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ON THE CONVERGENCE OF THE INTEGRALS OF A TRUNCATED HENSTOCK-KURZWEIL INTEGRABLE FUNCTION

Abstract

We deal with two ways to truncate a Henstock-Kurzweil integrable function and the convergence of their integrals. We also give an example to show the limitations to the convergence theorem.

The problem we deal with here is the following. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a measurable function. For M and N positive real numbers, we define two *truncations* of f:

$$f_{M,N}(x) = \begin{cases} f(x) & \text{if } -N \leq f(x) \leq M \\ M & \text{if } f(x) \geq M \\ -N & \text{if } -N \geq f(x) \end{cases}$$

$$\tilde{f}_{M,N}(x) = \begin{cases} f(x) & \text{if } -N \leq f(x) \leq M \\ 0 & \text{if } f(x) \geq M \\ 0 & \text{if } -N \geq f(x) \end{cases}$$

If M, N go to infinity both $f_{M,N}$ and $\tilde{f}_{M,N}$ converge pointwise to f. So if f is Lebesgue integrable, by the Dominated Convergence Theorem, their integrals converge to that of f. In particular if we take $M = N$. But this does not happen in the case of the Henstock-Kurzweil integral. In [2, Section 18, pp. 114–118] there is a study of the cases when $\int \tilde{f}_{M,M}$ converges to $\int f$. The

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class of such functions (called HL in [2]) contains properly the Lebesgue integrable functions but there are Henstock-Kurzweil integrable functions outside this class. (See [2, Example 18.3, p. 115].) We give here conditions on the truncations of the form $f_{M,N}$, such that the integrals of the truncated functions converge to the integral of the function. We also give an example showing that there is no such result for truncations of the form $\tilde{f}_{M,N}$.

More precisely, we prove here the following result.

Theorem 1. *Let $f : [a,b] \longrightarrow \mathbb{R}$. The following statements are equivalent:*

1. *f is Henstock-Kurzweil integrable and $\int_a^b f = A$;*
2. *f is measurable and there are increasing sequences of positive integers M_k, N_k and a decreasing sequence of gauges δ_k such that for all k and all δ_k-fine partition P, $|S(f,P) - S(f_k,P)| < 1/k$ and $|S(f_k,P) - A| < 1/k$, where $f_k = f_{M_k,N_k}$. Moreover, in this case, $\int_a^b f = A$ and $\int_a^b f_k \longrightarrow \int_a^b f$, as $k \longrightarrow \infty$.*

PROOF. We prove first that (2) implies (1). Let f_k and δ_k be as in (2). Let $m \le n$, and P be a δ_n-fine partition. Since $\delta_n \le \delta_m$, then P is also δ_m-fine. Thus

$$\begin{aligned} \left|\int_a^b f_n - \int_a^b f_m\right| \quad &\le \quad \left|\int_a^b f_n - S(f_n,P)\right| + \left|S(f_n,P) - S(f,P)\right| \\ &\quad + \left|S(f,P) - S(f_m,P)\right| + \left|S(f_m,P) - \int_a^b f_m\right| < 4/m. \end{aligned}$$

This means that the integrals $\int_a^b f_k$ form a Cauchy sequence and so converge to some $A \in \mathbb{R}$. Given $\varepsilon > 0$, choose n such that $|\int_a^b f_n - A| < \varepsilon/3$ and $1/n < \varepsilon/3$. Then for all δ_n-fine partition P we have

$$\left|S(f,P) - A\right| \le \left|S(f,P) - S(f_n,P)\right| + \left|S(f_n,P) - \int_a^b f_n\right| + \left|\int_a^b f_n - A\right| < \varepsilon.$$

That is, f is integrable to A and the integrals $\int_a^b f_n$ converge to $\int_a^b f$.

Now we prove that (1) implies (2). It is known that if f is Henstock-Kurzweil integrable then it is measurable (see [2, Section 5]). If $|f|$ is also integrable then it is Lebesgue integrable so the Dominated Convergence Theorem gives the desired result. So we can assume that $|f|$ is not integrable. It follows that neither $f^+ = \max(f,0)$ nor $f^- = \max(-f,0)$ is integrable.

For each M, N, $\min(f^+, M)$ and $\min(f^-, N)$ are integrable. But their integrals go to infinity as M and N go to infinity. On the other hand

$$\lim_{M\longrightarrow\infty}\int_a^b \big(\min(f^+,M+1)-\min(f^+,M)\big) \\ = \lim_{N\longrightarrow\infty}\int_a^b \big(\min(f^-,N+1)-\min(f^-,N)\big)=0,$$

because the Lebesgue measure of the sets $\{x\in[a,b]: N\le f^+(x)\le N+1\}$ and $\{x\in[a,b]: N\le f^-(x)\le N+1\}$ tend to zero as N tend to infinity.

Recall that $f_{M,N}=\min(f^+,M)-\min(f^-,N)$. We show now that we can find increasing sequences of integers M_k, N_k such that, for all k, $|\int_a^b f_{M_k,N_k}-\int_a^b f|<1/3k$.

Let $r_n=\int_a^b(\min(f^+,n+1)-\min(f^+,n))\ge 0$ and $s_n=\int_a^b(\min(f^-,n+1)-\min(f^-,n))\ge 0$. By the hypothesis on f, and the above, both r_n and s_n tend to zero as n goes to infinity, and both series $\sum r_n$ and $\sum s_n$ diverge; also $\int_a^b\min(f^+,N)=\sum_0^N r_n$, and $\int_a^b\min(f^-,N)=\sum_0^N s_n$. So, given $\varepsilon>0$, there is $n_0\in\mathbb{N}$ such that, for all $n\ge n_0$, $0\le r_n,s_n<\varepsilon$. Put $R_N=\sum_{n=0}^N r_n$ and $S_N=\sum_{n=0}^N s_n$, and $A=\int_a^b f$.

Either $R_{n_0}-S_{n_0}<A$ or $R_{n_0}-S_{n_0}\ge A$ can happen. In the first case, choose the first integer $n_1\ge n_0$ such that $R_{n_1}-S_{n_0}\ge A$. We have that $n_1>n_0$, and $R_{n_1-1}-S_{n_0}<A\le R_{n_1}-S_{n_0}$. Therefore $0\le(R_{n_1}-S_{n_0})-A<r_{n_1}<\varepsilon$. Letting $M=n_1$ and $N=n_0$, we have that $|\int_a^b f_{M,N}-\int_a^b f|<\varepsilon$. With the same type of argument, we can treat the case where $R_{n_0}-S_{n_0}\ge A$. Now it is a matter of applying this to $\varepsilon=1/3k$.

Put $f_k=f_{M_k,N_k}$. Let δ_k be a decreasing sequence of gauges such that, for all k and for all δ_k-fine partition P, $|S(f,P)-\int_a^b f|<1/3k$, and $|S(f_k,P)-\int_a^b f_k|<1/3k$. Then, for all k and all δ_k-fine P,

$$|S(f_k,P)-S(f,P)| \\ \le\Big|S(f,P)-\int_a^b f\Big|+\Big|\int_a^b f-\int_a^b f_k\Big|+\Big|S(f_k,P)-\int_a^b f_k\Big|\le\frac{1}{k},$$

which proves (2). □

One could conjecture that the same would be true for the $\tilde f_{M,N}$. But the following example shows that this is far from true.

Let $f,g:[-2,2]\longrightarrow\mathbb{R}$ be defined by:

$$f(x)=\begin{cases} \dfrac{2^k}{k} & \text{if } x\in\left(a_j+\dfrac{1}{2^{(k-1)}}+\dfrac{1}{2^k}\,,\,a_j+\dfrac{1}{2^{(k-2)}}\right) \\ -\dfrac{2^k}{k} & \text{if } x\in\left(a_j+\dfrac{1}{2^{(k-1)}}\,,\,a_j+\dfrac{1}{2^{(k-1)}}+\dfrac{1}{2^k}\right) \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x)=\begin{cases} \dfrac{2^{(k+1)}}{k+1} & \text{if } x\in\left(a_j+\dfrac{1}{2^{(k-2)}}-\dfrac{k+1}{k\,2^{(k+1)}}\,,\,a_j+\dfrac{1}{2^{(k-2)}}\right) \\ -\dfrac{2^k}{k} & \text{if } x\in\left(a_j+\dfrac{1}{2^{(k-1)}}\,,\,a_j+\dfrac{1}{2^{(k-1)}}+\dfrac{1}{2^k}\right) \\ 0 & \text{otherwise} \end{cases}$$

where $a_0=0$, $a_{j+1}=a_j-1/2^j$ and $k\geq j+1$. Let $h:[-2,6]\longrightarrow\mathbb{R}$ be such that $h(x)=f(x)$ if $x\in[-2,2]$ and $h(x)=g(x-4)$ if $x\in]2,6]$.

Notice that f is improper Riemann integrable in each interval $[a_{j+1},a_j]$ and $\int_{a_{j+1}}^{a_j} f=0$; and the same is true with g replaced for f. Then, for each $u\in(-2,6)$, h is integrable in $[u,6]$, and $\lim_{u\to-2}\int_u^6 h=0$. So h is integrable in $[-2,6]$ and $\int_{-2}^6 h=0$ (by [2, Corollary 7.10]).

But let B_k be the set where $h(x)=2^k/k$ and C_k the set where $h(x)=-2^k/k$, and $b_k=2^k m(B_k)/k=\int_{B_k} h$, $c_k=-2^k m(C_k)/k=\int_{C_k} h$.

Notice that $h(x)=2^{k+1}/(k+1)$ for all x in the intervals $(a_j+1/2^k+1/2^{k+1}\,,\,a_j+1/2^{(k-1)})$, and in the intervals $(4+a_j+1/2^{k-2}-(k+1)/k2^{k+1},4+a_j+1/2^{k-2})$, $0\leq j<k+1$, so $b_{k+1}=2$, $k\geq 1$. (There are $2k$ such intervals and the integral of h in each of these intervals is $1/k$.) And $h(x)=2^k/k=2$, for $k=1$, in the interval $(3/2,2)$, giving $b_1=1$. Similarly we can see that $c_n=-2$, for all $n\geq 1$.

Thus any truncation of h of the form $\tilde{h}_{M,N}$ would have integrals given by sums of some b_n and c_n (including $b_1=1$) resulting in an odd integer. These integrals will never be close to zero. So the best result known to me in this direction is Lu's Lemma, [3, Lemma 2].

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