Real Analysis Exchange
Vol. 23(1), 1998-99, pp. 161–174

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RESTRICTIONS TO CONTINUOUS AND POINTWISE DISCONTINUOUS FUNCTIONS

Abstract

We compare some of the restriction properties that can be found throughout the literature. For example, theorem 10 is a common generalization of three theorems: Blumberg's theorem [2], Baldwin's strengthening of Blumberg's theorem [1], and a related Brown-Prikry's result [8] on Marczewski's (s)-measurable functions.

1 Introduction

In 1922 Blumberg [2] proved that for every function $f : \mathbb{R} \to \mathbb{R}$ there exists a dense set $X \subseteq \mathbb{R}$, such that $f|_X$ is continuous. Since then many similar results involving domains and codomains other than $\mathbb{R}$ were obtained. Also many papers can be found, where "continuous" was changed to "differentiable" or "pointwise discontinuous" (i.e., $f : X \to \mathbb{R}$ is *pointwise discontinuous* (abbreviated PWD) if $\{x \in X : f \text{ is continuous at } x\}$ is dense in X, see [10] p.105). For a recent comprehensive account of these results see [6]. In this note we would like to compare some restriction properties of real functions defined on separable metric spaces. $\mathbb{R}$ is the set of all real numbers and $\mathbb{Q}$ is the set of rationals. For a set S and a cardinal κ, $[S]^\kappa = \{S' \subseteq S : |S'| = \kappa\}$. If $\mathcal{F} \subseteq \mathcal{P}(S)$ and $S' \subseteq S$, then $\mathcal{F}|_{S'} = \{F \cap S' : F \in \mathcal{F}\}$. If $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{P}(S)$, then $\mathcal{F}_1 \bigtriangleup \mathcal{F}_2 = \{F_1 \bigtriangleup F_2 : F_i \in \mathcal{F}_i, \text{ for } i = 1, 2\}$. Unless stated otherwise, X will always denote an uncountable, separable metric space, $\mathcal{J}$ will be a proper σ-ideal on X, and $\mathcal{A}$ will be a σ- algebra of subsets of X.

Our goal is to show that given a space X, σ-algebra $\mathcal{A}$, and a σ-ideal $\mathcal{J}$ then for every $\mathcal{A}$−measurable function $f : X \to \mathbb{R}$ there exists a "large"

Mathematical Reviews subject classification: Primary: 26A15, Secondary 26A03
Received by the editors January 2, 1997

*Most of this work was completed while the second author was a visiting professor at the University of Scranton.

set $W \subseteq X$ such that the restricted function $f|_W$ is continuous or pointwise discontinuous. The following six different notions of largeness associated with an ideal $\mathcal{J}$ can be found in restriction theorems stated in [6], [5], [1], [8], [14], and other papers. W is a subset of X.

$$
\begin{array}{ccccc}
 & & W \text{ is non-}\mathcal{J}\text{ dense in } X \ (\mathbf{D}) & & \\
 & \swarrow & & \searrow & \\
W \text{ is non-}\mathcal{J}\text{ dense in } W \ (\mathbf{DI}) & & & & \mathrm{cl}_X(W) \text{ is non-}\mathcal{J}\text{ dense in } X \ (\mathbf{WD}) \\
\downarrow & & \searrow & & \downarrow \\
W \notin \mathcal{J} \ (\mathbf{N}) & & & & \mathrm{cl}_X(W) \text{ is non-}\mathcal{J}\text{ dense in } \mathrm{cl}_X(W) \ (\mathbf{WDI}) \\
 & \searrow & & \swarrow & \\
 & & \mathrm{cl}_X(W) \notin \mathcal{J} \ (\mathbf{WN}) & &
\end{array}
$$

W is *non-$\mathcal{J}$ dense* in X if $W \cap U \notin \mathcal{J}$ for every nonempty open subset $U \subseteq X$. $\mathrm{cl}_X(W)$ stands for the closure of W in X. We shall refer to these properties using the bold abbreviations in parenthesis. Here is the key: **D**=non-$\mathcal{J}$-**D**ense, **DI=**non-$\mathcal{J}$-**D**ense in **I** tself, **N**=**N**ot in $\mathcal{J}$, **WN**=**W** eakly **N**ot in $\mathcal{J}$ (i.e., not in $\mathcal{J}$ after taking the closure of W), etc. In general all six are different classes of sets and the above diagram indicates all inclusions.

If $\mathfrak{L}$ is one of those properties (i.e. **D**, **DI**, ..., **WN**), we define a Continuous Restriction Property (C-RP) or a PointWise Discontinuous Restriction Property (PWD-RP) related to $\mathfrak{L}$. Namely, a function $f : X \to \mathbb{R}$ has a $\mathfrak{L}$-C-RP [resp. $\mathfrak{L}$-PWD-RP] if there exists a set $W \in \mathfrak{L}$ such that $f|_W$ is continuous [resp. PWD]. We shall say that a pair $(\mathcal{A}, \mathcal{J})$ has a $\mathfrak{L}$-C-RP [resp. $\mathfrak{L}$-PWD-RP] if every $\mathcal{A}$ -measurable function $f : X \to \mathbb{R}$ has the same property. $(\mathcal{A}, \mathcal{J})$ has $\mathcal{A}$-$\mathfrak{L}$-C-RP [resp. $\mathcal{A}$-$\mathfrak{L}$-PWD-RP] if the witness set W can be found in $\mathcal{A}$.

Let $\mathcal{B}(X)$ be the family of all Borel subsets of X and let $\mathcal{BR}(X)$ be the family of all sets with Baire property while $\mathcal{M}(X)$ is the ideal of all subsets of X meager in X. So for subsets $X_1 \subseteq X$, $\mathcal{M}(X_1)$ is the family of all relatively meager subsets of X_1. We have $\mathcal{M}(X_1) \subseteq \mathcal{M}(X)|_{X_1}$. For $X \subseteq \mathbb{R}$ let $\mathcal{L}(X)$ and $\mathcal{N}(X)$ be the Lebesgue measurable and null subsets of X. Classic theorems imply that $(\mathcal{BR}(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ has $\mathcal{BR}(\mathbb{R})$-**D**-C-RP, while the $(\mathcal{L}(\mathbb{R}), \mathcal{N}(\mathbb{R}))$ only has $\mathcal{L}(\mathbb{R})$-**DI-**C-RP. (See [8] for more details.)

2 Continuous Restrictions

For an arbitrary pair $(\mathcal{A}, \mathcal{J})$ on a separable metric space X we have the following implications.

$$\begin{array}{ccccc} & & \textbf{D}\text{-C-RP} & & \\ & \swarrow & & \searrow & \\ \textbf{DI}\text{-C-RP} & & & & \textbf{WD}\text{-C-RP} \\ \updownarrow (\star) & & \searrow & & \downarrow \\ \textbf{N}\text{-C-RP} & & & & \textbf{WDI}\text{-C-RP} \\ & \searrow & & \swarrow & \\ & & \textbf{WN}\text{-C-RP} & & \end{array}$$

Examples of pairs $(\mathcal{A}, \mathcal{J})$ indicating that, except for $(\star)$, none of these implications may be reversed, can be easily found.

2.1 $\mathcal{A} = \mathcal{P}(X)$

In 1923 W. Sierpinski and A. Zygmund [17] proved that whenever $|X| = \mathfrak{c}$, then there exists a function $z : X \to \mathbf{R}$ such that $z|_Y$ is not continuous for any $Y \in [X]^{\mathfrak{c}}$. This implies that under CH $(\mathcal{P}(X), \mathcal{J})$ can not have **N**-C-RP for any σ-ideal $\mathcal{J}$ containing all singletons. Without CH however $(\mathcal{P}(\mathbb{R}), [\mathbb{R}]^{\leq\omega})$ as well as $(\mathcal{P}(\mathbb{R}), \mathcal{M}(X))$ may have **D**-C-RP. (See [1], [15], and Theorem 2 below.) In ZFC Bradford and Goffman [3] proved that whenever an ideal $\mathcal{J}$ does not contain open sets, then $(\mathcal{P}(X),\ \mathcal{J})$ has **WD**-C-RP iff X has no meager open subsets. In general we have the following theorem.

Theorem 1. *$(\mathcal{P}(X), \mathcal{J})$ has **WDI**-C-RP.*

Proof. Let $f : X \to \mathbb{R}$ and suppose that $(\mathcal{P}(X), \mathcal{J})$ does not have the **WDI**-C-RP. By Brown's theorem 2, [5] p.132, we may assume that there exists a subset $X_1 \subseteq X$, $X_1 \notin \mathcal{J}$ such that $\mathcal{M}(X_1) \subseteq \mathcal{J}|_{X_1}$. Take $X_1' = X_1 \setminus \bigcup\{V \subseteq X_1 : V$ is open in X_1 and $V \in \mathcal{J}\}$. We have $\mathcal{M}(X_1') \subseteq \mathcal{M}(X_1) \subseteq \mathcal{J}$ and the last does not contain open subsets of X_1'. Hence we may apply the above mentioned Bradford-Goffman theorem, [3] p. 667, to X_1' and obtain a dense subset $W \subseteq X_1'$, such that $f|_W$ is continuous. Clearly $\mathrm{cl}_X(W) \supseteq X_1'$ and whenever U is open in X, $U \cap \mathrm{cl}_X(W) \neq \emptyset$, then $U \cap X_1' \notin \mathcal{J}$ by the definition of X_1'. □

It is known (see [5], p. 128) that for uncountable separable metric spaces X and any $f : X \to \mathbb{R}$ there exists a set $W \subseteq X$ such that $f|_W$ is continuous and

$|W \cap U| \geq \omega$ for every nonempty open set U. Observe that by taking $\mathcal{J} = [X]^{\leq\omega}$ in Theorem 1 above we obtain proposition (C) of [5] and additional property that $\mathrm{cl}_X(W)$ is uncountably dense in itself.

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals on a set X and $Y \subseteq X$, then we say that $\mathcal{J}_1$ *is orthogonal to* $\mathcal{J}_2$ *on* Y if $Y = Y_1 \cup Y_2$ where $Y_i \in \mathcal{J}_i$, $i = 1, 2$. We write "$\mathcal{J}_1 \perp \mathcal{J}_2$ on Y". Let us consider the following property of a space X and an ideal $\mathcal{J}$:

$$X = X_1 \cup X_2 \text{ where } X_1 \in \mathcal{M}(X) \text{ and } \mathcal{M}(X_2) \subseteq \mathcal{J}. \tag{1}$$

It follows from Theorem 1 of [5] that if open subsets of X do **not** have property (1), then $(\mathcal{P}(X), \mathcal{J})$ has **D**-PWD-RP. In this context the following theorem is somewhat surprising.

Theorem 2. *Suppose that X and $\mathcal{J}$ satisfy (1) and that $\mathcal{J} \not\perp \mathcal{M}(X)$ on any open set. Let $f : X \to \mathbb{R}$ be such that for every Borel set $B \in \mathcal{B}(X) \setminus \mathcal{J}$ the restricted function $f|_B$ has **N**-C-RP with respect to $\mathcal{J}|_B$. Then f has **D**-C-RP with respect to $\mathcal{J}$.*

PROOF. Let $X = X_1 \cup X_2$ be a partition as in (1). By enlarging X_1 to a Borel meager set we may assume that $X_1, X_2 \in \mathcal{B}(X)$. Let $\mathcal{U} = (U_n)_{n<\omega}$ be an open basis for X_2. Non-orthogonality of $\mathcal{J}$ and $\mathcal{M}(X)$ on open sets implies that $U_n \notin \mathcal{J}$. Since U_n is Borel in X, by the **N**-C-RP of $f|_{U_n}$ we obtain sets $A_n \subseteq U_n$, $A_n \notin \mathcal{J}$ such that $f|_{A_n}$ is continuous. Let $T_n = \{x \in U_n : \exists E_{\text{open}} \subset U_n (x \in E \text{ and } A_n \cap E \in \mathcal{J})\}$. X is separable so $T_n \cap A_n \in \mathcal{J}$ and $\mathrm{cl}_{U_n}(T_n) \setminus T_n \in \mathcal{M}(X_2) \subseteq \mathcal{J}$. Furthermore, since X_1 is meager, $\mathcal{J}$ and $\mathcal{M}(X)$ are non-orthogonal on U_n. Take $V_n = U_n \setminus \mathrm{cl}_{U_n}(T_n)$ and observe that $C_n = A_n \cap V_n$ is nonempty and non-$\mathcal{J}$ dense in V_n for all $n < \omega$.

Now let $W_n = C_n \setminus \bigcup_{k<n} \mathrm{cl}_X(V_k)$ and $W = \bigcup_{n<\omega} W_n$. Notice that $W_n = (V_n \cap W) \setminus \bigcup_{k<n} \mathrm{cl}_X(V_k)$. Hence W_n are open in W. $f|_{W_n}$ is continuous for all $n < \omega$ which implies that $f|_W$ is also continuous.

To see that W is non-$\mathcal{J}$ dense in X take an arbitrary nonempty open set $T \subseteq X$. Since X_2 is residual in X, $T_2 = X_2 \cap T$ contains some U_k. Let $k_0 = \min\{k : V_k \cap T_2 \neq \emptyset\}$. We clearly have $C_{k_0} \cap T_2 \notin \mathcal{J}$ but also $W_{k_0} \cap T_2 \notin \mathcal{J}$ as all sets of the form $\mathrm{cl}_X(V_k) \setminus V_k$ are nowhere dense in X_2 and are in $\mathcal{J}$ by (1). Naturally $W \cap T \notin \mathcal{J}$. □

For separable spaces Shelah's theorem 1.4, [15], p. 8, gives the following:

Theorem 3. *(Shelah [15]) It is relatively consistent with ZFC that for every function $f : 2^\omega \to 2^\omega$ there exists a non-meager subset of $A \subseteq 2^\omega$ such that $f|_A$ is continuous.*

Suppose that X is a complete space. Shelah's theorem 3 implies that whenever $B \notin \mathcal{M}(X)$ is a Borel subset of X,, then there exists a set $A \in$

$\mathcal{P}(B)\backslash\mathcal{M}(B)$ such that $f|_A$ is continuous. Theorems 3 and 2 yield the following fact.

Corollary 4. *It is consistent that for any complete space (or a Borel subset of a complete space without open meager sets) X the pair $(\mathcal{P}(X), \mathcal{M}(X))$ has* ***D****-C-RP.*

Remark 1. It is worth noting that ideals which are ccc in Borel sets have property (1). Suppose that $\mathcal{J}$ is any ccc in Borel sets ideal on X (i.e. $\mathcal{B}(X)\backslash\mathcal{J}$ does not contain uncountable pairwise disjoint subfamilies) and suppose that $\mathcal{M}(X) \nsubseteq \mathcal{J}$. Let $X_1^0 \in \mathcal{M}(X) \setminus \mathcal{J}$ be Borel. For an ordinal α try to find a set $X_1^\alpha \in \mathcal{B}(X \setminus \bigcup_{\beta<\alpha} X_1^\beta) \cap (\mathcal{M}(X) \setminus \mathcal{J})$. By the ccc property this attempt must fail after $\alpha_0 < \omega_1$ steps. Sets $X_1 = \bigcup_{\alpha<\alpha_0} X_1^\alpha$ and $X_2 = X \setminus X_1$ have the desired properties.

Corollary 4 shows that CH can not be eliminated from Theorem 1 of [5]. Namely in Shelah's model $(\mathcal{P}(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ has **D**-C-RP (in particular it has **D**-PWD-RP) and $\mathbb{R}$ does not satisfy condition (B') of [5] with property P= $\mathcal{M}(\mathbb{R})$.

2.2 $\mathcal{A}$-Measurable Functions

Now we would like to prove a theorem similar to 2 without assuming (1). To compensate for that we are going to work with $\mathcal{A}$-measurable functions and assume $\mathcal{A}$-**N**-C-RP of $f|_A$ for all $A \in \mathcal{A} \setminus \mathcal{J}$ i.e., assume that there exists a set $B \in \mathcal{A}|_A \setminus \mathcal{J}$ such that $f|_B$ is continuous. Following Bradford and Goffman [3] (see also [13]) we introduce the following definitions: Let $E \subseteq X$, and let $x \in X$. Then *x is non-$\mathcal{J}$ relative to E* if for every open $V \ni x$ we have $E \cap V \notin \mathcal{J}$. *$x$ is $\mathcal{J}$-heavy relative to E* if there exists an open set $U \ni x$ such that all $y \in U$ are non-$\mathcal{J}$ relative to E. The first two lemmas are straight forward generalizations of Lemmas 2 and 3 of [3].

Lemma 5. *Any subset $E \subseteq X$ can be written as a disjoint union of sets $E = A \cup B_1 \cup B_2$ such that all members of A are $\mathcal{J}$-heavy relative to E, $B_1 \in \mathcal{J}$, and B_2 is nowhere dense in X.*

Proof. Let us define $B_1 = \{x \in E : \exists U_{open} \subseteq X \quad x \in U \ \& \ (U \cap E) \in \mathcal{J}\}$. X is separable. Hence $B_1 \in \mathcal{J}$. Now let $B_2 = \{x \in E : x$ is non-$\mathcal{J}$ but not $\mathcal{J}$-heavy relative to $E\}$. Take an arbitrary open set $T \subseteq X$ and let $x \in B_2 \cap T$. Since x is not $\mathcal{J}$-heavy, there exists $y \in T$ which is not non-$\mathcal{J}$ relative to E. So there exists an open neighborhood V of y such that $E \cap V \in \mathcal{J}$. We must have $V \cap B_2 = \emptyset$, which shows that B_2 is nowhere dense. Clearly points of E that are not in B_1 nor B_2 are $\mathcal{J}$-heavy. □

For $f : X \to \mathbb{R}$ we define

$$H_f(X, \mathcal{J}) = \{x \in X : \forall K_{\text{open}} \ni f(x)(x \text{ is } \mathcal{J}\text{-heavy relative to } f^{-1}(K))\} \quad (2)$$

Properties of $H_f(X, \mathcal{J})$ were studied by Piotrowski [13] in a more general context.

Lemma 6. *Let $f : X \to \mathbb{R}$. There exist sets $B_1 \in \mathcal{J}$ and $B_2 \in \mathcal{M}(X)$ such that $H_f(X, \mathcal{J}) = X \setminus (B_1 \cup B_2)$.*

PROOF. Let $(G_n)_{n<\omega}$ be an open basis in $\mathbb{R}$ and let $S_n = f^{-1}(G_n) = A^n \cup B_1^n \cup B_2^n$ where the last union is like in lemma 5. Take $B_1 = \bigcup_{n<\omega} B_1^n$ and $B_2 = \bigcup_{n<\omega} B_2^n$. Now select an arbitrary $x \in X \setminus (B_1 \cup B_2)$ and an open set $K \ni f(x)$. Find $n < \omega$ such that $G_n \subseteq K$ and $f(x) \in G_n$. x is $\mathcal{J}$-heavy relative to S_n so in particular it is $\mathcal{J}$-heavy relative to the bigger set $f^{-1}(K)$. For the other inclusion take $K = G_n$, $n < \omega$ and it follows immediately. □

For any ideal $\mathcal{J}$ on a metric space X we define $\mathcal{J}^*$ to be the σ-ideal generated by $\mathcal{J}$ and $\mathcal{M}(X)$. The next lemma is easy to verify.

Lemma 7. *Let Z be a separable metric space and let $\mathcal{J}$ be an ideal on Z with $\mathcal{J} \not\perp \mathcal{M}(Z)$ on any open set. If $U \subseteq Z$ is open and $V \subseteq U$ is non-$\mathcal{J}^*$ dense in U,, then $\mathcal{J} \not\perp \mathcal{M}(V)$ on any open subset of V.*

Lemma 8. *Let Z be a zero-dimensional separable metric space. Assume that $\mathcal{J}$ is a σ-ideal and $\mathcal{A} \supseteq \mathcal{J} \cup \mathcal{B}(Z)$ is a σ-algebra on Z. Suppose that an $\mathcal{A}$ -measurable function $f : Z \to \mathbb{R}$ and $Y \in \mathcal{A}$ are such that*

1) $\mathcal{J} \not\perp \mathcal{M}(Z)$ on any open subset of Z

2) Y is non-$\mathcal{J}$ dense in itself

3) $Y \subseteq H_f(Z, \mathcal{J}^*)$

4) $f|_Y$ is continuous

5) $\forall A \in \mathcal{A} \setminus \mathcal{J} \exists B \in \mathcal{A}|_A \setminus \mathcal{J}$ $f|_B$ is continuous.

If $\varepsilon > 0$, then there exist pairwise disjoint open subsets $\mathcal{U} = (U_n)_{n<\omega}$ of Z and subsets $Y_n \subseteq V_n \subseteq U_n$, $Y_n \in \mathcal{A}$ such that

6) $diam(U_n) < \varepsilon$

7) $\bigcup \mathcal{U}$ is dense in Z

8) V_n are non-$\mathcal{J}^*$ dense in U_n

9) Y_n *are non-*$\mathcal{J}$ *dense in itself*

10) $f|_{Y_n}$ *is continuous*

11) $Y \subseteq \bigcup_{n<\omega} Y_n$

12) $Y_n \subseteq H_f(V_n, \mathcal{J}^*)$

13) $|f(x) - f(x')| < \varepsilon$ *whenever* $x, x' \in V_n$ *for some* $n < \omega$.

Proof. We shall first define the even numbered sets U_n, V_n, and Y_n to satisfy condition 11) and then define the odd numbered ones to satisfy 7). Select $y \in Y$. By 3) there exists an clopen neighborhood U'_y of y with $\mathrm{diam}(U'_y) < \varepsilon$ such that

$$f^{-1}\Big(\big(f(y) - \frac{\varepsilon}{2}, f(y) + \frac{\varepsilon}{2}\big)\Big) \text{ is non-}\mathcal{J}^* \text{ dense in } U'_y. \tag{3}$$

4) implies the existence of a clopen set $U''_y \ni y$ such that $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $x, x' \in Y \cap U''_y$. Let $U_y = U'_y \cap U''_y$ and observe that

$$Y \cap U_y \subseteq f^{-1}((f(y) - \frac{\varepsilon}{2}, f(y) + \frac{\varepsilon}{2})) \tag{4}$$

Then $(U_y)_{y \in Y}$ is a clopen cover of Y. There is a countable set $\{y_n : n < \omega\} \subseteq Y$ such that $(U_{y_n})_{n<\omega}$ is a subcover of Y. Set $G_n = U_{y_n} \setminus \bigcup_{k<n} U_k$. Then $(G_n)_{n<\omega}$ is a disjoint open cover of Y and by possibly deleting some sets we may assure that $G_n \cap Y \neq \emptyset$ for all $n < \omega$. For each $n < \omega$ we put $U_{2n} = G_n$, $V_{2n} = U_{2n} \cap f^{-1}((f(y_{2n}) - \frac{\varepsilon}{2}, f(y_{2n}) + \frac{\varepsilon}{2}))$, and $Y_{2n} = U_{2n} \cap Y$.

Assumption 2) implies that Y_{2n} is non-$\mathcal{J}$ dense in itself. 4) gives continuity of $f|_{Y_{2n}}$. Inclusion (4) shows that

$$Y_{2n} \subseteq V_{2n} \tag{5}$$

and condition (3) implies that V_{2n} is non-$\mathcal{J}^*$ dense in U_{2n}. Since $(G_n)_{n<\omega}$ was a cover of Y, the union $\bigcup_{n<\omega} Y_{2n} = Y$. 13) follows from the definition of V_{2n}. Hence we have verified all conditions except 7) and 12). While 7) will be taken care of by the odd U_n-s, 12) for even indices follows from the following claim

Claim: $Y_{2n} \subseteq H_f(V_{2n}, \mathcal{J}^*)$ for all $n < \omega$.

Let $x \in Y_{2n}$ and let $K \ni f(x)$ be an open subset of $\mathbb{R}$. Take $K_1 = K \cap (f(y_{2n}) - \frac{\varepsilon}{2}, f(y_{2n}) + \frac{\varepsilon}{2})$. By (5) $f(x) \in (f(y_{2n}) - \frac{\varepsilon}{2}, f(y_{2n}) + \frac{\varepsilon}{2})$; so $f(x) \in K_1$. By the assumption 3) there exists an open subset $U \subseteq Z$ such that $f^{-1}(K_1) \cap U$ is non-$\mathcal{J}^*$ dense in U. It follows that $f^{-1}(K_1) \cap U \cap G_n$ is non-$\mathcal{J}^*$ dense in

$\widehat{U} = U \cap G_n$. But since $f^{-1}(K_1) \cap G_n$ is a subset of V_{2n} we obtain that $f^{-1}(K) \cap V_{2n} \cap \widehat{U}$ is non-$\mathcal{J}^*$ dense in $\widehat{U}$. Thus $x \in H_f(V_{2n}, \mathcal{J}^*)$.

To define the odd $U_n \supseteq V_n \supseteq Y_n$ we proceed as in lemma 4 of [3]. Let $R = \{z_\alpha : \alpha < \kappa\}$ for some $\kappa \leq \mathfrak{c}$ be a well-ordering of $H_f(Z \setminus \mathrm{cl}_z(\bigcup_{n<\omega} U_{2n}), \mathcal{J}^*)$. Orthogonality of $\mathcal{J}$ and $\mathcal{M}(Z)$ on every open set and lemma 6 imply that R is dense in $Z \setminus \mathrm{cl}_z(\bigcup_{n<\omega} U_{2n})$. Suppose that we have defined sets $U'_\alpha \supseteq V'_\alpha \supseteq Y'_\alpha$ for all $\alpha < \beta < \omega_1$. Let z_{β_0} be the first element of $R \cap Z \setminus \mathrm{cl}_z(\bigcup_{n<\omega} U_{2n} \cup \bigcup_{\alpha<\beta} U'_\alpha)$. Let U'_β be a neighborhood of z_{β_0} disjoint from $\mathrm{cl}_z(\bigcup_{n<\omega} U_{2n} \cup \bigcup_{\alpha<\beta} U'_\alpha)$, of diameter less than ε, and such that $V'_\beta = f^{-1}(f(z_{\beta_0}) - \frac{\varepsilon}{2}, f(z_{\beta_0}) + \frac{\varepsilon}{2}) \cap U'_\beta$ is non-$\mathcal{J}^*$ dense in U'_β. It follows from Lemma 6 that $H_f(V'_\beta, \mathcal{J}^*)$ contains a subset $A \in \mathcal{A} \setminus \mathcal{J}$. By assumption 5), there exists a non-$\mathcal{J}$ dense in itself subset $Y'_\beta \in \mathcal{A}|_A \setminus \mathcal{J}$ such that $f|_{Y'_\beta}$ is continuous.

After countably many steps, say $\gamma < \omega_1$, the choice of z_{γ_0} will no longer be possible and this is when $\bigcup_{\alpha<\gamma} U'_\alpha$ will becomes dense in $Z \setminus \mathrm{cl}_z(\bigcup_{n<\omega} U_{2n})$. It suffices to renumerate sets U'_α, V'_α, and Y'_α, $\alpha < \gamma$ as U_{2n+1}, V_{2n+1}, and Y_{2n+1}, $n < \omega$. □

Lemma 9. *If $\mathcal{A}$ contains all Borel subsets of X and an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}$ has **N**-C-RP with respect to $\mathcal{J}$, then it also has $\mathcal{A}$-**N**-C-RP with respect to the same ideal.*

PROOF. Suppose that $W \in \mathcal{P}(X) \setminus \mathcal{J}$ is such that $f|_W$ is continuous. There exists a G_δ subset $G \subseteq X$ and a continuous function $g : G \to \mathbb{R}$ such that $f|_W \subseteq g$, see [9]. Since the difference $f|_G - g$ is also $\mathcal{A}$ -measurable, the set $W_1 = (f|_G - g)^{-1}(\{0\})$ is in $\mathcal{A} \setminus \mathcal{J}$. Clearly $f|_{W_1}$ is continuous. □

Theorem 10. *Let X be a separable metric space and let $\mathcal{J}$ a σ-ideal on X, $\mathcal{J} \not\perp \mathcal{M}(X)$ on every open set. Suppose that $\mathcal{A} \supseteq \mathcal{J} \cup \mathcal{B}(X)$ is a σ-algebra on X. If $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable and $f|_A$ has **N**-C-RP whenever $A \in \mathcal{A} \setminus \mathcal{J}$, then f has $\mathcal{A}$- **D**-C-RP.*

PROOF. Without loss of generality we may assume that X is zero-dimensional. (It suffices to remove a meager set to assure that.) Let $\Delta = \omega^{<\omega} \setminus \{\emptyset\}$. We construct three trees of subsets of X : $(U_\tau)_{\tau\in\Delta}$, $(V_\tau)_{\tau\in\Delta}$, and $(Y_\tau)_{\tau\in\Delta}$.

Claim. There exists an $\mathcal{A}$-measurable non-$\mathcal{J}$ dense in itself subset $Y \subseteq H_f(X, \mathcal{J}^*)$ such that $f|_Y$ is continuous.

$\mathcal{A} \supseteq \mathcal{J} \cup \mathcal{B}(X)$ and Lemma 6 applied to $\mathcal{J}^*$ imply that $H_f(X, \mathcal{J}^*) \in \mathcal{A} \setminus \mathcal{J}^*$. $f|_{H_f(X,\mathcal{J}^*)}$ has **N**-C-RP so there exists a set $Y' \subseteq H_f(X, \mathcal{J}^*)$, $Y' \notin \mathcal{J}$ such that $f|_{Y'}$ is continuous. Since $\mathcal{A} \supseteq \mathcal{B}(X)$, Lemma 9 may be used to extend Y' to a subset of $H_f(X, \mathcal{J}^*)$, $Y'' \in \mathcal{A}$ such that $f|_{Y''}$ is still continuous. Define $Y = Y'' \setminus \{x \in Y'' : \exists E \subseteq_{\text{open}} X (x \in E \text{ and } E \cap Y'' \in \mathcal{J})\}$. $Y \in \mathcal{A}$ is nonempty because X is separable and non-$\mathcal{J}$ dense in itself.

To obtain the first level of the three trees apply Lemma 8 with $Z = X$, $Y = Y$, and $\varepsilon = 1$. Now let $k > 0$. To obtain sets $(U_{\tau\hat{}n})_{\tau\in\omega^k,n\in\omega}$, $(V_{\tau\hat{}n})_{\tau\in\omega^k,n\in\omega}$, and sets $(Y_{\tau\hat{}n})_{\tau\in\omega^k,n\in\omega}$, from level $k+1$ we apply Lemma 8 for each $\tau \in \omega^k$ with $Z = V_\tau$, $Y = Y_\tau$ and $\varepsilon = \frac{1}{k+1}$. Then simply put $U_{\tau\hat{}n} = U_n$, $V_{\tau\hat{}n} = V_n$, and $Y_{\tau\hat{}n} = Y_n$, where U_n, V_n, and Y_n are from the Lemma and satisfy conditions 6)-13). Observe that the assumption 1) is preserved from one step to another due to Lemma 7.

Now let $W = \bigcup_{k\in\omega}\bigcup_{\tau\in\omega^k} Y_\tau$. It is easy to see that for every $k \in \omega$ the union $\bigcup_{\tau\in\omega^k} U_\tau$ is dense in X. To show that W is non-$\mathcal{J}$ dense in X let T be a nonempty open subset on X. Due to decreasing diameters of U_τ there exists a $k \in \omega$ and a $\tau \in \omega^k$ such that $U_\tau \subseteq T$. This implies that $T \cap W \supseteq Y_\tau \notin \mathcal{J}$.

It suffices to verify that $f|_W$ is continuous. Let $x \in W$. For almost all $k \in \omega$ there exist sequences $\tau \in \omega^k$ such that $x \in Y_\tau$. $Y_\tau \subseteq V_\tau \cap W$ and $V_\tau \cap W$ is open in W with $\operatorname{diam}(f(V_\tau)) < \frac{1}{k}$. □

The following applications illustrate the strength of theorem 10.

Corollary 11. *(H. Blumberg [2]) If X is a Baire space, then $(\mathcal{P}(X), \{\emptyset\})$ has **D**-C-RP.*

Proof. Apply Theorem 10 with $\mathcal{A} = \mathcal{P}(X)$ and $\mathcal{J} = \{\emptyset\}$. □

Let $\omega \le \kappa < \mathfrak{c}$. It is well known (see [16]) that if $X \in [\mathbb{R}]^\kappa$ and $f : X \to \mathbb{R}$, then, under Martin's Axiom, there exists a set $Y \in [X]^\kappa$ such that $f|_Y$ is continuous. Theorem 10 gives the following.

Corollary 12. *(S. Baldwin [1]) Assume Martin's Axiom. Let $\omega < \kappa < \mathfrak{c}$, $cf(\kappa) > \omega$. Suppose that $X \subseteq \mathbb{R}$ contains no meager open subsets and $f : X \to \mathbb{R}$. Then $(\mathcal{P}(X), [X]^{<\kappa})$ has **D**-C-RP.*

Proof. Clearly, under Martin's Axiom $[X]^{<\kappa} \not\angle \mathcal{M}(X)$ on every open set. Apply theorem 10 with $\mathcal{A} = \mathcal{P}(X)$ and $\mathcal{J} = [X]^{<\kappa}$. □

A set $S \subseteq X$ is called (s)-*measurable* if for every perfect set $P \subseteq X$ there exists a perfect subset $P' \subseteq P$ such that either $P' \cap S = \emptyset$ or $P' \subseteq S$. (s_0) is the ideal of hereditarily (s)-measurable sets. It is well known (see Marczewski [11]) that whenever X is complete, then $f : X \to \mathbb{R}$ is (s)-measurable iff for every perfect set $P \subseteq X$ there exists a perfect subset $Q \subseteq P$ such that $f|_Q$ is continuous. It follows that f is (s)-measurable iff $f|_A$ has (s)-**N**-C-RP whenever $A \in (s) \setminus (s_0)$. The following corollary follows from the more general theorem 3 of [8]:

Corollary 13. *(Brown and Prikry [8]) If X is a complete space without isolated points, then $((s), (s_0))$ has (s)-**D**-C-RP.*

PROOF. It is well known that $(s_0) \measuredangle \mathcal{M}(X)$ on any open subset of X and that (s) contains all Borel subsets. Theorem 10 completes the proof. $\square$

3 PWD Restrictions

Now we would like to look at the diagram of pointwise discontinuous restriction properties.

D-PWD-RP

$\swarrow$ (1) $\searrow$ (2)

DI-PWD-RP **WD**-PWD-RP

$\downarrow$ (3) $\searrow$ (4) $\downarrow$ (5)

N-PWD-RP **WDI**-PWD-RP

$\searrow$ (6) $\swarrow$ (7)

WN-PWD-RP

Clearly for any class $\mathfrak{L}$ the $\mathfrak{L}$-C-RP implies the corresponding $\mathfrak{L}$-PWD-RP. In addition to that, the following properties are equivalent.

- **WD**-C-RP $\iff$ **WD**-PWD-RP
- **WDI**-C-RP $\iff$ **WDI**-PWD-RP
- **WN**-C-RP $\iff$ **WN**-PWD-RP

Hence, we are going to focus on the left side of the diagram. The original Blumberg's theorem [2] implies that $(\mathcal{P}(\mathbb{R}), \mathcal{J})$ has **WD**-C-RP for any ideal $\mathcal{J}$ without open sets. The following theorem shows that **WD**-C-RP$\not\Rightarrow$ **N**-PWD-RP ((4) can not be turned by $-90°$).

Theorem 14. *Let $\mathcal{J} = \{M \cup C : M \in \mathcal{M}(\mathbb{R})$ and $C \in [\mathbf{R}]^{<\mathfrak{c}}\}$. Then $(\mathcal{P}(\mathbb{R}), \mathcal{J})$ does not have the **N**-PWD-RP.*

PROOF. Let $z : \mathbb{R} \to \mathbb{R}$ be the Zygmund-Sierpinski function [17]. Let $A \subseteq \mathbb{R}$ and suppose that $G = \{x \in A : z|_A$ is continuous at $x\}$ is dense in A. Since G is a relative G_δ subset of A, $A \setminus G \in \mathcal{M}(A) \subseteq \mathcal{J}$. z is the Zygmund-Sierpinski function; so $G \in [A]^{<\mathfrak{c}} \subseteq \mathcal{J}$. It follows that $A \in \mathcal{J}$.

Example 15. Implication (1) can not be reversed. Let $K \subseteq \mathbb{R}$ be a nowhere dense perfect set. Take $X = K \cup \mathbb{Q}$ and $\mathcal{J} = [X]^{\leq\omega}$. $X = \bigcup_{n<\omega} X_n$ where X_n are pairwise disjoint and nowhere dense in X. Assume $X_1 = K$. Let $f : X \to \mathbb{R}$, $f(x) = n$ for $x \in X_n$. Well known arguments (see [3] p. 667) shows that $(\mathcal{P}(X), \mathcal{J})$ does not have **D**-PWD-RP. On the other hand $(\mathcal{P}(K), [K]^{\leq\omega})$ has **D**-PWD-RP (see [4]); so $(\mathcal{P}(X), \mathcal{J})$ has **DI**-PWD-RP.

Remark 2. Assume that $\mathcal{J}$ contains all singletons. Under CH $(\mathcal{P}(X), \mathcal{J})$ has **DI**-PWD-RP iff it has **N**-PWD-RP. Suppose that $(\mathcal{P}(X), \mathcal{J})$ does not have **DI** -PWD-RP. By Brown's theorem 2 of [5] $X = \bigcup_{n<\omega} X_n$ where $\mathcal{M}(X_n) \subseteq \mathcal{J}$. Take $z : X \to \mathbf{R}$ to be the Zygmund-Sierpinski function on X. Suppose that $z|_A$ is pointwise discontinuous for some $A \in \mathcal{P}(X) \setminus \mathcal{J}$. There exists an $n < \omega$ such that $A \cap X_n \notin \mathcal{J}$. We can find a set B, $A \supseteq B \supseteq A \cap X_n$ such that $z|_B$ is PWD and $|B \setminus (A \cap X_n)| \leq \omega$. The set $G = \{x \in B : z|_B \text{ is continuous at } x\}$ is a dense G_δ in B. Hence $B \setminus G \in \mathcal{M}(B) \subseteq \mathcal{J}$. Since $B \notin \mathcal{J}$, $|G| > \omega$ which contradicts the Zygmund-Sierpinski property under CH.

4 Baire, Lebesgue, and Other Measurable Functions

If $\mathcal{A}$ is the Baire, Lebesgue, universally measurable, or other classic σ-algebra of sets, then restriction properties for $(\mathcal{A}, \mathcal{J}_\mathcal{A})$, where $\mathcal{J}_\mathcal{A}$ is the ideal of sets hereditarily in $\mathcal{A}$, are discussed in [8]. Here we look at restriction properties for $\mathcal{A}$ with arbitrary ideals $\mathcal{J}$ other than $\mathcal{J}_\mathcal{A}$.

If X is meager on itself, then $\mathcal{BR}(X) = \mathcal{P}(X)$ and this case has been discussed above. It follows from a well known theorem of Nikodym [12] that if X does not contain nonempty meager open subsets, then $(\mathcal{BR}(X), \mathcal{M}(X))$ has $\mathcal{BR}(X)$-**D**-C-RP. Using the same technique we can show that $(\mathcal{BR}(X) \triangle \mathcal{J}, \mathcal{J}^*)$ has $(\mathcal{BR}(X) \triangle \mathcal{J})$-**D**-C-RP as long as $\mathcal{J} \not\perp \mathcal{M}(X)$ on every open set. It remains to examine pairs $(\mathcal{BR}(X), \mathcal{J})$ where $\mathcal{J} \perp \mathcal{M}(X)$. In such case there exists a nowhere dense set $F \notin \mathcal{J}$. It is easy to find a discrete set $D \subseteq X \setminus F$, such that $\mathrm{cl}_X(D) \supseteq F$. This last observation may be applied in a more general situation and yields the following facts.

Proposition 16. *If $\mathcal{M}(X) \nsubseteq \mathcal{J}$, then the pair $(\mathcal{P}(X), \mathcal{J})$ has $\mathcal{BR}(X)$-**N**-PWD-RP.*

Corollary 17. *$(\mathcal{BR}(X), \mathcal{J})$ has **N**-PWD-RP for any σ-ideal $\mathcal{J}$.*

In general **N**-PWD-RP is the best restriction property that we can hope for $(\mathcal{BR}(X), \mathcal{J})$ where $\mathcal{J} \perp \mathcal{M}(X)$.

Example 18. Let $X = \mathbb{R} = P \dot\cup S$ where P is some nowhere dense perfect set. Let $\mathcal{MC}(P) = \{P' \cup P'' : P' \in \mathcal{M}(P) \text{ and } P'' \in [P]^{<\mathfrak{c}}\}$. Define $\mathcal{J} = \{P' \cup S' :$

$P' \in \mathcal{MC}(P)$ and $S' \subseteq S\}$. Clearly $\mathcal{M}(X) \perp \mathcal{J}$ on X. $(\mathcal{BR}(X), \mathcal{J})$ does not have **DI**-PWD-RP because of the following $\mathcal{BR}(X)$-measurable function

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in S \\ z(x) & \text{if} \quad x \in P, \end{cases}$$

where z is the Zygmund-Sierpinski function on P. If W was a non-$\mathcal{J}$ dense in itself, then $W \subseteq P$. If $f|_W$ was PWD, then the set $G = \{x \in W : f|_W$ is continuous at $x\}$ is a dense G_δ subset of W so $W \setminus G \in \mathcal{M}(P)$. This implies that $G \notin \mathcal{J}$ and in particular $|G| = \mathfrak{c}$, but that contradicts the Zygmund-Sierpinski property.

From Proposition 16 we easily obtain

Corollary 19. *If X has positive outer measure, then $(\mathcal{P}(X), \mathcal{N}(X))$ has **N**-PWD-RP.*

Here also no stronger restriction property is provable due to Example 18. Corollary 19 also follows from Theorem E of [7] on points of differentiability. More couterexamples for other σ-algebras follow from the next theorem.

Theorem 20. *Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$ and assume that there exists a set $X \notin \mathcal{MC} = \{M \cup C : M \in \mathcal{M}(\mathbb{R})$ and $C \in [\mathbb{R}]^{<\mathfrak{c}}\}$ such that $\mathcal{P}(X) \subseteq \mathcal{A}$. If we define $\mathcal{MC}_X = \{A \subseteq \mathbb{R} : A \cap X \in \mathcal{MC}\}$, then $(\mathcal{A}, \mathcal{MC}_X)$ does not have the **N**-PWD-RP.*

Proof. Follow Example 18 with $P = X$. □

Corollary 21. *There exists a σ-ideal $\mathcal{J}$ on $\mathbb{R}$ such that $(\mathcal{L}(\mathbb{R}), \mathcal{J})$ does not have **N**-PWD-RP.*

Proof. Use Theorem 20 with X being a second category measure zero set. □

Corollary 22. *Assume CH. If $\mathcal{A}$ is one of the following σ-algebras: (s) - measurable, universally measurable, or $\mathcal{B}(\mathbb{R}) \triangle \mathcal{UN}(\mathbb{R}) = \{B \triangle N : B \in \mathcal{B}(X)$ and N is universally null$\}$, then there exists a σ-ideal $\mathcal{J}$ such that $(\mathcal{A}, \mathcal{J})$ does not have the **N**-PWD-RP.*

Proof. Use Theorem 20 with X being a Lusin set. □

Remark 3. In the random real model $(\mathcal{B}(\mathbb{R}) \triangle \mathcal{UN}(\mathbb{R}), \mathcal{J})$ has **N**-PWD-RP for all $\mathcal{J}$. Recall that in this model $\mathcal{UN}(\mathbb{R}) \subseteq [\mathbb{R}]^{\leq \omega_1} \subseteq \mathcal{M}(\mathbb{R})$. It follows that $\mathcal{B}(\mathbb{R}) \triangle \mathcal{UN}(\mathbb{R}) \subseteq \mathcal{BR}(\mathbb{R})$ and Corollary 17 implies **N**-PWD-RP.

The authors are very grateful to Zbyszek Piotrowski for interesting discussions. We also would like to thank Jack Brown. His comments eliminated several errors and clarified some of our proofs.

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