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AN ω-LIMIT SET FOR A LIPSCHITZ FUNCTION WITH ZERO TOPOLOGICAL ENTROPY

Abstract

Let Q be the middle thirds Cantor set in $[0,1]$, and take C to be the countable set containing the midpoints of the intervals complementary to Q, together with $\{-\frac{1}{6}\}$. We develop a Lipschitz function $f:[-\frac{1}{4},1]\to[-\frac{1}{4},1]$ that possesses zero topological entropy, and for which $Q\cup C$ – an uncountable set with isolated points – is an ω-limit set of f.

1 Introduction

The iterative properties of continuous functions have received considerable attention in recent years. In particular, much has been learned about the structure of the ω-limit sets that various classes of continuous functions possess. Bruckner and Smítal have characterized the structure of ω-limit sets for the class of continuous functions as well as those continuous functions with zero topological entropy [2], [3].

Theorem 1. *Let F be a nonempty closed set. Then F is an ω-limit set for a continuous function if and only if F is either nowhere dense, or F is a union of finitely many nondegenerate closed intervals.*

Theorem 2. *Let $F\subset(0,1)$ be a nonempty infinite closed set. Then F is an ω-limit set for a continuous function $f:[0,1]\to[0,1]$ with zero topological entropy if and only if $F=Q\cup C$, where Q is a Cantor set, and C is countable, dense in F if nonempty, and such that for any interval J contiguous to Q, $\operatorname{card}(J\cap C)\le 1$ if 0 or 1 is in J, and $\operatorname{card}(J\cap C)\le 2$ otherwise.*

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In [4] we investigate how the structure of possible ω-limit sets is affected by considering classes of functions better behaved than the typical continuous function. We endow the class of closed sets $\mathcal{K}$ contained in $[0,1]$ with the Hausdorff metric ρ, and show that the typical closed set in $\{\mathcal{K}, \rho\}$ cannot be an ω-limit set for any Lipschitz function. This is in marked contrast to the continuous case since all of these typical sets are Cantor sets, and therefore ω-limit sets of non-Lipschitz continuous functions. The main result of [8] shows, however, that every nowhere dense compact set is homeomorphic to an ω-limit set for a differentiable function with bounded derivative. The significant cleavage between the class of ω-limit sets for continuous functions and the class of ω-limit sets for Lipschitz functions must, then, be measure based. In [9] we make some progress towards characterizing ω-limit sets for Lipschitz functions, but many questions remain. In this note we answer one of those queries by showing that a Lipschitz function possessing zero topological entropy can have an infinite ω-limit set with isolated points.

We proceed through a couple sections. In section two we develop some notation, give a few definitions and review those previously known results that will be useful in the course of our construction. Section three is dedicated to the development of the Lipschitz function $f : [-\frac{1}{4}, 1] \to [-\frac{1}{4}, 1]$ possessing zero topological entropy that also has an infinite ω-limit set with isolated points.

2 Preliminaries

In the ensuing section we will develop a Lipschitz function $f : [-\frac{1}{4}, 1] \to [-\frac{1}{4}, 1]$ that possesses zero topological entropy as well as an uncountable ω-limit set with isolated points. We call a function $f : [a,b] \to [a,b]$ Lipschitz if there exists a real number M such that $| f(x) - f(y) | < M | x - y |$ for all x and y in $[a,b]$. A set E is called an ω-limit set for a continuous function f mapping a compact interval I into itself if there exists an x in I such that $E = \omega(x, f)$ is the cluster set of the sequence $\{f^n(x)\}_{n=0}^{\infty} = \{x, f(x), f(f(x)), \dots\}$. There are many ways in which one can characterize those continuous functions $f : I \to I$ that possess zero topological entropy. In [6] one finds a comprehensive list of such characterizations. For our purposes, however, it suffices to note that the topological entropy $\mathbf{h}(f)$ of a continuous function f is zero if and only if the period of every periodic point of f is a power of two.

An important tool in the development of our function is the following theorem due to Smítal [7].

Theorem 3. *Let $f : I \to I$ be a continuous function with zero topological entropy, and let E be an infinite ω-limit set of f. Then there is a sequence $\{J_k\}_{k=1}^{\infty}$ of f-periodic intervals so that, for any k,*

- *J_k has period 2^k;*
- *$J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$;*
- *$E \subset \cup_{i=1}^{2^k} f^i(J_k)$;*
- *$E \cap f^i(J_k) \neq \emptyset$ for every i.*

To simplify our work with the f-periodic intervals $f^i(J_k)$, we code them with finite tuples of zeros and ones using a device found in [5]. Let $\mathbb{N}$ denote the natural numbers, and take $\mathcal{N}$ to be the set of sequences composed of zeros and ones. If $\mathbf{n} \in \mathcal{N}$ and $\mathbf{n} = \{n_i\}_{i=1}^{\infty}$, we let $\mathbf{n} \mid k = (n_1, n_2, \ldots, n_k)$. Set $\mathbf{0} = \{0, 0, 0, \ldots.\}$ and $\mathbf{1} = \{1, 1, 1, \ldots.\}$. Now, define a function $\mathcal{A} : \mathcal{N} \to \mathcal{N}$ given by $\mathcal{A}(\mathbf{n}) = \mathbf{n} + 1\mathbf{0}$, where addition is modulus two from left to right. For each $k \in \mathbb{N}$ and $i \in \mathbb{N}$ put $F_{\mathbf{1}|k} = J_k$ and $F_{\mathcal{A}^i(\mathbf{1})|k} = f^i(J_k)$. Thus, for every $\mathbf{m}$ and $\mathbf{n}$ in $\mathcal{N}$ and $k \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that $\mathcal{A}^j(\mathbf{m} \mid k) = \mathbf{n} \mid k$; the above relations define $F_{\mathbf{n}|k}$ for all $\mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$. In the construction that follows, we will take $F_{\mathbf{n}|k,1}$ to lie to the left of $F_{\mathbf{n}|k,0}$ for all $\mathbf{n} \in \mathcal{N}$ and $k \in \mathbb{N}$.

We let Q be the middle thirds Cantor set contained in $[0, 1]$, and take C to be the set comprised of the midpoints of the complementary intervals of Q contained in $[0, 1]$, together with the point $\{-\frac{1}{6}\}$.

3 Example

A brief discussion of the ideas behind our construction may prove helpful. Our intention is to reverse Smítal's Theorem, and let the sets $F_{\mathbf{n}|k}$ determine our function f, rather than the other way around. We start with the compact interval $[-\frac{1}{4}, 1]$, and split it so that each midpoint c of an interval (a, b) complementary to Q is always contained in the same periodic interval $F_{\mathbf{n}|k}$ as the right endpoint b, for each k. We also take $F_{\mathbf{n}|k,1} \subset \operatorname{int}(F_{\mathbf{n}|k})$ for each k (see Theorem 3.1,[1]). Moreover, if we set $F_{\mathbf{n}} = \cap_{k=1}^{\infty} F_{\mathbf{n}|k} = \cap_{k=1}^{\infty} [a_{\mathbf{n}|k}, b_{\mathbf{n}|k}] = [a_{\mathbf{n}}, b_{\mathbf{n}}]$, then $F_{\mathbf{n}} = [a_{\mathbf{n}}, b_{\mathbf{n}}]$ for an $a_{\mathbf{n}} \in C$ and $b_{\mathbf{n}} = \max\{x : x \in Q, x < a_{\mathbf{n}}\}$ whenever $\mathbf{n}$ has a tail of ones, and $F_{\mathbf{n}} = \{x\}$ is a singleton otherwise. The trick is not so much to find sets $F_{\mathbf{n}|k}$ for which we can do this, but to insure that the function f to which they give rise is Lipschitz and has zero topological entropy.

We begin our construction by defining inductively the f-periodic intervals $F_{\mathbf{n}|k}$. Let $F_1 = [-\frac{1}{4}, \frac{1}{3}]$ and $F_0 = [\frac{2}{3} - \frac{1}{4}, 1]$, and suppose $F_{\mathbf{n}|k-1} = [a_{\mathbf{n}|k-1}, b_{\mathbf{n}|k-1}]$. If $\mathbf{n} \mid k-1 \neq \mathbf{1} \mid k-1$, set $F_{\mathbf{n}|k-1,0} = [b_{\mathbf{n}|k-1} - (\frac{1}{3})^k - (\frac{3}{4})(\frac{1}{3})^k, b_{\mathbf{n}|k-1}]$ and $F_{\mathbf{n}|k-1,1} = [a_{\mathbf{n}|k-1} + \frac{1}{2^{j+1}}[\frac{1}{4}(\frac{1}{3})^l], b_{\mathbf{n}|k-1} - 2(\frac{1}{3})^k]$, where j is the length of the string of ones in which $\mathbf{n} \mid k-1$ terminates, and $l = (k-1) - j$. If $\mathbf{n} \mid k-1 = \mathbf{1} \mid k-1$, we define $F_{\mathbf{n}|k-1,0}$ as we did above, but in this case set $F_{\mathbf{1}|k-1,1} = [a_{\mathbf{1}|k-1} + \frac{1}{2^{k-1}}[\frac{1}{4}(\frac{1}{3})], b_{\mathbf{n}|k-1} - 2(\frac{1}{3})^k]$. We let $G = (\max F_1, \min F_0) = (b_1, a_0)$, and in general, take $G_{\mathbf{n}|k} = (\max F_{\mathbf{n}|k,1}, \min F_{\mathbf{n}|k,0}) = (b_{\mathbf{n}|k,1}, a_{\mathbf{n}|k,0})$. For each $\mathbf{n} \in \mathcal{N}$, let $F_{\mathbf{n}} = \cap_{k=1}^{\infty} F_{\mathbf{n}|k} = \cap_{k=1}^{\infty}[a_{\mathbf{n}|k}, b_{\mathbf{n}|k}] = [a_{\mathbf{n}}, b_{\mathbf{n}}]$. It follows that $C = \{a_{\mathbf{n}} : \mathbf{n} \in \mathcal{M}\}$, where $\mathcal{M}$ consists of all the elements of $\mathcal{N}$ having a tail of ones. If we let S be composed of all x such that $\{x\} = F_{\mathbf{n}}$ for some $\mathbf{n} \in \mathcal{N}$, then S consists of all the elements of Q except for those that are the right endpoint of an interval complementary to Q, and $\{0\}$. If we take $B = \{b_{\mathbf{n}} : \mathbf{n} \in \mathcal{M}\}$, then $Q = S \cup B$.

Now, set $L = Q \cup C \cup \{a_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in \mathbb{N}\}$. We begin to define our function $f : [-\frac{1}{4}, 1] \to [-\frac{1}{4}, 1]$ by describing its behavior on L. If $x \in S$, define $f(x)$ so that $\{f(x)\} = F_{\mathcal{A}(\mathbf{n})}$ when $\{x\} = F_{\mathbf{n}}$. On $C \cup B$ we define f so that $f(a_{\mathbf{n}}) = a_{\mathcal{A}(\mathbf{n})}$ and $f(b_{\mathbf{n}}) = b_{\mathcal{A}(\mathbf{n})}$, and when $\mathbf{n} \mid k \neq \mathbf{1} \mid k$, we let $f(a_{\mathbf{n}|k}) = a_{\mathcal{A}(\mathbf{n})|k}$. We complete our definition of f on L by setting $f(a_{\mathbf{1}|k}) = a_{\mathbf{0}|k+1}$.

Our map $f : L \to L$ is continuous. To show this, it suffices to establish the continuity of f at each point of C, B and S as the remaining points of L are isolated. We show that f is continuous at $a_{\mathbf{1}}$ and $b_{\mathbf{1}}$; the proofs for the other points of $Q \cup C$ are similar. To show that f is continuous at $a_{\mathbf{1}}$, let U be a neighborhood of $f(a_{\mathbf{1}}) = a_{\mathbf{0}} = \sup(Q \cup C) = 1$. There exists $k \in \mathbb{N}$ such that $F_{\mathbf{0}|k} \subset U$. Let $V = (a_{\mathbf{1}|k+1}, b)$ where $b \in (a_{\mathbf{1}}, b_{\mathbf{1}}) = (-\frac{1}{6}, 0)$. Then $V \cap L = \{a_{\mathbf{1}|j} : j > k+1\} \cup \{a_{\mathbf{1}}\}$, and $f(V \cap L) = \{a_{\mathbf{0}|j+1} : j > k+1\} \cup \{a_{\mathbf{0}}\} \subset F_{\mathbf{0}|k} \subset U$. We conclude that f is continuous at $a_{\mathbf{1}}$. Now, let U be a neighborhood of $f(b_{\mathbf{1}}) = b_{\mathbf{0}} = \sup(Q \cup C) = 1$. Choose $k \in \mathbb{N}$ so that $F_{\mathbf{0}|k} \subset U$. Let $V = (b, b_{\mathbf{1}|k+1})$ where $b \in (a_{\mathbf{1}}, b_{\mathbf{1}})$. Then $\sup\{f(x) : x \in V \cap L\} = 1$, and $\inf\{f(x) : x \in V \cap L\} = f(a_{\mathbf{1}|k+1,0}) = a_{\mathbf{0}|k+1,1} \subset F_{\mathbf{0}|k} \subset U$, so that f is continuous at $b_{\mathbf{1}}$, too.

We now extend our function linearly to the intervals contiguous to the closed set L obtaining a function also denoted by f that is continuous on all of $[-\frac{1}{4}, 1]$. Our next task is to verify that $f : [-\frac{1}{4}, 1] \to [-\frac{1}{4}, 1]$ does indeed have zero topological entropy. From our development of f, one sees that $f(G_{\mathbf{n}|k}) = G_{\mathcal{A}(\mathbf{n})|k}$ for $\mathbf{n} \mid k \neq \mathbf{1} \mid k$, $\overline{G} \subset f(G)$, and $\overline{G_{\mathbf{0}|k}} \subset f(G_{\mathbf{1}|k})$. From this we conclude that f^{2^k} is linear on $G_{\mathbf{0}|k}$ and has a slope greater than one. Thus, G contains exactly one periodic point, which is necessarily a repelling fixed point, and each $G_{\mathbf{n}|k}$ contains exactly one periodic point of period 2^k, which is also repelling (1). We also note that $f(F_{\mathbf{n}|k}) = F_{\mathcal{A}(\mathbf{n})|k}$ whenever $\mathbf{n} \mid k \neq \mathbf{1} \mid k$, and $f(F_{\mathbf{1}|k})$ is a proper subset of $F_{\mathbf{0}|k}$ (2). From (1) and (2), Bruckner and Ceder are able to conclude that for each $x \in [-\frac{1}{4}, 1]$, either $\omega(x, f)$ is a 2^k cycle for some k, or $\omega(x, f) \subset \cap_{k=1}^{\infty} \cup_{\mathbf{n} \in \mathcal{N}} F_{\mathbf{n}|k}$ ([1], proof of Theorem 4.3). Since $\cap_{k=1}^{\infty} \cup_{\mathbf{n} \in \mathcal{N}} F_{\mathbf{n}|k}$ contains no cycles, it follows that f must be a 2^{∞} function, and that $\mathbf{h}(f)$ is zero.

We now establish that our function $f : [-\frac{1}{4}, 1] \to [-\frac{1}{4}, 1]$ is Lipschitz.

We begin by noting that f is increasing on $\overline{\text{conv}}(F_{\mathbf{1}|k,0})$ for each $k \in \mathbb{N} \cup \{0\}$, and since $f(F_{\mathbf{1}|k,0}) = F_{\mathbf{0}|k,1}$, one sees that $f \mid F_{\mathbf{1}|k,0}$ is Lipschitz with growth constant three. Since $f(\overline{G}) = \overline{\text{conv}}(F_1 \cup G \cup F_{01})$ and, in general, $f(\overline{G_{\mathbf{1}|k}}) = \overline{\text{conv}}(F_{\mathbf{0}|k,1} \cup G_{\mathbf{0}|k} \cup F_{\mathbf{0}|k+1,1})$, it follows from the similarity within our construction that $f \mid \overline{G_{\mathbf{1}|k}}$ is Lipschitz for a particular constant M that works for all $k \in \mathbb{N} \cup \{0\}$. In fact, we can take $M = 16\frac{5}{6}$, as a tedious but not terribly difficult calculation shows. In a rather straightforward way one also shows that f is Lipschitz of constant $3\frac{1}{9}$ on $[-\frac{1}{4}, -\frac{1}{6}]$. To establish that f is Lipschitz at $x = 0$, we note that $f(0) = 1$, and $\frac{|1-f(y)|}{|0-y|}$ is largest when $y = a_{\mathbf{1}|k,0}$. For all $k \geq 1$, $\frac{|1-f(a_{\mathbf{1}|k,0})|}{|0-a_{\mathbf{1}|k,0}|} = \frac{1-a_{\mathbf{0}|k,1}}{a_{\mathbf{1}|k,0}} < \frac{\frac{2}{3^k}}{\frac{1}{3^{k+1}}} = \frac{3}{2}$. We conclude, then, that $f : [-\frac{1}{4}, 1] \to [-\frac{1}{4}, 1]$ is indeed a Lipschitz function.

That $Q \cup C$ is an ω-limit set of f follows from the observation that the orbit of a_0 is $\{a_{\mathbf{n}|k} : \mathbf{n} \in \mathcal{N}, k \in \mathbb{N}\}$, so that $\omega(a_0, f) = Q \cup C$. It is worth noting that while taking Q to be the middle thirds Cantor set, and choosing C to be the midpoints of its complementary intervals, somewhat simplified our construction, we clearly did not minimize the Lipschitz constant of the resulting function. In fact, for any $d > 1$ we can take a Cantor set Q and an appropriate countable set of points C so that $Q \cup C$ is an ω -limit set for a Lipschitz function f with zero topological entropy and Lipschitz constant less than d.

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