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OMEGA–LIMIT SETS AND NON–CONTINUOUS FUNCTIONS

Abstract

We investigate the dense mapping property introduced by Keller in connection with iteration in Newton's method. Various kinds of functions are shown to have the dense mapping property. We show that a function has the dense mapping property iff it is bilaterally quasicontinuous. We also present an invariance theorem and other results on omega–limit sets.

Introduction and preliminaries

For a general background and notation, we refer the reader to [1], [4], [8]. We will let $f : \mathbb{R} \to \mathbb{R}$ represent a real-valued function on the real line. The iterates of f are defined inductively such that $f(f^n(x)) = f^{n+1}(x)$, where f^n is the n-fold composition of f. The *trajectory* of x in X is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where $f^0(x) = x$. The *orbit* of x is the point set $\{f^n(x) : n \geq 0\}$. The *omega–limit set* of f at x (denoted by $\omega(x, f)$) is the limit set of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. Therefore

$$\omega(x, f) = \bigcap_{m \geq 0} \mathrm{Cl}\big(\cup_{n \geq m} f^n(x)\big)$$

where Cl( ) denotes the closure operator.

We investigate ω-limit sets of Darboux-Baire 1 functions and related functions, because of the familiar application to Newton's method of finding the zeros of a function which is differentiable, but not C^1.

A function $f : X \to Y$, where X and Y are topological spaces, is *quasi-continuous* at a point x in X if for any open set V containing $f(x)$, and for any open set U containing x, there exists an open nonempty set $G \subset U$ such

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that $f(G) \subset V$ [10]. Darboux-Baire 1 (D-B_1) functions are not quasicontinuous [10]. However, Darboux-Baire 1 functions which have Banach's condition T_2, are quasicontinuous. See [3, p. 123] and [11, p. 277; chapters VII, IX]. A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy *Banach's condition* T_2 if almost every value taken by f is taken at most a denumerable number of times. Functions which are both Darboux and quasicontinuous are bilaterally quasicontinuous, but bilaterally quasicontinuous functions are not necessarily Darboux [2]. A function $f : \mathbb{R} \to \mathbb{R}$ is *bilaterally quasicontinuous* if f is both right-hand-sided quasicontinuous and left-hand-sided quasicontinuous; and f is *left-hand-sided* (*right-hand-sided*) *quasicontinuous* at x if for every $\delta > 0$ and for every open neighborhood V of $f(x)$ there exists an open nonempty set

$$W \subset (x - \delta, x) \cap f^{-1}(V) \quad \big(W \subset (x, x + \delta) \cap f^{-1}(V)\big)\,.$$

A subset S of a topological space X is called *semi-open* if there is an open set G in X such that $G \subset S \subset \mathrm{Cl}(G)$. See [5] and [6]. A subset C of X is called *semi-closed* if its complement is semi-open. For a set $E \subset X$, the *semi-closure* of E, denoted by $\mathrm{SCl}(E)$, is defined to be the intersection of all semi-closed sets containing E. For any set E, the set $\mathrm{SCl}(E)$ is semi-closed. Arbitrary intersections of semi-closed sets are semi-closed, and a set C is semi-closed iff $\mathrm{Int}(\mathrm{Cl}(C)) \subset C$. A function is quasicontinuous iff the inverse image of every open (closed) set is semi-open (semi-closed). The function $f : X \to Y$ is quasicontinuous iff, for any subset E of X, $f(\mathrm{SCl}(E)) \subset \mathrm{Cl}(f(E))$.

We recall that the *boundary* of a set S in a topological space X, denoted by $\mathrm{Bd}(S)$, is defined as follows: $\mathrm{Bd}(S) = \mathrm{Cl}(S) \cap \mathrm{Cl}(X \setminus S)$. A function $f : \mathbb{R} \to \mathbb{R}$ has a *perfect road* if for each $x \in \mathbb{R}$, there exists a perfect set P having x as a bilateral limit point such that $f_{|P}$ is continuous at x.

Finally, a function $f : \mathbb{R} \to \mathbb{R}$ has the *Young property* if for each $x \in \mathbb{R}$, there exist sequences $x_n \uparrow x$ and $y_n \downarrow x$ such that

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)\,.$$

The dense mapping property and quasicontinuity

In [8] Keller introduced the concept of "dense mapping property" (DMP). A function $f : X \to Y$, where X and Y are topological spaces, has *the dense mapping property* (DMP) if for every subset D of X, with $\mathrm{Cl}(D)$ connected, $f(\mathrm{Cl}(D)) \subset \mathrm{Cl}(f(D))$. Keller shows that D-B_1 functions do not have the DMP. The DMP is more general than continuity, but any Darboux function with the DMP, can have only those ω-limit sets which are possible for continuous functions [8]. In particular, Keller proves (Theorem 2.5) that if $f : \mathbb{R} \to \mathbb{R}$

is Darboux and has the DMP, then for any x in $\mathbb{R}$, the set $\omega(x,f)$ is either nowhere dense, or is a finite union of nonsingleton connected closed sets.

Theorem 1. *Let $f : X \to Y$ be a function, where $X = Y = \mathbb{R}$. Then f is bilaterally quasicontinuous iff f has the* DMP*.*

Proof. For the sufficiency, assume that f is not left-hand-sided quasicontinuous at some point x_o in X. Then there is an open set V containing $f(x_o)$ and there exists $\delta > 0$ such that there is no open nonempty set $G \subset (x_o - \delta, x_o)$ such that $f(G) \subset V$. Evidently, the set

$$D = \{x \in (x_o - \delta, x_o) \,:\, f(x) \notin V\}$$

is dense in $(x_o - \delta, x_o)$. By the DMP, $f(\mathrm{Cl}(D)) \subset \mathrm{Cl}(f(D))$. Since $x_o \in \mathrm{Cl}(D)$, $V \cap f(D) \neq \emptyset$, a contradiction. Similarly, we can show that f is right-hand-sided quasicontinuous. Hence, f is bilaterally quasicontinuous.

For the necessity, let D be a subset of X such that $\mathrm{Cl}(D)$ is connected. We may suppose that $\mathrm{Cl}(D)$ is an interval. Therefore, $\mathrm{SCl}(D)$ is also an interval, since $\mathrm{Int}(\mathrm{Cl}(D)) \subset \mathrm{SCl}(D) \subset \mathrm{Cl}(D)$. Since f is quasicontinuous $f(\mathrm{SCl}(D)) \subset \mathrm{Cl}(f(D))$. The claim now is that

$$f(\mathrm{Cl}(D)) \subset \mathrm{Cl}(f(D))\,.$$

Assume, to the contrary, that there exists x in $\mathrm{Cl}(D)$ such that $f(x) \notin \mathrm{Cl}(f(D))$. Therefore, there exists an open set H containing $f(x)$ such that $H \cap f(D) = \emptyset$. Now, by bilateral quasicontinuity, for any open set U containing x, $f^{-1}(H) \cap U$ contains an open nonempty set G such that $f(G) \subset H$. If x is an endpoint, the set G can be chosen on the left side or right side of x, so that G meets the set D. Hence $H \cap f(D)$ is not empty, and we have a contradiction. □

Corollary 2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a D-B_1 function. Then if f has Banach's condition T_2, f has the* DMP*.*

Proof. Since f is Darboux and quasicontinuous, f is bilaterally quasicontinuous, and hence has the DMP. □

Corollary 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a D-B_1 function satisfying Banach's condition T_2. Then for any x in $\mathbb{R}$, $\omega(x,f)$ is either nowhere dense, or is a finite union of nonsingleton connected closed sets.*

Corollary 4. *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable derivative. Then for any x in $\mathbb{R}$, $\omega(x,f)$ is either nowhere dense, or is a finite union of nonsingleton connected closed sets.*

PROOF. Riemann integrable derivatives are quasicontinuous [10]. Since we also have the Darboux property, the result follows. $\square$

In [10] Marcus proves the more general result that a derivative which is continuous almost everywhere, is quasicontinuous. Therefore, Corollary 4 is also true for this class of functions.

Corollary 5. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is Darboux and has the* DMP. *Then* f *has a perfect road. Moreover,* f *has the Young property.*

PROOF. Since f is Darboux and quasicontinuous, f has a perfect road [7] and also has the Young property [2]. $\square$

Of course, fixed points are one kind of omega-limit set. Since Darboux quasicontinuous functions of the form $f : \mathbb{R} \to \mathbb{R}$ are not necessarily connectivity functions [7], we cannot necessarily conclude that DMP functions with reasonable properties are connectivity or have a fixed point, although either is a possibility. On the other hand, it is well known that D-B_1 functions of the form $f : I \to I$ have a fixed point. We can prove a slight variation of this result:

Theorem 6. *Let* $f : I \to I$ *be a* D-B_1 *function, where* I *is a closed bounded interval. If* $J = [a, b]$ *is a compact subinterval of* I *such that* $f(J) \supset J$*, then* f *has a fixed point in* J*.*

PROOF. Define $g(x) = f(x) - x$. Then g is Darboux, because the continuous functions form the maximum additive family for the set of D-B_1 functions. Since $f(J) \supset J$, there are points x_1 and x_2 in J such that $f(x_1) = a$ and $f(x_2) = b$. It is easy to show that $g(x_1) \leq 0$ and $g(x_2) \geq 0$. Hence, by the intermediate value property, $g(y) = 0$ for some y in J. $\square$

Invariance of the ω-limit set

It is well known that ω-limit sets for continuous functions are invariant. That is, for a continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(\omega(x, f)) \subset \omega(x, f)$. This can be shown as follows:

$$f\big(\omega(x,f)\big) = f\Big(\bigcap_{m\geq 0}\mathrm{Cl}\big(\cup_{n\geq m} f^n(x)\big)\Big) \subset \bigcap_{m\geq 0} f\Big(\mathrm{Cl}\big(\cup_{n\geq m} f^n(x)\big)\Big) \subset$$

$$\subset \bigcap_{m\geq 0}\mathrm{Cl}\Big(f\big(\cup_{n\geq m} f^n(x)\big)\Big) \subset \bigcap_{m\geq 0}\mathrm{Cl}\Big(\cup_{n\geq m} f^n(x)\Big) = \omega(x,f)\,.$$

Corollary 2.6 of [8], applied to a real-valued function $f : \mathbb{R} \to \mathbb{R}$, states that if f is Darboux and has the DMP, then for any component K of $\omega(x, f)$, with nonempty interior, $f^n(K)$ is also a component of $\omega(x, f)$ for any nonnegative integer n.

In this section, we investigate the question of invariance for certain kinds of noncontinuous functions, including quasicontinuous functions. We remark that, although bilaterally quasicontinuous functions of the form $f : \mathbb{R} \to \mathbb{R}$ have the DMP, quasicontinuous functions do not, as shown by the following simple example:

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x+1 & \text{if } x \geq 1 \end{cases}$$

The next result presents a kind of invariance theorem for quasicontinuous functions.

Theorem 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be quasicontinuous on $\mathbb{R}$. Let $x \in \mathbb{R}$. Suppose that there is a nonempty open set $G \subset \omega(x, f)$. Then $f(G) \subset \omega(x, f)$.*

PROOF. Let $I = (a, b)$ be any component of G. Let $K = \mathrm{Cl}(I)$. Then

$$f(K) = f\Big(\bigcap\nolimits_{m\geq 0} \mathrm{Cl}\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big) \subset$$

$$\subset \bigcap\nolimits_{m\geq 0} f\Big(\mathrm{Cl}\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big) = \bigcap\nolimits_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big) \cup A\Big),$$

where

$$A = \mathrm{Cl}\big(I \cap \cup_{n\geq m} f^n(x)\big) \setminus \mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big).$$

The claim now is that $A \subset \big\{\{a\}, \{b\}\big\}$; that is, that A is contained in the boundary of K. In order to see this, we recall that, by the definition of an omega-limit set, the orbit of x is evidently dense in I. Therefore since $\mathrm{Cl}(I) \setminus \mathrm{SCl}(I)$ is contained in $\big\{\{a\}, \{b\}\big\}$, then

$$A = \mathrm{Cl}\big(I \cap \cup_{n\geq m} f^n(x)\big) \setminus \mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big)$$

is also contained in $\big\{\{a\}, \{b\}\big\}$. Since $f(K) = f(I) \cup f(a) \cup f(b)$, and since

$$f(K) \subset \bigcap\nolimits_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big) \cup \big\{\{a\}, \{b\}\big\}\Big) \subset$$

$$\subset \bigcap\nolimits_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big) \cup f(a) \cup f(b)\Big) \subset$$

$$\subset \bigcap\nolimits_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big) \cup f(a) \cup f(b),$$

then

$$f(I) \cup f(a) \cup f(b) \;\subset\; \bigcap_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big) \cup f(a) \cup f(b)\,.$$

It follows that

$$f(I) \;\subset\; \bigcap_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big)\,.$$

Since f is quasicontinuous, then

$$\bigcap_{m\geq 0} f\Big(\mathrm{SCl}\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big) \;\subset\; \bigcap_{m\geq 0} \mathrm{Cl}\Big(f\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big)\,.$$

Hence

$$f(I) \subset \bigcap_{m\geq 0} \mathrm{Cl}\Big(f\big(I \cap \cup_{n\geq m} f^n(x)\big)\Big) \;\subset\; \bigcap_{m\geq 0} \mathrm{Cl}\big(\cup_{n\geq m} f^n(x)\big) = \omega(x,f)\,.$$

Since I was an arbitrary component of G, then $f(G) \subset \omega(x,f)$. $\square$

Corollary 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be quasicontinuous on $\mathbb{R}$. Let $x \in \mathbb{R}$. Then $f(\mathrm{Int}(\omega(x,f))) \subset \omega(x,f)$.*

As indicated above, for continuous real-valued functions, invariance can be proved using the property that

$$f\big(\mathrm{Cl}(E)\big) \;\subset\; \mathrm{Cl}\big(f(E)\big) \quad \text{for } \textbf{any} \text{ subset } E\,.$$

Of course, for noncontinuous functions, such is not possible. In fact, as the next result shows, for a nowhere dense set N, if $f(\mathrm{Cl}(N)) \subset \mathrm{Cl}(f(N))$, then f is continuous.

Theorem 9. *Let $f : X \to Y$ be a function, where $X = Y = \mathbb{R}$. Suppose that for any nowhere dense subset N of X, $f(\mathrm{Cl}(N)) \subset \mathrm{Cl}(f(N))$. Then f is continuous.*

PROOF. Assume f is not continuous at some point x in X, and let V be an open set containing $f(x)$. Let U be any open set containing x. Then U contains a point x_1 such that $f(x_1) \notin V$. Choose a positive integer n such that $1/n < |x - x_1|$. Choose $\delta > 0$ such that $\delta < 1/n$. Then the interval $(x-\delta, x+\delta)$ contains a point x_2 such that $f(x_2) \notin V$. Continuing in this way, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x, such that the set $N = \{x_n : n \geq 1\}$ is nowhere dense in X. Since there is no subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{f(x_{n_k}\}_{k=1}^{\infty}$ converges to $f(x)$, then

$$f\big(\mathrm{Cl}(N)\big) \;\not\subset\; \mathrm{Cl}\big(f(N)\big)\,,$$

a contradiction. $\square$

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