Real Analysis Exchange
Vol. 23(2), 1998-99, pp. 223–234

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PERTURBED TYPE RANDOM CANTOR SET*

Abstract

We define the perturbed type random Cantor set which is a variation of Falconer's random Cantor set, and calculate its Hausdorff dimension by using game fairer with time. It is a generalization of a Falconer's result about random Cantor set.

1 Introduction

Recently many authors ([3], [5], [7]) investigated random fractals associated with random Cantor sets and computed their Hausdorff dimensions. We generalize some of their results related to random Cantor sets. In this paper we deal with only random Cantor sets.

First we introduce a probability space $(\Omega, \mathcal{F}, P)$ [4] such that the sample space Ω is the class of all decreasing sequences of sets [0,1]=$E_0 \supset E_1 \supset E_2 \supset \dots$ satisfying the following conditions.

1) Each E_n consists 2^n disjoint closed intervals I_j, where $j \in \{1, 2\}^n$.
2) Each interval I_j of E_n contains the two intervals $I_{j,1}$ and $I_{j,2}$ of E_{n+1} with the left endpoints of I_j and $I_{j,1}$ and the right endpoints of I_j and $I_{j,2}$ coinciding.
3) For fixed numbers a and b such that $0 < a \le b < \frac{1}{2}$, we write

$$C_{i_1,\dots,i_n} = |I_{i_1,\dots,i_n}|/|I_{i_1,\dots,i_{n-1}}|, C_1 = |I_1|, C_2 = |I_2|$$

and require $a \le C_{i_1,\dots,i_n} \le b$ for all $i_1, \dots, i_n$, where $|I|$ denotes the diameter of the interval I, and a probability measure P is defined on a

Mathematical Reviews subject classification: Primary 28A80, Secondary 60B05
Received by the editors December 18,1996
*Supported in part by TGRC and BSRI-96-1401

suitably large σ-field $\mathcal{F}$ of subsets of Ω such that the ratios $C_{i_1,\ldots,i_n}$ are random variables. (The number b is considered for a simpler calculation.)

We consider an increasing sequence of sub σ-fields of $\mathcal{F}$, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ such that

$$\mathcal{F}_0 = \{\emptyset, \Omega\},$$

$$\mathcal{F}_n = \sigma(\mathcal{F}_{n-1}; C_j, j \in \{1,2\}^n), \quad n = 1, 2, \ldots.$$

In fact, $\mathcal{F}$ contains $\cup_{n=0}^{\infty}\mathcal{F}_n$. We assume that P is a probability measure of $\mathcal{F}$ such that $C_{j,1}$ has the same distribution as $C_{i,1}$ and $C_{j,2}$ has the same distribution as $C_{i,2}$ respectively where $j \in \{1,2\}^n$ and i is the element of singleton $\{1\}^n, n = 0, 1, 2, \ldots$.

We note that Falconer [4] assumed that $C_{j,1}$ and $C_{j,2}$ have the same distribution as C_1 and C_2 respectively for all j.

From now on, we write $C_{i,1}$, $C_{i,2}$ as L_n, R_n respectively for each $n = 0, 1, 2, \ldots$ and i is the element of singleton $\{1\}^n$. We assume that the C_j s are independent random variables, except that for each j we do not require $C_{j,1}$ and $C_{j,2}$ to be independent.

If the above conditions are satisfied, the set $F = \cap_{n=0}^{\infty} E_n$ is called a perturbed type random Cantor set, where $\{E_n\}_{n=1}^{\infty} \in \Omega$.

In this paper, we do not require the (L_n, R_n) to be identically distributed, but we will impose a condition on these random variables to obtain some results. Note that there is a unique solution $s_n \in [0,\infty)$ of $E(L_n^{s_n} + R_n^{s_n}) = 1$ for each $n = 1, 2, \ldots$. It is not difficult to show if $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges for some number s, then s_n converges to s. We note that if $\{s_n\}_{n=1}^{\infty}$ converges to some number s, then $E(L_n^s + R_n^s)$ converges to 1.

Now we attempt to find the Hausdorff dimension of the perturbed type random Cantor set satisfying the condition that $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges for some number s by studying the values of $E(L_n^s + R_n^s)$.

We define a sequence of random variables $X_n = \sum_{I \in E_n} |I|^s$ for each $n = 1, 2, \ldots$. (Each E_n has 2^n disjoint closed intervals I_j where $j \in \{1,2\}^n$. Thus $I \in E_n$ means that I is one of 2^n such intervals.) Then $\{X_n\}_{n=1}^{\infty}$ forms an adapted sequence of random variables with respect to $[\Omega, \mathcal{F}, P; (\mathcal{F}_n, n \in N)]$ (cf. [6]).

Using the independence of the C_j s, we obtain the conditional expectation of X_n with respect to $\mathcal{F}_m$, $E(X_n|\mathcal{F}_m) = \prod_{k=m+1}^{n} E(L_k^s + R_k^s) X_m$ for $n \geq m+1$. We recall that an adapted sequence $\{X_n\}_{n=1}^{\infty}$ is a game fairer with time [2], if given $\varepsilon > 0, \delta > 0$, there exists $M > 0$ such that $n > m \geq M$,

$$P(|E(X_n|\mathcal{F}_m) - X_m| < \delta) \geq 1 - \varepsilon.$$

Also we recall the s-dimensional Hausdorff outer measure $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F)$, where $\mathcal{H}_\delta^s(F) = \inf\{\sum_{n=1}^{\infty} |U_n|^s | \{U_n\}$ is a δ-cover of $F\}$ [4].

2 The Hausdorff Dimension

Henceforth we write $X_n = \sum_{I\in E_n} |I|^s$ for each $n = 1, 2, \dots$ and $\{E_n\}_{n=1}^{\infty} \in \Omega$. We use the following lemma to find the Hausdorff dimension of the perturbed type random Cantor set. The proof of the lemma is easy, so it is omitted.

Lemma 1. *If $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges, then*

(1) $\prod_{n=1}^{\infty} E(L_n^s + R_n^s)$ converges in $(0,\infty)$,

(2) $\prod_{k=m_1}^{m_2} E(L_k^s + R_k^s)$ have an upper bound $U(<\infty)$ and a lower bound $L(>0)$ for every $m_2 > m_1$, and

(3) $\sup_{n>m} |\prod_{k=m+1}^{n} E(L_k^s + R_k^s) - 1|$ converges to 0 as $m \to \infty$.

Proposition 2. *If $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges, then $\{X_n\}_{n=1}^{\infty}$ is L^2-bounded. Hence $\{X_n\}_{n=1}^{\infty}$ is uniformly integrable.*

PROOF. Fix n. Let v_k be the variance of $L_k^s + R_k^s$, i.e. $v_k = E[(L_k^s + R_k^s)^2] - [E(L_k^s + R_k^s)]^2$ with $v_0 = 1$, and $\alpha_k = [E(L_k^s + R_k^s)]^2$ with $\alpha_{n+1} = 1$, $\beta_k = E(L_k^{2s} + R_k^{2s})$ with $\beta_0 = 1$, $\beta_{-1} = 1$, where $k = 1, 2, \cdots, n$. Then

$$\begin{aligned}
E(X_n^2|\mathcal{F}_{n-1}) &= E\big(\sum_{j,j'\in\{1,2\}^{n-1}} (C_{j,1}^s + C_{j,2}^s)(C_{j',1}^s + C_{j',2}^s)|I_j|^s|I_{j'}|^s \,|\, \mathcal{F}_{n-1}\big)\\
&= E\big(\sum_{j\in\{1,2\}^{n-1}} (C_{j,1}^s + C_{j,2}^s)^2|I_j|^{2s}\\
&\quad + \sum_{\substack{j\neq j'\\ j,j'\in\{1,2\}^{n-1}}} (C_{j,1}^s + C_{j,2}^s)(C_{j',1}^s + C_{j',2}^s)|I_j|^s|I_{j'}|^s \,|\, \mathcal{F}_{n-1}\big)\\
&= \sum_{j\in\{1,2\}^{n-1}} E[(L_n^s + R_n^s)^2]|I_j|^{2s}\\
&\quad + \sum_{\substack{j\neq j'\\ j,j'\in\{1,2\}^{n-1}}} [E(L_n^s + R_n^s)]^2|I_j|^s|I_{j'}|^s\\
&= \sum_{j,j'\in\{1,2\}^{n-1}} E[(L_n^s + R_n^s)]^2|I_j|^s|I_{j'}|^s + v_n \sum_{j\in\{1,2\}^{n-1}} |I_j|^{2s}\\
&= \alpha_n \sum_{j,j'\in\{1,2\}^{n-1}} |I_j|^s|I_{j'}|^s + v_n \sum_{j\in\{1,2\}^{n-1}} |I_j|^{2s}\\
&= \alpha_n X_{n-1}^2 + v_n \sum_{j\in\{1,2\}^{n-1}} |I_j|^{2s}.
\end{aligned}$$

Thus

$$\begin{aligned} E(X_n^2|\mathcal{F}_{n-2}) &= \alpha_n E(X_{n-1}^2|\mathcal{F}_{n-2}) + v_n E\big(\sum_{j\in\{1,2\}^{n-1}} |I_j|^{2s}|\mathcal{F}_{n-2}\big) \\ &= \alpha_n E\big(\sum_{j,j'\in\{1,2\}^{n-2}} (C_{j,1}^s + C_{j,2}^s)(C_{j',1}^s + C_{j',2}^s)|I_j|^s|I_{j'}|^s \,|\, \mathcal{F}_{n-2}\big) \\ &\quad + v_n E(\sum_{j\in\{1,2\}^{n-2}} (C_{j,1}^{2s} + C_{j,2}^{2s})|I_j|^{2s}|\mathcal{F}_{n-2}) \\ &= \alpha_n(\alpha_{n-1}X_{n-2}^2 + v_{n-1}\sum_{j\in\{1,2\}^{n-2}} |I_j|^{2s}) \\ &\quad + v_n E(L_{n-1}^{2s} + R_{n-2}^{2s}) \sum_{j\in\{1,2\}^{n-2}} |I_j|^{2s} \\ &= \alpha_n\alpha_{n-1}X_{n-2}^2 + \alpha_n v_{n-1} \sum_{j\in\{1,2\}^{n-2}} |I_j|^{2s} \\ &\quad + v_n\beta_{n-1} \sum_{j\in\{1,2\}^{n-2}} |I_j|^{2s}. \end{aligned}$$

From the above calculation, we easily obtain the following equality.

$$E(X_n^2) = E(X_n^2|\mathcal{F}_0) = \sum_{m=-1}^{n-1} (\prod_{i=m+2}^{n+1} \alpha_i) v_{m+1} (\prod_{j=-1}^{m} \beta_j)$$

(Note that $\alpha_{n+1} = 1$ and $\beta_0 = \beta_{-1} = 1$).

Since $\prod_{i=1}^{\infty}[E(L_i^s + R_i^s)]^2$ converges, $\prod_{i=m+2}^{n+1} \alpha_i$ is bounded above by some bound $B(> 1)$ which is independent of the choice of m and n with $n \geq m+1$(cf. Lemma 1). Since

$$\beta_j = E(L_j^{2s} + R_j^{2s}) \leq E(b^s[L_j^s + R_j^s]) = b^s E(L_j^s + R_j^s),$$

we have

$$\prod_{j=-1}^{m} \beta_j \leq (b^s)^m \prod_{j=1}^{m} E(L_j^s + R_j^s) \leq (b^s)^m B^{\frac{1}{2}},$$

where $m \geq 1$. Now for $m \geq 1$,

$$\begin{aligned} v_m &= E[(L_m^s + R_m^s)^2] - [E(L_m^s + R_m^s)]^2 \\ &\leq (2b^s)^2 - (2a^s)^2 = 4(b^{2s} - a^{2s}). \end{aligned}$$

Hence

$$E(X_n^2) \le B + \sum_{m=0}^{n-1} 4(b^{2s} - a^{2s}) B^{\frac{3}{2}} (b^s)^m$$
$$= B + \frac{1-(b^s)^n}{1-b^s} 4(b^{2s} - a^{2s}) B^{\frac{3}{2}}.$$

Thus

$$\sup_n E(X_n^2) \le B + \frac{4(b^{2s} - a^{2s}) B^{\frac{3}{2}}}{1-b^s}.$$

(We note that the constants a, b are the numbers given in Introduction.)

Lemma 3. *([2]). If $\{X_n\}_{n=1}^{\infty}$ is a uniformly integrable game fairer with time, then $\{X_n\}$ converges in L^1.*

From now on, we write

$$X_n(I_j) = \sum_{I \in I_j \cap E_n} |I|^s \text{ for } j \in \{1,2\}^k,$$

where an integer $k \ge 0$ and $n = k, k+1, \dots$.

Corollary 4. *Assume that $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges. Then $\{X_n\}_{n=1}^{\infty}$ is a uniformly integrable game fairer with time and hence converges in L^1. Further, the sequence of random variables $\{X_n(I_j)\}_{n=k}^{\infty}$ is also a uniformly integrable game fairer with time for each $j \in \{1,2\}^k$ and hence converges in L^1.*

Proof. Fix an integer $k \ge 0$ and $j \in \{1,2\}^k$. We note that $X_0(I_\phi) = 1$. Now, $[X_n(I_j)]^2 \le X_n^2$, so $\{X_n(I_j)\}_{n=k}^{\infty}$ is L^2-bounded and hence, uniformly integrable. Now, fix $\epsilon, \delta > 0$. Since $\{X_n(I_j)\}_{n=k}^{\infty}$ is uniformly integrable, we can find $\alpha > 1$ such that

$$\int_{(X_m(I_j) > \alpha)} X_m(I_j \, dP < \epsilon$$

for all $m \ge k$. Then $P(X_m(I_j) > \alpha) < \epsilon/\alpha < \epsilon$ for all $m \ge k$. By Lemma 1, we can choose $M \ge k$ such that for all $n > m > M$,

$$\Big| \prod_{i=m+1}^{n} E(L_i^s + R_i^s) - 1 \Big| < \delta/\alpha.$$

Hence in $(X_m(I_j) \le \alpha)$ for all $n > m > M$,

$$|E(X_n(I_j)|\mathcal{F}_m) - X_m(I_j)| = \Big|\prod_{i=m+1}^{n} E(L_i^s + R_i^s) - 1)\Big| X_m(I_j)$$

$$\le \Big|\prod_{i=m+1}^{n} E(L_i^s + R_i^s) - 1)\Big|\alpha < \delta.$$

Remark 5. Because $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges, $\{X_n(I_j)\}_{n=k}^{\infty}$ is a uniformly integrable game fairer with time whereas the condition of Falconer that $L_n = L$ and $R_n = R$ for all n makes $\{X_n(I_j)\}_{n=k}^{\infty}$ a uniformly integrable martingale.

Now we compute an upper bound for the Hausdorff dimension of a certain perturbed type random Cantor set.

Theorem 6. *Suppose that $\{X_n\}_{n=1}^{\infty}$ is a uniformly integrable game fairer with time and $\lim_n s_n = s$. Then the Hausdorff dimension of the perturbed type random Cantor set F is equal to or less than s for almost all F (i.e., for P-almost all $\{E_n\}_{n=1}^{\infty} \in \Omega$).*

PROOF. By Lemma 3, $\{X_n\}$ has an L^1-limit X. Since $E(X) < \infty$, $X < \infty$ a.s. on Ω. Since $\{X_n\}_{n=1}^{\infty}$ converges to X in L^1, there is a subsequence $\{X_{n_k}\}_{k=1}^{\infty}$ that converges to X a.s. In particular, there is a random variable M such that

$$X_{n_k} = \sum_{I \in E_{n_k}} |I|^s \le M < \infty \text{ for all } k \text{ a.s..}$$

Then $\mathcal{H}_\delta^s(F) \le \sum_{I \in E_{n_k}} |I|^s \le M$ if $n_k \ge -\log\delta/\log 2$ a.s.. Hence $\mathcal{H}^s(F) \le M$ a.s..

Corollary 7. *If $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges, then the Hausdorff dimension of the perturbed type random Cantor set F is equal to or less than s for almost all F.*

We will use the energy of some mass distribution([4]) μ on F to find a lower bound for the Hausdorff dimension of the perturbed type random Cantor set F. To attain the goal, we first generate a mass distribution μ on F by using the L^1-convergence of $\{X_n\}_{n=1}^{\infty}$.

Lemma 8. *Assume that $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges. If we define random variables*

$$\mu(I_j) = \operatorname*{Lim}_{n\to\infty} \sum_{I \in I_j \cap E_n} |I|^s, \textit{ for } j \in \{1,2\}^k \textit{ and } k = 0,1,2,\ldots,$$

(Here, Lim denote the L^1-limit) then μ extends to a mass distribution on the perturbed type random Cantor set F for P-almost all F in Ω.

PROOF. In fact, $\mu(I_j) = \mathrm{Lim}_{n\to\infty} X_n(I_j)$ and its L^1-limit exists by Corollary 4. Since

$$\mu(I_j) = \mathop{\mathrm{Lim}}_{n\to\infty} X_n(I_j) = \mathop{\mathrm{Lim}}_{n\to\infty} (X_n(I_{j,1}) + X_n(I_{j,2}))$$

and

$$\mu(I_{j,1}) = \mathop{\mathrm{Lim}}_{n\to\infty} X_n(I_{j,1}), \mu(I_{j,2}) = \mathop{\mathrm{Lim}}_{n\to\infty} X_n(I_{j,2}),$$

we have

$$\mu(I_j) = \mu(I_{j,1}) + \mu(I_{j,2}) \text{ a.s. on } \Omega.$$

Clearly $\mu([0,1]) = \mathrm{Lim}_{n\to\infty} X_n$ and

$$\begin{aligned} E(\mathop{\mathrm{Lim}}_{n\to\infty} X_n) &= \lim_{n\to\infty} E(X_n) = \lim_{n\to\infty} E(X_n|\mathcal{F}_0) \\ &= \lim_{n\to\infty} \prod_{k=1}^{n} E(L_k^s + R_k^s) = \prod_{n=1}^{\infty} E(L_n^s + R_n^s). \end{aligned}$$

Thus $\mu([0,1]) < \infty$ a.s., and $\mu([0,1]) > 0$ with positive probability. Let $\mu([0,1]) = 0$ with probability $q < 1$. Observing the construction of the perturbed type random Cantor set, we easily see that $\mu(I_j) = 0$ with probability $q^{1/2^n}$ where $j \in \{1,2\}^n$ and $\frac{\mu(I_j)}{|I_j|^s} = 0$ with the same probability. Now we show that q must be zero. For $j \in \{1,2\}^n$,

$$\begin{aligned} E\Big(\frac{\mu(I_j)}{|I_j|^s}\Big) &= E\Big(\frac{\mu(I_j)}{|I_j|^s}|\mathcal{F}_n\Big) \\ &= \Big(\lim_{k\to\infty} \prod_{i=n+1}^{k} E(L_i^s + R_i^s)|I_j|^s\Big)/|I_j|^s \\ &= \prod_{i=n+1}^{\infty} E(L_i^s + R_i^s). \end{aligned}$$

Using agruments similar to those in the proof of Proposition 2, we obtain for fixed $k \geq n$ and $j \in \{1,2\}^n$,

$$E\Big[\Big(\frac{X_k(I_j)}{|I_j|^s}\Big)^2\Big] = \sum_{m=n-1}^{k-1} \Big(\prod_{i=m+2}^{k+1} \alpha_i\Big) v_{m+1} \Big(\prod_{l=n-1}^{m} \beta_l\Big),$$

where $\alpha_i = [E(L_i^s + R_i^s)]^2$ with $\alpha_{k+1} = 1$, $v_n = 1$, $\beta_l = E(L_l^{2s} + R_l^{2s})$ with $\beta_{n-1} = \beta_n = 1$. It is not difficult to show that

$$E[(\frac{X_k(I_j)}{|I_j|^s})^2] \leq B + \frac{4(b^{2s} - a^{2s})B^{\frac{3}{2}}}{1 - b^s} \text{ for some } B > 1.$$

By Fatou's theorem, for all n and every $j \in \{1,2\}^n$,

$$E\Big[\Big(\frac{\mu(I_j)}{|I_j|^s}\Big)^2\Big] \leq \underline{\lim}_{k\to\infty} E\Big[(\frac{X_k(I_j)}{|I_j|^s})^2\Big] \leq B + \frac{4(b^{2s} - a^{2s})B^{\frac{3}{2}}}{1 - b^s}$$

for some $B > 1$. This implies $\{\frac{\mu(I_j)}{|I_j|^s} | j \in \{1,2\}^n\}_{n=1}^{\infty}$ is uniformly integrable. Hence we find $\alpha > 0$ such that $\int_{(\frac{\mu(I_j)}{|I_j|^s} > \alpha)} \frac{\mu(I_j)}{|I_j|^s}\, dP \leq \frac{1}{3}$ for all $j \in \{1,2\}^n$ and all n. Assume that $0 < q < 1$. Then there is an integer N such that $(1 - q^{1/2^n})\alpha \leq \frac{1}{3}$ for all $n \geq N$. Thus for all $n \geq N$ and every $j \in \{1,2\}^n$,

$$\begin{aligned} E\Big(\frac{\mu(I_j)}{|I_j|^s}\Big) &= \int_{(\frac{\mu(I_j)}{|I_j|^s} > \alpha)} \frac{\mu(I_j)}{|I_j|^s}\, dP + \int_{(0 < \frac{\mu(I_j)}{|I_j|^s} \leq \alpha)} \frac{\mu(I_j)}{|I_j|^s}\, dP \\ &\leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

But noting that $E\big(\frac{\mu(I_j)}{|I_j|^s}\big) = \prod_{i=n+1}^{\infty} E(L_i^s + R_i^s)$ for every $j \in \{1,2\}^n$ and that $\lim_{n\to\infty} \prod_{i=n+1}^{\infty} E(L_i^s + R_i^s) = 1$, we have a contradiction. Hence $\mu([0,1]) = 0$ with probability $q = 0$. By Proposition 1.7[4], μ is uniquely extendable to the Borel sets in $[0,1]$ and the support of μ is contained in $F = \cap_{n=0}^{\infty} E_n \subset [0,1]$, for P-almost all F in Ω.

Lemma 9. *For $n \geq k+2$ and $0 < t < s$,*

$$\begin{aligned} &E(|I_j|^{-t} \sum_{I \in I_{j,1} \cap E_n} \sum_{I' \in I_{j,2} \cap E_n} |I|^s |I'|^s | \mathcal{F}_{k+1}) \\ &= \Big[\prod_{i=k+2}^{n} E(L_i^s + R_i^s)\Big]^2 |I_j|^{-t} |I_{j,1}|^s |I_{j,2}|^s \end{aligned}$$

for $j \in \{1,2\}^k$.

Proof. Let $j \in \{1,2\}^k$.

$$\begin{aligned}
&E(|I_j|^{-t} \sum_{I \in I_{j,1} \cap E_n} \sum_{I' \in I_{j,2} \cap E_n} |I|^s |I'|^s | \mathcal{F}_{n-1}) \\
&= E(|I_j|^{-t} \sum_{I_i \in I_{j,1} \cap E_{n-1}} \sum_{I_{i'} \in I_{j,2} \cap E_{n-1}} (C_{i,1}^s + C_{i,2}^s)|I_i|^s (C_{i',1}^s \\
&\quad + C_{i',2}^s)|I_{i'}|^s | \mathcal{F}_{n-1}) \\
&= |I_j|^{-t} \sum_{I_i \in I_{j,1} \cap E_{n-1}} \sum_{I_i' \in I_{j,2} \cap E_{n-1}} [E(L_n^s + R_n^s)]^2 |I_i|^s |I_{i'}|^s.
\end{aligned}$$

By the independence of $(C_{i,1}, C_{i',1})$, $(C_{i,1}, C_{i',2})$ $(C_{i,2}, C_{i',1})$ and $(C_{i,2}, C_{i',2})$.

$$\begin{aligned}
&E(|I_j|^{-t} \sum_{I \in I_{j,1} \cap E_n} \sum_{I' \in I_{j,2} \cap E_n} |I|^s |I'|^s | \mathcal{F}_{n-2}) \\
&= |I_j|^{-t} [E(L_n^s + R_n^s)]^2 E\Big(\sum_{I_i \in I_{j,1} \cap E_{n-2}} \\
&\sum_{I_{i'} \in I_{j,2} \cap E_{n-2}} (C_{i,1}^s + C_{i,2}^s)|I_i|^s \\
&(C_{i',1}^s + C_{i',2}^s)|I_{i'}|^s | \mathcal{F}_{n-2}\Big) \\
&= |I_j|^{-t} [E(L_n^s + R_n^s)]^2 [E(L_{n-1}^s + R_{n-1}^s)]^2 \\
&\quad \sum_{I_i \in I_{j,1} \cap E_{n-2}} \sum_{I_{i'} \in I_{j,2} \cap E_{n-2}} |I_i|^s |I_{i'}|^s.
\end{aligned}$$

Continuing in this fashion, we obtain the conclusion.

Theorem 10. *If $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges, then the Hausdorff dimension of the perturbed type random Cantor set F is equal to or greater than s for almost all F.*

Proof. Let $x, y \in F$. Let $x \wedge y$ denote the interval in E_n contining x and y with the greatest possible integer n. We note that the gap of $I_{j,1}$ and $I_{j,2}$ is equal to or greater than $d|I_j|$, where $d = 1 - 2b$. Let $0 < t < s$, fix $j \in \{1,2\}^k$ and consider the mass distribution μ in Lemma 8. Then

$$\begin{aligned}
&E\Big(\iint_{x \wedge y = I_j} |x-y|^{-t} \, d\mu(x) \, d\mu(y) \Big) \\
&\le 2d^{-t} E(|I_j|^{-t} \mu(I_{j,1}) \mu(I_{j,2}))
\end{aligned}$$

$$= 2d^{-t}E(|I_j|^{-t}\operatorname*{Lim}_{n\to\infty} X_n(I_{j,1})\operatorname*{Lim}_{n\to\infty} X_n(I_{j,2}))$$

Since $X_n(I_{j,1}) \xrightarrow{L^1} \mu(I_{j,1})$ and $X_n(I_{j,2}) \xrightarrow{L^1} \mu(I_{j,2})$, there exists $\{n_k\}_{k=1}^{\infty}$ such that $X_{n_k}(I_{j,1}) \xrightarrow{a.s} \mu(I_{j,1})$ and there exists $\{n_{k_\ell}\}_{\ell=1}^{\infty}$, a subsequence of $\{n_k\}_{k=1}^{\infty}$ such that $X_{n_{k_\ell}}(I_{j,2}) \xrightarrow{a.s} \mu(I_{j,2})$. Clearly $X_{n_{k_\ell}}(I_{j,1}) \xrightarrow{a.s} \mu(I_{j,1})$. Hence

$$\begin{aligned}
&E\Big(\iint_{x\wedge y=I_j} |x-y|^{-t}\,d\mu(x)\,d\mu(y)\Big)\\
&\le 2d^{-t}E(|I_j|^{-t}\lim_{\ell\to\infty} X_{n_{k_\ell}}(I_{j,1})\lim_{\ell\to\infty} X_{n_{k_\ell}}(I_{j,2}))\\
&= 2d^{-t}E(|I_j|^{-t}\lim_{\ell\to\infty}\sum_{I\in I_{j,1}\cap E_{n_{k_\ell}}}|I|^s\lim_{\ell\to\infty}\sum_{I'\in I_{j,2}\cap E_{n_{k_\ell}}}|I'|^s)\\
&= 2d^{-t}E(|I_j|^{-t}\lim_{\ell\to\infty}\sum_{\substack{I\in I_{j,1}\cap E_{n_{k_\ell}}\\ I'\in I_{j,2}\cap E_{n_{k_\ell}}}}|I|^s|I'|^s)\\
&\le 2d^{-t}\underline{\lim}_{\ell\to\infty}E(|I_j|^{-t}\sum_{\substack{I\in I_{j,1}\cap E_{n_{k_\ell}}\\ I'\in I_{j,2}\cap E_{n_{k_\ell}}}}|I|^s|I'|^s)\ \text{[Fatou's Lemma]}\\
&= 2d^{-t}\underline{\lim}_{\ell\to\infty}\Big[\prod_{i=k+2}^{n_{k_\ell}}E(L_i^s+R_i^s)\Big]^2E(|I_j|^{-t}|I_{j,1}|^s|I_{j,2}|^s)\ \text{[Lemma 9]}\\
&\le 2d^{-t}BE(|I_j|^{2s-t})
\end{aligned}$$

for some $B<\infty$ [Lemma 1(2)]. We note that B is independent of k.

Now we choose $\varepsilon>0$ such that $b^{s-t}<1-\frac{2\varepsilon}{1+\varepsilon}$. Then there exists a large number N such that $E(L_k^s+R_k^s)<1+\varepsilon$ for all $k\ge N$. Therefore for all $k\ge N$,

$$\begin{aligned}
E\Big(\sum_{I\in E_k}|I|^{2s-t}\Big) &\le E(b^{s-t}(L_1^s+R_1^s))\cdots E(b^{s-t}(L_k^s+R_k^s)\Big)\\
&= \prod_{i=1}^{N-1}b^{s-t}E(L_i^s+R_i^s)\prod_{i=N}^{k}b^{s-t}E(L_i^s+R_i^s)\\
&\le \prod_{i=1}^{N-1}b^{s-t}E(L_i^s+R_i^s)(1-\varepsilon)^{k-N}.
\end{aligned}$$

Let $\prod_{i=1}^{N-1} b^{s-t} E(L_i^s + R_i^s) = \alpha$. Then

$$\sum_{k=N}^{\infty} E\big(\sum_{I \in E_k} |I|^{2s-t}\big) \le \sum_{k=N}^{\infty} \alpha(1-\epsilon)^{k-N} = \alpha/\varepsilon < \infty.$$

Hence for $k \ge N$,

$$\begin{aligned} &\sum_{k=N}^{\infty} E\Big(\sum_{I \in E_k} \iint_{x \wedge y = I} |x-y|^{-t}\, d\mu(x)\, d\mu(y)\Big) \\ &\le 2d^{-t} B \sum_{k=N}^{\infty} E(\sum_{I \in E_k} |I|^{2s-t}) \\ &\le 2d^{-t} B\alpha/\varepsilon. \end{aligned}$$

Now

$$\begin{aligned} &E\Big(\int_F \int_F |x-y|^{-t}\, d\mu(x) d\mu(y)\Big) \\ &= E\Big(\sum_{k=0}^{\infty} \sum_{j \in \{1,2\}^k} \iint_{x \wedge y = I_j} |x-y|^{-t}\, d\mu(x)\, d\mu(y)\Big) \\ &= E\Big(\sum_{k=0}^{N-1} \sum_{j \in \{1,2\}^k} \iint_{x \wedge y = I_j} |x-y|^{-t}\, d\mu(x), d\mu(y)\Big) \\ &\quad + E\Big(\sum_{k=N}^{\infty} \sum_{j \in \{1,2\}^k} \iint_{x \wedge y = I_j} |x-y|^{-t}\, d\mu(x)\, d\mu(y)\Big) \\ &\le 2d^{-t} B E\Big(\sum_{k=0}^{N-1} \sum_{j \in \{1,2\}^k} |I_j|^{2s-t}\Big) + 2d^{-t} B\varepsilon/\alpha \\ &\le 2d^{-t} B\Big(\sum_{k=0}^{N-1} 2^k b^{2s-t} + \frac{\varepsilon}{\alpha}\Big) < \infty. \end{aligned}$$

We note that μ is a mass distribution on F for almost all F (cf. Lemma 8). Thus the Hausdorff dimension of $F = \cap_{n=0}^{\infty} E_n$ is greater than or equal to t for almost all F since the F has finite t-energy of μ (cf. Theorem 4.13 (a)[4]).

Corollary 11. *If $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges, then the Hausdorff dimension of the perturbed type random Cantor set F is s for almost all F.*

Corollary 12. *([4]) If $L_n = L$ and $R_n = R$ for all n, then the Hausdorff dimension of the perturbed type random Cantor set F is s for almost all F, where s is the solution of the expectation equation $E(L^s + R^s) = 1$.*

Remark 13. We see that there is a close connection between the Hausdorff dimension s of the perturbed type random Cantor set and the values of $E(L_n^s + R_n^s)$. We note that its Hausdorff dimension does not depend on the similarities of the distribution functions (L_n, R_n), but on their expectation values $E(L_n^s + R_n^s)$. For there are many examples such that $L_{2n-1} = L$, $L_{2n} = L'$, $R_{2n-1} = R$, $R_{2n} = R'$ with $E(L_n^s + R_n^s) = 1$ for each n for some number s and (L, R) and (L', R') are not identically distributed.

In particular, if we consider a specific Ω consisting of only one member, the perturbed type random Cantor set is in fact a perturbed Cantor set([1]). Then we see that the Hausdorff dimension of the perturbed type random Cantor set (in fact, a perturbed Cantor set) is s, where $\log E(L_n^s + R_n^s)$ converges to 0 as n increases to ∞ without the condition that $\sum_{n=1}^{\infty} \log E(L_n^s + R_n^s)$ converges (Corollary 8 [1]).

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