote by $\omega_1$ the least uncountable ordinal number. Now, assuming the Continuum Hypothesis ($2^\omega = \omega_1$), it is not difficult to define a subset

$$E = \{f_\xi \ : \ \xi < \omega_1\}$$

of $F$ satisfying the following two conditions:

a) if $f$ is an arbitrary function from $F$, then there exists an ordinal $\xi < \omega_1$ such that $f \prec f_\xi$ (in other words, $E$ is cofinal with $F$);

b) for any ordinals $\xi$ and $\zeta$ such that $\xi < \zeta < \omega_1$, the relation $f_\zeta \prec f_\xi$ is not true.

Notice that each of the conditions a) and b) implies the equality

$$\mathrm{card}\,(E) = \omega_1.$$

Further, for any two natural numbers $m$ and $n$, we put

$$E_{m,n} = \{f_\xi \in E \ : \ f_\xi(m) \leq n\}.$$

In this way, we get a countable double family of sets

$$(E_{m,n})_{m<\omega,\ n<\omega},$$

which is usually called the Banach-Kuratowski matrix over the basic set $E$. Each set $E_{m,n}$ is called a member of this matrix.

It can easily be checked that, for every natural number $m$, we have the inclusions

$$E_{m,0} \subset E_{m,1} \subset \cdots \subset E_{m,n} \subset \cdots$$

and the equality

$$E = \cup\{E_{m,n} \ : \ n < \omega\}.$$

Also, conditions a) and b) immediately imply, for an arbitrary function $f$ from $F$, that the set

$$E_{0,f(0)} \cap E_{1,f(1)} \cap \cdots \cap E_{m,f(m)} \cap \cdots$$

is at most countable.

Starting with these properties of the Banach-Kuratowski matrix, one can easily obtain that there does not exist a nonzero $\sigma$-finite diffused measure on $E$ defined simultaneously for all the sets

$$E_{m,n} \qquad (m < \omega,\ n < \omega).$$

This is the classical result of Banach and Kuratowski (see [1]). It is important for our further considerations that an analogous result remains true for many other functionals on $E$ which are much more general than ordinary measures. In particular, the above-mentioned result holds true for certain set functions which we call admissible functionals on $E$. Let us give the precise definition of an admissible functional.

Let $\nu$ be a real-valued function defined on some class of subsets of $E$. We say that $\nu$ is an admissible functional on $E$ if the following conditions are fulfilled:

1) $\mathrm{dom}\,(\nu)$ is closed under finite intersections;

2) $0 \le \nu(Z) < +\infty$, for all $Z \in \mathrm{dom}\,(\nu)$;

3) if $\{Z_k \ : \ k < \omega\}$ is an increasing (with respect to inclusion) family of sets belonging to $\mathrm{dom}\,(\nu)$, then the set $\cup\{Z_k \ : \ k < \omega\}$ also belongs to $\mathrm{dom}\,(\nu)$ and

$$\nu(\cup\{Z_k \ : \ k < \omega\}) \le \sup\{\nu(Z_k) \ : \ k < \omega\};$$

4) if $\{Z_k \ : \ k < \omega\}$ is a decreasing (with respect to inclusion) family of sets belonging to $\mathrm{dom}\,(\nu)$, then the set $\cap\{Z_k \ : \ k < \omega\}$ also belongs to $\mathrm{dom}\,(\nu)$ and

$$\nu(\cap\{Z_k \ : \ k < \omega\}) \ge \inf\{\nu(Z_k) \ : \ k < \omega\}.$$

In addition, we say that an admissible functional $\nu$ is diffused if the family of all countable subsets of $E$ is contained in $\mathrm{dom}\,(\nu)$ and, for each countable set $Z \subset E$, we have $\nu(Z) = 0$.

Obviously, if $\nu$ is a finite measure on $E$, then $\nu$ satisfies conditions 1) – 4). But, in general, an admissible functional $\nu$ need not have any additive properties similar to the corresponding properties of ordinary measures. However, it is not difficult to see that the Banach-Kuratowski method works for such functionals, too, and we can conclude that there does not exist a nonzero diffused admissible functional on $E$ defined simultaneously for all members of a Banach-Kuratowski matrix. Namely, we have the following statement.

**Lemma 1.** *Let $(E_{m,n})_{m<\omega,\ n<\omega}$ be a Banach-Kuratowski matrix over $E$ and let $\nu$ be a diffused admissible functional on $E$ such that*

$$\{E_{m,n} \ : \ m < \omega,\ n < \omega\} \subset \mathrm{dom}\,(\nu).$$

*Then $\nu$ is identically equal to zero.*

Proof. Suppose to the contrary that there exists a set $X \in \mathrm{dom}\,(\nu)$ with $\nu(X) = t > 0$. Obviously, we can find a natural number $n(0)$ for which $\nu(X \cap E_{0,n(0)}) > t/2$. Let us put $X_0 = X \cap E_{0,n(0)}$. Then we can find a natural number $n(1)$ for which $\nu(X_0 \cap E_{1,n(1)}) > t/2$. Let us put $X_1 =$

$X_0 \cap E_{1,n(1)}$. Proceeding in this manner we are able to construct a sequence of sets $\{X_m \ : \ m < \omega\}$ satisfying the conditions:

a) $X_0 \supset X_1 \supset \cdots \supset X_m \supset \cdots$;

b) $X_m \in \mathrm{dom}\,(\nu)$ and $\nu(X_m) > t/2$, for all $m < \omega$;

c) $X_m \subset E_{m,n(m)}$, for all $m < \omega$.

Now, from a) and b), we get $\nu(\cap\{X_m \ : \ m < \omega\}) \geq t/2 > 0$. On the other hand, c) immediately implies that $card(\cap\{X_m \ : \ m < \omega\}) \leq \omega$. Therefore, $\nu(\cap\{X_m \ : \ m < \omega\}) = 0$. The contradiction completes the proof of Lemma 1. $\square$

Now, we wish to construct an analogue of the Banach-Kuratowski matrix for our basic set $E$ equipped with a certain group of its transformations. In order to do it, we need the following auxiliary proposition (see, e.g., [4]).

**Lemma 2.** *Let $E$ be a basic set with $card(E) = \omega_1$ and let $G$ be a group of transformations of $E$, such that*

*(1)* $\mathrm{card}\,(G) = \mathrm{card}\,(E) = \omega_1$*;*

*(2) $G$ acts transitively in $E$.*

*Then there exists a family $\{Y_\xi \ : \ \xi < \omega_1\}$ of subsets of $E$ satisfying the subsequent four relations:*

*(a)* $\mathrm{card}\,(Y_\xi) \leq \omega$*, for each ordinal $\xi < \omega_1$;*

*(b) $Y_\xi \cap Y_\zeta = \emptyset$, for any two ordinals $\xi < \omega_1, \ \zeta < \omega_1, \ \xi \neq \zeta$;*

*(c) $\cup\{Y_\xi \ : \ \xi < \omega_1\} = E$;*

*(d) for each subset $\Xi$ of $\omega_1$, the set*

$$Y_\Xi = \cup\big\{Y_\xi \ : \ \xi \in \Xi\big\}$$

*is almost invariant with respect to $G$, i.e. we have*

$$\mathrm{card}\,(g(Y_\Xi) \triangle Y_\Xi) \leq \omega,$$

*for all transformations $g$ from $G$ (where the symbol $\triangle$ denotes, as usual, the operation of symmetric difference of sets).*

Proof. We can represent the given group $G$ in the form

$$G = \cup\big\{G_\xi \ : \ \xi < \omega_1\big\},$$

where a family $\{G_\xi \ : \ \xi < \omega_1\}$ has the following properties:

i) $\mathrm{card}\,(G_\xi) \leq \omega$, for each $\xi < \omega_1$;

ii) $G_\xi$ is a subgroup of $G$, for each $\xi < \omega_1$;

iii) $G_\xi \subset G_\zeta$, for any two ordinals $\xi$ and $\zeta$ such that $\xi < \zeta < \omega_1$.

Further, let us fix a point $e \in E$ and, for each $\xi < \omega_1$, let us put

$$Y_\xi = G_\xi(e) \setminus \cup\big\{G_\zeta(e) \ : \ \zeta < \xi\big\}.$$

Then it is not hard to check that the family of sets

$$\{Y_\xi \ : \ \xi < \omega_1\}$$

satisfies the relations (a) – (d). □

**Lemma 3.** *Suppose that the Continuum Hypothesis holds. Let $E$ be a basic set with* $\mathrm{card}\,(E) = 2^\omega = \omega_1$ *and let $G$ be a group of transformations of $E$, such that*

*(1)* $\mathrm{card}\,(G) = \mathrm{card}\,(E) = 2^\omega = \omega_1$*;*

*(2) $G$ acts transitively in $E$.*

*Then there exists a Banach-Kuratowski matrix*

$$(E_{m,n})_{m<\omega,\ n<\omega}$$

*over $E$, having the property that, for any two natural numbers $m$ and $n$, the set $E_{m,n}$ is almost invariant with respect to $G$, i.e.*

$$\mathrm{card}\,(g(E_{m,n}) \triangle E_{m,n}) \leq \omega,$$

*for all transformations $g$ from $G$.*

Proof. As we know, the Continuum Hypothesis implies the existence of a Banach-Kuratowski matrix

$$(\Xi_{m,n})_{m<\omega,\ n<\omega}$$

over the set $\omega_1$. Let $\{Y_\xi \ : \ \xi < \omega_1\}$ be a family of subsets of $E$, satisfying the relations (a) – (d) of Lemma 2. For any pair $(m, n)$ of natural numbers, we put

$$E_{m,n} = \cup\{Y_\xi \ : \ \xi \in \Xi_{m,n}\}.$$

One can easily check that

$$(E_{m,n})_{m<\omega,\ n<\omega}$$

is the required Banach-Kuratowski matrix. □

In [2], assuming Martin's Axiom, two $\sigma$-algebras $S_1$ and $S_2$ of subsets of the real line $\mathbf{R}$ were constructed satisfying the following conditions:

(1) $S_1$ and $S_2$ are countably generated;

(2) the Borel $\sigma$-algebra $B(\mathbf{R})$ is contained in $S_1 \cap S_2$;

(3) there exists a measure $\mu_1$ on $S_1$ extending the standard Borel measure on $\mathbf{R}$;

(4) there exists a measure $\mu_2$ on $S_2$ extending the standard Borel measure on $\mathbf{R}$;

(5) there is no nonzero $\sigma$-finite diffused measure defined on the $\sigma$-algebra generated by $S_1 \cup S_2$.

Notice also that a result similar (in some sense) to the result mentioned above was obtained in paper [3], without the aid of additional set-theoretical axioms.

Assuming the Continuum Hypothesis, it was proved in [6] that the $\sigma$-algebras $S_1$ and $S_2$ and the measures $\mu_1$ and $\mu_2$ can be taken invariant under the group $\Gamma$ of all isometric transformations of $\mathbf{R}$. The next statement yields a more general result.

**Theorem.** *Suppose that the Continuum Hypothesis holds. Then there are two $\sigma$-algebras $S_1$ and $S_2$ of subsets of $\mathbf{R}$, having the following properties:*

*(1) $S_1$ and $S_2$ are countably generated and invariant under the group $\Gamma$;*

*(2) $B(\mathbf{R}) \subset S_1 \cap S_2$;*

*(3) there exists a $\Gamma$-invariant measure $\mu_1$ on $S_1$ extending the standard Borel measure on $\mathbf{R}$;*

*(4) there exists a $\Gamma$-invariant measure $\mu_2$ on $S_2$ extending the standard Borel measure on $\mathbf{R}$;*

*(5) there is no nonzero diffused admissible functional defined on the $\sigma$-algebra generated by $S_1 \cup S_2$.*

Proof. We denote by $\lambda$ the standard Borel measure on $\mathbf{R}$, which is invariant with respect to the group $\Gamma$. Obviously, $\Gamma$ acts transitively in $\mathbf{R}$ and satisfies the equalities

$$\mathrm{card}\,(\Gamma) = \mathrm{card}\,(\mathbf{R}) = 2^{\omega} = \omega_1.$$

Therefore, taking into account Lemma 3, we can find a countable family

$$\{X_k \ : \ k < \omega\}$$

of subsets of $\mathbf{R}$, such that

(a) $\mathrm{card}\,(g(X_k) \triangle X_k) \leq \omega$, for each $k < \omega$ and for each $g \in \Gamma$;

(b) there is no nonzero diffused admissible functional on $\mathbf{R}$ defined simultaneously for all sets $X_k$ $(k < \omega)$.

In particular, it immediately follows from relation (b) that at least one set $X_k$ is nonmeasurable with respect to the measure $\lambda$. We may assume, without loss of generality, that $X_0$ is nonmeasurable with respect to $\lambda$. On the other hand, according to relation (a), the set $X_0$ is almost invariant with respect to $\Gamma$. Consequently, applying the metrical transitivity of $\lambda$, we obtain that $X_0$ and $\mathbf{R} \setminus X_0$ are $\lambda$-thick subsets of $\mathbf{R}$, i.e.

$$\lambda_*(X_0) = \lambda_*(\mathbf{R} \setminus X_0) = 0.$$

Now, let us put

$S_1$ = the $\sigma$-algebra generated by

$$B(\mathbf{R}) \cup \{X_0\} \cup \{(\mathbf{R} \setminus X_0) \cap X_k \ : \ 0 < k < \omega\};$$

$S_2$ = the $\sigma$-algebra generated by

$$B(\mathbf{R}) \cup \{\mathbf{R} \setminus X_0\} \cup \{X_0 \cap X_k \ : \ 0 < k < \omega\}.$$

It can easily be seen that $S_1$ and $S_2$ are countably generated and invariant under the group $\Gamma$.

Let us define

$$\mu_1(X) = \lambda^*(X \cap X_0) \qquad (X \in S_1),$$

$$\mu_2(X) = \lambda^*(X \cap (\mathbf{R} \setminus X_0)) \qquad (X \in S_2).$$

Then it is not difficult to check that $\mu_1$ and $\mu_2$ are $\Gamma$-invariant measures extending $\lambda$.

Finally, the $\sigma$-algebra $S$ generated by $S_1 \cup S_2$ satisfies the relation

$$\{X_k \ : \ k < \omega\} \subset S.$$

Thus, there is no nonzero diffused admissible functional on $S$. □

**Remark 1.** *Obviously, an analogous result holds true for a finite-dimensional Euclidean space (sphere) equipped with the group of all its isometric transformations and with the standard Borel measure. Also, we have an analogous result for an arbitrary uncountable locally compact Polish topological group equipped with the group of all its left (right) translations and with the left (right) invariant Haar measure.*

**Remark 2.** *It is worth noting here that a number of facts from measure theory are true for admissible functionals. For instance, let $E$ be an arbitrary set and let $\nu$ be an admissible functional on $E$. Suppose, in addition, that $dom(\nu)$ is closed under countable unions and $E \in dom(\nu)$. As usual, we say that a function $\phi \ : \ E \to \mathbf{R}$ is $\nu$-measurable if, for each open interval $U \subset \mathbf{R}$, we have $\phi^{-1}(U) \in dom(\nu)$. Now, let $\{f_n \ : \ n < \omega\}$ be a sequence of $\nu$-measurable functions convergent pointwise to a function $f$. Then one can easily see that $f$ is $\nu$-measurable, too, and, for every $\varepsilon > 0$, there exists a set $X \in \mathrm{dom}\,(\nu)$ such that $\nu(X) > \nu(E) - \varepsilon$ and the sequence of functions $\{f_n|X \ : \ n < \omega\}$ converges uniformly to the function $f|X$. Obviously, this fact generalizes the classical theorem of Egorov. Notice also that the assumption $\nu(Z) \geq 0$, for all $Z \in \mathrm{dom}\,(\nu)$, is superfluous here and hence can be omitted.*

**Remark 3.** *There are certain relationships between admissible functionals and Choquet capacities. For instance, let $\Phi$ be a class of subsets of a basic set $E$, closed under countable unions and countable intersections. Suppose also that $E \in \Phi$. Let $\nu$ be an arbitrary increasing admissible functional defined on the class $\Phi$ (i.e., for any two sets $X \in \Phi$ and $Y \in \Phi$ such that $X \subset Y$, we have $\nu(X) \leq \nu(Y)$). For each subset $Z$ of $E$, let us define*

$$\nu^*(Z) = \inf\{\nu(X) \ : \ X \in \Phi, \quad Z \subset X\}.$$

*Then it is not difficult to check that $\nu^*$ is a Choquet capacity on $E$ with respect to the given class $\Phi$. In particular, according to the well-known Choquet theorem, we have the equality*

$$\nu^*(Z) = \sup\{\nu(X) \ : \ X \in \Phi, \quad X \subset Z\},$$

*for every set $Z$ analytic over the class $\Phi$ (i.e. for every set $Z \in A(\Phi)$).*

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