Real Analysis Exchange
Vol. 23(1), 1998-99, pp. 217–221

Marianna Csörnyei*, Eötvös University, Department of Analysis, Budapest, Múzeum Krt. 6-8, H-1088, Hungary, e-mail: `csornyei@cs.elte.hu`

ON THE NON-EXISTENCE OF CERTAIN BOUNDED LINEAR PROJECTIONS

Abstract

It is known that there is a bounded linear operator A from the space of bounded real functions to the subspace of bounded Lebesgue-measurable functions such that for any Lebesgue-measurable function f we have $Af = f$ for a.e. $x \in \mathbb{R}$. S. A. Argyros proved that A could not be a projection; i.e. we can always find a bounded measurable function f and a point $x \in \mathbb{R}$ for which $(Af)(x) \neq f(x)$.

We give an independent proof and in particular we prove that there does not exist a projection to the space of functions with the Baire property, either.

S. A. Argyros proved in [AR] that there does not exist a bounded linear projection from the space of all bounded real functions to the subspace of all bounded Lebesgue-measurable functions and the subspace of all bounded Borel-measurable functions.

In this paper we give an independent proof, and our proof covers more general cases as well. In particular, we prove that such a projection does not exist to the subspace of functions with the Baire property, either.

More precisely, we show that if $\mathcal{M} \subseteq P(\mathbb{R})$ is a σ-algebra, if there is a σ-ideal $\mathcal{K} \subset \mathcal{M}$, $\mathcal{K} \neq \mathcal{M}$ and if $\mathcal{P} \subseteq \mathcal{N} \stackrel{\text{def}}{=} \mathcal{M} \setminus \mathcal{K}$ such that

(0) $\{x\} \in \mathcal{M}$ for all $x \in \mathbb{R}$;

(1) for every $N \in \mathcal{N}$ there exists $P \subseteq N, P \in \mathcal{P}$;

(2) given more than ω elements of $\mathcal{N}$ there exist infinitely many of them with non-empty intersection;

(3) $|\mathcal{P}| \leq 2^\omega$ and $|P| = 2^\omega$ for all $P \in \mathcal{P}$,

Mathematical Reviews subject classification: 28A05, 28A20,47B38
Received by the editors November 15, 1997
*Research supported by the Hungarian National Foundation for Scientific Research, Grant No. T 019476.

then there is no bounded linear projection from the space of bounded functions to the subspace of bounded $\mathcal{M}$-measurable functions.

The special cases of measurable image functions or functions with the Baire property follow by putting $\mathcal{M}$ = Lebesgue measurable sets, $\mathcal{K}$ = sets of 0 measure, or $\mathcal{M}$ = sets with the Baire property, $\mathcal{K}$ = first category sets, respectively, and in both cases we may choose $\mathcal{P}$ = Borel elements of $\mathcal{N}$.

In what follows let $\mathcal{B}$ be the space of all bounded real functions, let $\mathcal{F}$ denote the subspace of bounded $\mathcal{M}$-measurable functions, and suppose we are given a linear projection $P : \mathcal{B} \to \mathcal{F}$. Pf is also denoted by $\widetilde{f}$, and $\| f \|$ is the usual sup norm of f.

Our main result is the following theorem.

Theorem. *Making use of the notations above, for every $K > 0$ there exists an $f \in \mathcal{B}$ such that $\| \widetilde{f} \| > K \| f \|$.*

PROOF. The proof is based on a series of lemmas. Suppose that for some K, $\| \widetilde{f} \| \leq K \| f \|$ holds for every $f \in \mathcal{B}$.

Lemma 1. *For every $f \in \mathcal{B}$ and $m \in \mathcal{F}$ we have $\| f - \widetilde{f} \| \leq (K-1) \| f + m \|$.*

PROOF. Let $f \in \mathcal{B}$, $a \in \mathbb{R}$ be arbitrary, put $d = \| f \| - f(a)$ and let

$$n(x) \stackrel{\text{def}}{=} \begin{cases} d & \text{if } x = a \\ 0 & \text{otherwise.} \end{cases}$$

Then $n \in \mathcal{F}$ and hence

$$\widetilde{f}(a) + d = \widetilde{f}(a) + n(a) = \widetilde{f}(a) + \widetilde{n}(a) \leq \| \widetilde{f + n} \| \leq$$

$$\leq K \| f + n \| = K \| f \| = (K-1) \| f \| + f(a) + d.$$

That is $\widetilde{f}(a) - f(a) \leq (K-1) \| f \|$ holds for every $a \in \mathbb{R}$ and $f \in \mathcal{B}$.

Applying this for $-f$ we obtain $-\widetilde{f}(a) + f(a) \leq (K-1) \| f \|$. Therefore $\| f - \widetilde{f} \| \leq (K-1) \| f \|$ $(f \in \mathcal{B})$. Finally replacing f by $f + m$ $(m \in \mathcal{F})$ we have

$$\| f - \widetilde{f} \| = \| (f+m) - (\widetilde{f+m}) \| \leq (K-1) \| f + m \|$$

for every $m \in \mathcal{F}$. $\square$

Definition. A subset $A \subseteq \mathbb{R}$ is called
–*large*, if there is no $N \in \mathcal{N}$ such that $N \subseteq (\mathbb{R} \setminus A)$ (i.e. A intersects every element of $\mathcal{P}$),
–*good*, if $\{x | \widetilde{\chi_A}(x) \neq 0\} \in \mathcal{K}$, and *bad*, if it is not good.

Lemma 2. *There are at most countably many pairwise disjoint bad sets.*

Proof. Suppose that the sets $A_\gamma\,(\gamma \in \Gamma)$ are pairwise disjoint, bad, and $|\Gamma| > \omega$.

For the sake of brevity we put $\chi_\gamma = \chi_{A_\gamma}$. For every $\gamma \in \Gamma$, $n \in \mathbb{N}$ and $\varepsilon = \pm 1$ let

$$A_{\gamma,n,\varepsilon} \stackrel{\text{def}}{=} \{x : \varepsilon(\widetilde{\chi_\gamma}(x)) > 1/n\}.$$

The sets A_γ are all bad. Hence for each γ there exists a pair (n, ε) such that $A_{\gamma,n,\varepsilon} \in \mathcal{N}$. Since $|\Gamma| > \omega$, we may choose (n, ε) such that $|\{\gamma \in \Gamma | A_{\gamma,n,\varepsilon} \in \mathcal{N}\}| > \omega$.

Referring to (2) there exists an x such that $x \in A_{\gamma,n,\varepsilon}$ holds for infinitely many $\gamma \in \Gamma$, say, for $\gamma_1, \gamma_2, ..., \gamma_N, ...$. Consider now $f = \sum_{i=1}^N \chi_{\gamma_i}$. Then

$$\| f \| = 1, \quad |\widetilde{f}(x)| = \Big|\sum_{i=1}^N \widetilde{\chi_{\gamma_i}}(x)\Big| \geq N/n > K,$$

if N is large enough. This contradiction proves the lemma. □

Definition. A subset $B \subseteq \mathbb{R}$ is called *very good*, if it is large, good, and $\widetilde{(\chi_B)}\big|_B \equiv 0$ holds.

Lemma 3. *Every large good set A contains a very good subset.*

Proof. A is a good set, therefore $\widetilde{(\chi_A)}\big|_A$ admits a support $A^* \subseteq A$, $A^* \in \mathcal{K}$. Let $B \stackrel{\text{def}}{=} A \setminus A^*$. Then B is obviously good, $\widetilde{(\chi_A)}\big|_B \equiv 0$ and $(\chi_{A^*})\big|_B \equiv 0$. Now by $\chi_{A^*} \in \mathcal{F}$ we have

$$\widetilde{\chi_B} = (\widetilde{\chi_A - \chi_{A^*}}) = \widetilde{\chi_A} - \chi_{A^*}.$$

Hence $\widetilde{(\chi_B)}\big|_B \equiv 0$. Finally, since A meets every set of the form $N \setminus A^*$ $(N \in \mathcal{N})$, B intersects every set $N \in \mathcal{N}$. □

Lemma 4. *If we are given continuum many pairwise disjoint large sets A_γ $(\gamma \in \Gamma)$, $\quad |\Gamma| = 2^\omega$, then for every γ we can choose an element $\gamma^* \in A_\gamma$ such that the set $A^* = \{\gamma^* | \gamma \in \Gamma\}$ is large as well. (The set A^* is called a diagonal of our family A_γ $(\gamma \in \Gamma)$.)*

Proof. Since the cardinal number of $\mathcal{P}$ is at most 2^ω, we can index the elements of $\mathcal{P}$ by Γ: $\{P_\gamma | \gamma \in \Gamma\} = \mathcal{P}$. (An element of $\mathcal{P}$ may have several indices.) Choose $\gamma^* \in A_\gamma \cap P_\gamma$ arbitrarily. Now the complement of $A^* = \{\gamma^* | \gamma \in \Gamma\}$ does not contain any element of $\mathcal{P}$ as a subset. Thus it does not contain any element of $\mathcal{N}$, either. □

Definition. A system of sets $\mathcal{A} = \{A; A_i^\alpha | i = 1, ..., N, \alpha \in A\}$ is called a *comb of degree* N, if

(1) $A \cap A_i^\alpha = \alpha$

(2) $A_i^\alpha \cap A_j^\alpha = \alpha \quad (i \neq j)$

(3) $A_i^\alpha \cap A_j^\beta = \emptyset \quad (\alpha \neq \beta)$.

A comb is *good*, if all the sets it contains are good, and it is *very good*, if the sets it contains are very good.
The *points* of a comb are the points of the sets it contains, and two combs are *disjoint*, if they have not got a common point.

Lemma 5. *For every N there exist continuum many pairwise disjoint very good combs of degree N.*

PROOF. We proceed by induction.

In the case $N = 0$ we have to show that there exist continuum many pairwise disjoint very good sets. For that it is sufficient to show, that there exist continuum many pairwise disjoint large sets, since by Lemma 2 there are continuum many good sets among them, thus we can omit the bad sets. Finally, by Lemma 3 every good set contains a very good subset.

We define the pairwise disjoint large sets by transfinite recursion.

Let P_α $(\alpha < \Omega)$ be a well ordering of $\mathcal{P}$ to the initial ordinal number Ω of continuum. (If $|\mathcal{P}| < 2^\omega$, then we use one or more members of $\mathcal{P}$ repeatedly.) For every ordinal $\alpha < \Omega$ we choose a point $p_{\alpha,\beta} \in P_\beta$ $(\beta \leq \alpha)$, and for every $\alpha < \Omega$ we choose additional points $p_{\beta,\alpha} \in P_\alpha$ $(\beta < \alpha)$ such that all selected points are different.

Before the α-th step less than continuum many points have been chosen. Thus by $|P_\alpha| = 2^\omega$ we can always continue choosing different points. Hence the sets $A_\alpha = \{p_{\alpha,\beta} : \beta < \Omega\}$ are certainly pairwise disjoint and they are obviously large as well since $p_{\alpha,\beta} \in A_\alpha \cap P_\beta$; that is, A_α intersects every element of $\mathcal{P}$.

$N \to N + 1$

Suppose that we have proved the lemma for some N and let

$$\mathcal{A}_{\gamma\delta} \quad (\gamma, \delta \in \Gamma \times \Gamma,\ |\Gamma| = 2^\omega)$$

be a system of pairwise disjoint very good combs of degree N. For every γ let B_γ be a diagonal of the sets $A_{\gamma\delta}$. Continuum many of these diagonals are good sets. Every large good diagonal has a very good subset, say $A_\gamma = \{(\gamma_\delta)^* | \delta \in \Gamma_\gamma \subseteq \Gamma\}$. Define the comb

$$\mathcal{A}_\gamma = \{A_\gamma; (A_\gamma)_i^{(\gamma_\delta)^*} | i = 1, ..., N+1, \delta \in \Gamma_\gamma\},$$

where

$$(A_\gamma)_i^{(\gamma_\delta)^*} \stackrel{\text{def}}{=} \begin{cases} (A_{\gamma_\delta})_i^{(\gamma_\delta)^*} & \text{if } i = 1, ..., N, \\ A_{\gamma_\delta} & \text{if } i = N+1. \end{cases}$$

These are pairwise disjoint very good combs of degree $N+1$ completing the proof of the lemma. $\square$

Now we return to the actual proof of our main result.

Take a comb $\mathcal{A} = \{A; A_i^\alpha | i = 1, ..., N, \alpha \in A\}$ of degree N, where $N \geq K$. Choose a point $\alpha \in A$ arbitrarily and consider $f \stackrel{\text{def}}{=} \sum_{i=1}^N \chi_{A_i^\alpha}$. Then we have

$$f(x) = \begin{cases} N & \text{if } x = \alpha, \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

Let

$$m(x) = \begin{cases} -N & \text{if } x = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $m \in \mathcal{F}$, and $\| f + m \| \leq 1$. On the other hand, since the sets A_i^α are all very good, we have $\widetilde{f}(\alpha) = 0$. Therefore $(f - \widetilde{f})(\alpha) = N$. Referring to Lemma 1 we obtain

$$N \leq \| f - \widetilde{f} \| \leq (K-1) \| f + m \| \leq K - 1 < N.$$

This contradiction proves the Theorem.

References

[AR] S. A. Argyros, *On the space of bounded measurable functions*, Quart. J. Math. Oxford (2), **34** (1983), 129–132.