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CONVERGENCE AND APPROXIMATE DIFFERENTIATION

Abstract

The main result of this paper is Theorem 1, which states the following: Let $F, F_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \dots$ be Lebesgue measurable functions such that $\{F_n\}_n$ converges pointwise to F on $[a, b]$. If each F_n is approximately derivable *a.e.* on $[a, b]$, $\{F_n\}_n$ is uniformly absolutely continuous on a set $P \subset [a, b]$, and $\{(F_n)'_{ap}\}_n$ converges in measure to a measurable function g, finite *a.e.* on $[a, b]$, then F is approximately derivable *a.e.* on P and $F'_{ap}(x) = g(x)$ *a.e.* on P. An immediate consequence of this result is the famous theorem of Džvaršeĭšvili on the passage to the limit for the Denjoy and Denjoy* integrals (see Theorem 47, p. 40 of [3]). As was pointed out by Bullen in [3] (p. 309), "the $\mathcal{D}^*$ integral case of Theorem 47 of [3] was rediscovered by Lee P. Y." (see also Theorem 7.6 of [7]).

1 Preliminaries

We shall denote the Lebesgue measure of the set A by $m(A)$, whenever $A \subset \mathbb{R}$ is Lebesgue measurable. If $f : [a, b] \to \mathbb{R}$ and $[\alpha, \beta] \subseteq [a, b]$, then let $\mathcal{O}(F; [\alpha, \beta] = \sup\{|F(y) - F(x)| : x, y \in [\alpha, \beta]\}$. Let $\mathcal{C}$ denote the class of all continuous functions and $\mathcal{C}_{ap}$ the class of all approximately continuous functions. A function $f : P \to \mathbb{R}$ is said to satisfy Lusin's condition (N), if $m(F(Z)) = 0$, whenever $m(Z) = 0$. For the definitions of AC, AC^*, VB and VB^* see [11].

Definition 1. *([11], p. 221).* Let $F : P \to \mathbb{R}$, and $Q \subseteq P$. We denote by $V(F; Q)$ the upper bound of the numbers $\sum_i |F(b_i) - F(a_i)|$, where $\{[a_i, b_i]\}_i$ is any sequence of nonoverlapping closed intervals with endpoints in Q. (We may suppose without loss of generality that $\{[a_i, b_i]\}_i$ is a finite set.)

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Definition 2. Let $E \subseteq [a,b]$. A function $F : [a,b] \to \mathbb{R}$ is said to be ACG (respectively AC^*G, VBG, VB^*G, $\mathcal{C}G$) on E if there exists a sequence of sets $\{E_n\}$ with $E = \cup_n E_n$, such that F is AC (respectively AC^*, VB, VB^*, $\mathcal{C}$) on each E_n. If in addition the sets E_n are supposed to be closed we obtain the classes $[ACG]$, $[AC^*G]$, $[VBG]$, $[VB^*G]$, $[\mathcal{C}G]$. Note that ACG and AC^*G used here differ from those of [11] (because in our definitions the continuity is not assumed).

2 Main Theorem

Lemma 1. *Let P be a subset of $[a,b]$ and $F : P \to \mathbb{R}$ an AC function. Then there exists a function $G : \overline{P} \to \mathbb{R}$, $G \in AC$ such that $G_{|P} = F$. Moreover, if for $\epsilon > 0$, $\delta_\epsilon > 0$ is given by the fact that $F \in AC$ on P, then $\delta_{\frac{\epsilon}{3}}$ satisfies the definition of G being AC on $\overline{P}$ for ϵ. As a consequence, if F is measurable, then F is approximately derivable a.e. on P.*

PROOF. Let $x_o \in \overline{P}$. For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $F \in AC$ on P. Then $|F(x) - F(y)| < \epsilon$ whenever $x, y \in (x_o - \delta_\epsilon/2, x_o + \delta_\epsilon/2) \cap P$. By the Cauchy criterion, the following limits exist and are finite:

$$\lim_{x \nearrow x_o, x \in P} F(x), \quad \lim_{x \searrow x_o, x \in P} F(x), \quad \lim_{x \to x_o, x \in P} F(x),$$

whenever x_o is a left, right or bilateral accumulation point of P respectively. If $x_o \in P$ any of the three limits equals $F(x_o)$, provided they exist.
Let $G : \overline{P} \to \mathbb{R}$ be defined by

$$G(x) = \begin{cases} F(x) & \text{if } x \text{ is an isolated point of } P \\ \lim_{\substack{x \nearrow x_o \\ x \in P}} F(x) & \text{if } x \text{ is a right accumulation point of } P \\ \lim_{\substack{x \searrow x_o \\ x \in P}} F(x) & \text{if } x \text{ is a left accumulation point of } P. \end{cases}$$

Let $\{[a_i, b_i]\}$, $i = 1, 2, \dots, n$ be a finite set of closed intervals with endpoints in $\overline{P}$, $a_1 < b_1 \le a_2 < b_2 \le \dots \le a_n < b_n$, such that $\sum_{i=1}^n (b_i - a_i) < \delta_{\frac{\epsilon}{3}}$. Let $\mathcal{A}_1 = \{i : i = \text{odd}, i \le n\}$ and $\mathcal{A}_2 = \{i : i = \text{even}, i \le n\}$. Then there exists a finite set $\{[x_i, y_i]\}_{i \in \mathcal{A}_1}$ of nonoverlapping closed intervals with endpoints in P such that

$$|F(x_i) - G(a_i)| < \frac{\epsilon}{6(n+1)}, \quad |F(y_i) - G(b_i)| < \frac{\epsilon}{6(n+1)}$$

and

$$\sum_{i \in \mathcal{A}_1} (y_i - x_i) < \delta_{\frac{\epsilon}{3}}.$$

It follows that

$$\sum_{i\in\mathcal{A}_1} |G(b_i) - G(a_i)| \leq \sum_{i\in\mathcal{A}_1} |G(a_i) - F(x_i)| +$$

$$+ \sum_{i\in\mathcal{A}_1} |F(x_i) - F(y_i)| + \sum_{i\in\mathcal{A}_1} |F(y_i) - G(b_i)| < \frac{\epsilon}{12} + \frac{\epsilon}{3} + \frac{\epsilon}{12} = \frac{\epsilon}{2}\,.$$

Similarly it follows that

$$\sum_{i\in\mathcal{A}_2} |G(b_i) - G(a_i)| < \epsilon/2\,.$$

Therefore $\sum_{i=1}^{n} |G(b_i) - G(a_i)| < \epsilon$. The last assertion follows from Theorem 4.2 of [11], p. 222 and by the fact that an AC function on a set is VB on that set. □

Definition 3. *([3], p. 38).* Let P be a real set and $F_n : P \to \mathbb{R}$, $n = 1, 2, \ldots$.

- The sequence $\{F_n\}_n$ is said to be UAC on P if it has the following property: for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ such that $\sum_{k=1}^{m} |F_n(\beta_k) - F_n(\alpha_k)| < \epsilon$ for all $n = 1, 2, \ldots$, whenever $\{[\alpha_k, \beta_k]\}$, $k = 1, 2, \ldots, m$ is a finite set of nonoverlapping closed intervals with endpoints in P and $\sum_{k=1}^{m} (\beta_k - \alpha_k) < \delta_\epsilon$.
- The sequence $\{F_n\}_n$ is said to be $UACG$ on P, if $P = \cup P_k$ and $\{F_n\}_n$ is UAC on each P_k. If in addition each P_k is supposed to be closed, then $\{F_n\}_n$ is said to be $[UACG]$ on P.

Remark 1. If P is a closed set, then $[UACG]$ is in fact Džvaršeĭšvili's condition "UACG" of [3], p. 38 (this follows using the technique of the proof of Theorem 9.1 of [11], p. 233). This fact, for $P = [a, b]$, was pointed out by Bullen (see [3], p. 308).

Corollary 1. *Let P be a subset of $[a, b]$ and let $F_n : P \to \mathbb{R}$, $n = 1, 2, \ldots$. If $\{F_n\}_n$ is UAC on P, then there exist $G_n : \overline{P} \to \mathbb{R}$, $n = 1, 2, \ldots$, $(G_n)_{/P} = F_n$, such that $\{G_n\}_n$ is UAC on $\overline{P}$. Moreover, if for $\epsilon > 0$, $\delta_\epsilon > 0$ is given by the fact that $\{F_n\}_n$ is UAC on P, then $\delta_{\epsilon/3}$ satisfies the definition of $\{G_n\}_n$ being UAC on $\overline{P}$ for ϵ.*

Lemma 2. *Let P be a closed subset of $[a, b]$, $a, b \in P$ and let $F : [a, b] \to \mathbb{R}$ be a function which is linear on the closure of each interval contiguous to P. Then $V(F; [a, b]) = V(F; P)$.*

PROOF. Clearly $V(F;P) \leq V(F;[a,b])$. Therefore we only need to show that $V(F;[a,b]) \leq V(F;P)$. Let

$$\Delta : a = a_o < a_1 < \ldots < a_m = b$$

be a division of $[a,b]$. Let $\Delta_1 := \Delta \cup \{$the endpoints of those intervals contiguous to P which contain points of $\Delta\}$. Suppose that

$$\Delta_1 : a = \alpha_o < \alpha_1 < \ldots < \alpha_n = b\,.$$

Let $\Delta_2 = \Delta_1 \cap P$. Suppose that

$$\Delta_2 : a = \beta_1 < \ldots < \beta_p = b\,.$$

Then

$$\sum_{i=1}^{m} |F(a_i) - F(a_{i-1})| \leq \sum_{i=1}^{n} |F(\alpha_i) - F(\alpha_{i-1})| = \sum_{i=1}^{p} |F(\beta_i) - F(\beta_{i-1})|.$$

(The equality follows by the fact that F is linear on the closure of each interval contiguous to P.) Therefore $V(F;[a,b]) \leq V(F;P)$. □

Lemma 3. *Let P be a subset of $[a,b]$ and let $F : P \to \mathbb{R}$, $F \in AC$. For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $F \in AC$ on P. Then there exists a function $\tilde{F} : [a,b] \to \mathbb{R}$, $\tilde{F} \in AC$ such that $\tilde{F}_{/P} = F$ and*

$$(\mathcal{L}) \int_A |\tilde{F}'(t)|dt < \epsilon$$

whenever A is a measurable subset of $\overline{P}$ with $m(A) < \delta_{\epsilon/6}$.

PROOF. For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $F \in AC$ on P. Let $c_o = \inf(P)$, $d_o = \sup(P)$, and let (c_k, d_k), $k = 1, 2, \ldots$ be the intervals contiguous to $\overline{P}$. By Lemma 1 there exists $G : \overline{P} \to \mathbb{R}$ such that $G \in AC$ on $\overline{P}$, $G_{/P} = F$ and for ϵ, the number $\delta_{\frac{\epsilon}{3}}$ is the δ given by the fact that $G \in AC$ on $\overline{P}$. Let $\tilde{F} : [a,b] \to \mathbb{R}$ be defined by

$$\tilde{F}(x) = \begin{cases} G(c_o) & \text{if } x \in [a, c_o] \\ G(x) & \text{if } x \in \overline{P} \\ \textit{linearly} & \text{on each } [c_k, d_k] \\ G(d_o) & \text{if } x \in [d_o, b]. \end{cases}$$

Then $\tilde{F} \in AC$ on $[a,b]$. (See for example Theorem 2.11.1 (xviii) of [4].) Let A be a measurable subset of $\overline{P}$ with $m(A) < \delta_{\epsilon/6}$. Then there exists a sequence $\{(\alpha_i,\beta_i)\}_i$ such that $(\alpha_i,\beta_i)\cap A\neq\emptyset$ for each i, $A\subset\cup_{i=1}^{\infty}(\alpha_i,\beta_i)$ and $\sum_{i=1}^{\infty}(\beta_i-\alpha_i)<\delta_{\epsilon/6}$. Let $a_i=\inf(\alpha_i,\beta_i)\cap\overline{P}$ and $b_i=\sup(\alpha_i,\beta_i)\cap\overline{P}$. Then $a_i,b_i\in\overline{P}$ and

$$\begin{aligned}(\mathcal{L})\int_A|\tilde{F}'(t)|dt &\le \sum_{i=1}^{\infty}(\mathcal{L})\int_{a_i}^{b_i}|\tilde{F}'(t)|dt = \\ &= \sum_{i=1}^{\infty}V(\tilde{F};[a_i,b_i]) = \sum_{i=1}^{\infty}V(G;[a_i,b_i]\cap\overline{P}).\end{aligned} \tag{1}$$

(The first equality follows by Theorem 8 of [8], p. 259, and the second equality follows by Lemma 2.)
For each i there exists a division

$$\Delta_i : a_i = a_{i,0} < a_{i,1} \ldots < a_{i,j_i} = b_i\,,$$

with each point in $\overline{P}$ such that

$$V(G;[a_i,b_i]\cap\overline{P}) < \frac{\epsilon}{2^{i+1}} + \sum_{k=1}^{j_i}|G(a_{i,k})-G(a_{i,k-1})|\,. \tag{2}$$

By (1) and (2), it follows that

$$(\mathcal{L})\int_A|\tilde{F}'(t)|dt \le \sum_{i=1}^{\infty}V(G;[a_i,b_i]\cap\overline{P}) < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\,.$$

□

Corollary 2. (An extension of Lemma 2 of [3], p. 38). *Let $P\subseteq[a,b]$ and let $\{F_n\}_n$ be a UAC sequence on P. For $\epsilon>0$ let $\delta_\epsilon>0$ be given by the latter fact. Then there exist $\tilde{F}_n:[a,b]\to\mathbb{R}$ such that $\tilde{F}_n\in AC$, $(\tilde{F}_n)_{/P}=F_n$ and*

$$(\mathcal{L})\int_A|\tilde{F}_n'(t)|dt<\epsilon\,,$$

for all $n=1,2,\ldots$, whenever A is a measurable subset of $\overline{P}$ with $m(A)<\delta_{\epsilon/6}$.

PROOF. Apply Lemma 3 to each F_n. □

Corollary 3. *Let $\{F_n\}_n$ be an UAC sequence on $[a,b]$, and let $x_o\in[a,b]$ such that $\lim_{n\to\infty}F_n(x_o)=\ell\in\mathbb{R}$. Let $g:[a,b]\to\overline{\mathbb{R}}$ be finite a.e. such that $\{F_n'\}_n$ converges to g in measure. Then g is Lebesgue integrable on $[a,b]$. Moreover, if $G(x)=\ell+(\mathcal{L})\int_{x_o}^{x}g(t)dt$, then $\{F_n\}_n$ converges uniformly to G on $[a,b]$ and $G'(x)=g(x)$ a.e. on $[a,b]$.*

PROOF. By Corollary 2, the summable functions $F_n^{'}$, $n = 1, 2, \ldots$ have equi-absolutely continuous integrals. (The functions of a family $\mathcal{M}$ of summable functions defined on a set E, are said to have equi-absolutely continuous integrals, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|\int_Q f(x)dx| < \epsilon$, $(\forall) f \in \mathcal{M}$, whenever Q is a measurable subset of E with $m(Q) < \delta$; [8], p. 151.) By Vitali's theorem ([8], p. 152), g is Lebesgue integrable on $[a, b]$, and by the proof of the same theorem it follows that

$$\lim_{n\to\infty} (\mathcal{L}) \int_a^b |F_n^{'}(t) - g(t)|dt = 0\,.$$

For $\epsilon > 0$ there exists a positive integer n_ϵ such that

$$(\mathcal{L}) \int_a^b |F_n^{'}(t) - g(t)|dt < \frac{\epsilon}{2} \quad \text{and} \quad |F_n(x_o) - \ell| < \frac{\epsilon}{2}$$

whenever $n \geq n_\epsilon$. Suppose that $x \geq x_o$. Then

$$|F_n(x) - G(x)| = \left| F_n(x_o) + (\mathcal{L}) \int_{x_o}^x F_n^{'}(t)dt - \ell - (\mathcal{L}) \int_{x_o}^x g(t)dt \right|$$

$$\leq |F_n(x_o) - \ell| + (\mathcal{L}) \int_{x_o}^x |F_n^{'}(t) - g(t)|dt < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,,$$

whenever $n > n_\epsilon$. Similarly, if $x < x_o$, then we obtain $|F_n(x) - G(x)| < \epsilon$, whenever $n > n_\epsilon$. Therefore $\{F_n\}_n$ converges uniformly to G on $[a, b]$.
That $G^{'}(x) = g(x)$ *a.e.* on $[a, b]$ is obvious. □

Definition 4. ([3], p. 38). Let $P \subset [a, b]$ and $F_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$.

- The sequence $\{F_n\}_n$ is said to be UAC^* on P if it has the following property: for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that

$$\sum_{k=1}^m \mathcal{O}(F_n; [\alpha_k, \beta_k]) < \epsilon\,, \quad n = 1, 2, \ldots\,,$$

  whenever $\{[\alpha_k, \beta_k]\}$, $k = 1, 2, \ldots, m$ is a set of nonoverlapping closed intervals with endpoints in P and $\sum_{k=1}^m (\beta_k - \alpha_k) < \delta_\epsilon$.

- The sequence $\{F_n\}_n$ is said to be UAC^*G on P, if $P = \cup P_k$ and $\{F_n\}_n$ is UAC^* on each P_k. If in addition each P_k is supposed to be closed, then $\{F_n\}_n$ is said to be $[UAC^*G]$ on P.

Remark 2. If P is a closed set, then $[UAC^*G]$ is in fact Džvaršeĭšvili's condition "$UACG^*$" of [3], p. 38. (This follows using the technique of the proof of Theorem 9.1 of [11], p. 233.) If $P = [a, b]$ and each F_n is supposed to be continuous on $[a, b]$, then $[UAC^*G]$ on $[a, b]$ is identical with P. Y. Lee's Definition 7.4, (ii) of [7], p. 39.

Lemma 4. *Let $P \subset [a, b]$ and $F, F_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$.*

(i) If $\{F_n\}_n$ is UAC on P and converges pointwise to F on P, then $F \in AC$ on P.

(ii) If $\{F_n\}_n$ is UAC^* on P and converges pointwise to F on $[a, b]$, then $F \in AC^*$ on P.

PROOF. (i) For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $\{F_n\}_n$ is UAC on P. Let $\{[a_i, b_i]\}$, $i = 1, 2, \ldots, m$ be a set of nonoverlapping closed intervals with endpoints in P such that $\sum_{i=1}^m (b_i - a_i) < \delta_{\epsilon/2}$. Then for each $n = 1, 2, \ldots$ we have

$$\sum_{i=1}^{m} |F_n(b_i) - F_n(a_i)| < \frac{\epsilon}{2}\,.$$

Passing to the limit, we obtain that

$$\sum_{i=1}^{m} |F(b_i) - F(a_i)| \le \frac{\epsilon}{2} < \epsilon\,.$$

Hence $F \in AC$ on P.

(ii) For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $\{F_n\}_n$ is UAC^* on P. Let $\{[a_i, b_i]\}$, $i = 1, 2, \ldots, m$, be a set of nonoverlapping closed intervals with endpoints in P, such that $\sum_{i=1}^m (b_i - a_i) < \delta_{\epsilon/3}$. Then for each $n = 1, 2, \ldots$

$$\sum_{i=1}^{m} \mathcal{O}(F_n; [a_i, b_i]) < \frac{\epsilon}{3}\,.$$

Since $\{F_n\}_n$ converges pointwise to F on $[a, b]$, it follows that for each $i = 1, 2, \ldots, m$ we have $\mathcal{O}(F; [a_i, b_i]) \le \epsilon < +\infty$. Thus, for each $i = 1, 2, \ldots, m$, there exists $[\alpha_i, \beta_i] \subseteq [a_i, b_i]$ such that

$$\mathcal{O}(F; [a_i, b_i]) < |F(\beta_i) - F(\alpha_i)| + \frac{\epsilon}{2^i}\,.$$

Let n be a positive integer such that

$$|F_n(\alpha_i) - F(\alpha_i)| < \frac{\epsilon}{6m} \quad \text{and} \quad |F_n(\beta_i) - F(\beta_i)| < \frac{\epsilon}{6m} \quad i = 1, 2, \ldots, m.$$

(This is possible because $\{F_n\}_n$ converges pointwise to F on $[a,b]$.) Then

$$\sum_{i=1}^{m} \mathcal{O}(F;[a_i,b_i]) < \frac{\epsilon}{3} + \sum_{i=1}^{m} |F(\beta_i) - F(\alpha_i)| \le \frac{\epsilon}{3} + \sum_{i=1}^{m} |F(\beta_i) - F_n(\beta_i)|$$

$$+ \sum_{i=1}^{m} |F_n(\beta_i) - F_n(\alpha_i)| + \sum_{i=1}^{m} |F_n(\alpha_i) - F(\alpha_i)| < \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon\,.$$

Hence $F \in AC^*$ on P. □

Lemma 5. *Let P be a closed subset of $[a,b]$, $a,b \in P$, and let $F, F_n : [a,b] \to \mathbb{R}$, $n = 1,2,\dots$, be such that F and each F_n are linear on the closure of each interval contiguous to P. If $\{F_n\}_n$ is UAC on P and converges pointwise to F on P, then $\{F_n\}_n$ is UAC on $[a,b]$ and $F \in AC$ on $[a,b]$. Consequently F and F_n are derivable a.e. on $[a,b]$. Moreover, if $\{F_n'\}_n$ converges in measure to an a.e. finite function g on P, then $F'(x) = g(x)$ a.e. on P.*

PROOF. We consider for example the case when the set of all intervals contiguous to P is infinite. Let $\{(c_k,d_k)\}$, $k = 1,2,\dots$ be the intervals contiguous to P. Since $\{F_n\}_n$ converges pointwise to F on P, it follows that $\{F_n\}_n$ converges pointwise to F on $[a,b]$. For $\epsilon > 0$ let $\delta_\epsilon > 0$ be given by the fact that $\{F_n\}_n$ is UAC on P. Let k_ϵ be a positive integer such that $\sum_{k=1+k_\epsilon}^{\infty}(d_k - c_k) < \delta_\epsilon$. Since $\{F_n\}_n$ converges pointwise to F on P, there exists a positive integer n_ϵ such that

$$\frac{|F_n(d_k) - F_n(c_k)|}{d_k - c_k} < 1 + \frac{|F(d_k) - F(c_k)|}{d_k - c_k} \qquad \text{for each } k = 1,2,\dots,k_\epsilon, \tag{3}$$

whenever $n \ge 1 + n_\epsilon$. Let

$$M_\epsilon = 1 + \max_{\substack{n=1,\dots,n_\epsilon \\ k=1,\dots,k_\epsilon}} \left\{ \frac{|F_n(d_k) - F_n(c_k)|}{d_k - c_k}, \frac{|F(d_k) - F(c_k)|}{d_k - c_k} \right\}, \tag{4}$$

$$\eta_\epsilon = \min\left\{ \frac{\epsilon}{M_\epsilon}, \delta_\epsilon \right\}. \tag{5}$$

Let $\{[\alpha_i,\beta_i]\}$, $i = 1,2,\dots,m$ be a finite set of nonoverlapping closed subintervals of $[a,b]$ with $\sum_{i=1}^{m}(\beta_i - \alpha_i) < \eta_\epsilon$. If $(\alpha_i,\beta_i) \cap P \ne \emptyset$, let $\alpha_i' = \inf((\alpha_i,\beta_i) \cap P)$ and $\beta_i' = \sup((\alpha_i,\beta_i) \cap P)$. Then $[\alpha_i,\beta_i] = [\alpha_i,\alpha_i'] \cup [\alpha_i',\beta_i'] \cup [\beta_i',\beta_i]$. Therefore $\cup_{i=1}^{m}[\alpha_i,\beta_i]$ can also be written as the union of a finite set $\{[a_j,b_j]\}$, $j = 1,2,\dots,p$, $p \le 3m$, of nonoverlapping nondegenerate closed intervals such that either $[a_j,b_j] \subseteq [c_k,d_k]$ for some k, or both a_j and b_j belong to P. Let

$\mathcal{A}_1 = \{j \,:\, a_j, b_j \in P\}\,;$

$\mathcal{A}_2 = \{j \,:\, [a_j, b_j] \subset \cup_{k=1+k_\epsilon}^{\infty}[c_k, d_k]\}\,;$

$\mathcal{A}_3 = \{j \,:\, [a_j, b_j] \subset \cup_{k=1}^{k_\epsilon}[c_k, d_k]\}\,.$

By (5) we have

$$\sum_{j \in \mathcal{A}_1} |F_n(b_j) - F_n(a_j)| < \epsilon\,, \quad \text{for each } n\,. \tag{6}$$

Because F_n is linear on each $[c_k, d_k]$ it follows that

$$\sum_{j \in \mathcal{A}_2} |F_n(b_j) - F_n(a_j)| \leq \sum_{k=1+k_\epsilon}^{\infty} |F_n(d_k) - F_n(c_k)| < \epsilon\,. \tag{7}$$

Let $i \in \mathcal{A}_3$ and $k \leq k_\epsilon$ such that $[a_i, b_i] \subset [c_k, d_k]$.
If $n \geq 1 + n_\epsilon$, then by (3) and (4) it follows that

$$\frac{|F_n(b_j) - F_n(a_j)|}{b_j - a_j} = \frac{|F_n(d_k) - F_n(c_k)|}{d_k - c_k} < 1 + \frac{|F(d_k) - F(c_k)|}{d_k - c_k} < M_\epsilon\,.$$

If $n \leq n_\epsilon$, then by (4) it follows that

$$\frac{|F_n(b_j) - F_n(a_j)|}{b_j - a_j} = \frac{|F_n(d_k) - F_n(c_k)|}{d_k - c_k} < M_\epsilon\,.$$

By (5), for each n we have

$$\sum_{j \in \mathcal{A}_3} |F_n(b_j) - F_n(a_j)| < M_\epsilon \cdot \sum_{j \in \mathcal{A}_3} (b_j - a_j) < M_\epsilon \cdot \frac{\epsilon}{M_\epsilon} = \epsilon\,. \tag{8}$$

By (6), (7) and (8) it follows that

$$\sum_{i=1}^{m} |F_n(\beta_i) - F_n(\alpha_i)| \leq \sum_{j=1}^{p} |F(b_j) - F(a_j)| < 3\epsilon\,,$$

for each n. Therefore $\{F_n\}_n$ is UAC on $[a, b]$. That $F \in AC$ on $[a, b]$ follows by Lemma 4, (i). Clearly F and F_n are derivable $a.e.$ on $[a, b]$.

We prove the second part. Let $x \in (c_k, d_k)$ for some k. Then

$$F_n^{'}(x) = \frac{F_n(d_k) - F_n(c_k)}{d_k - c_k} \longrightarrow \frac{F(d_k) - F(c_k)}{d_k - c_k} \quad \text{if } n \to \infty\,.$$

Let

$$g_o(x) = \begin{cases} g(x) & \text{if } x \in P \\ \frac{F(d_k)-F(c_k)}{d_k-c_k} & \text{if } x \in (c_k, d_k) \text{ for each } k \\ 0 & \text{if } x \in [a, c_o] \cup [d_o, b] \end{cases}$$

Since $\{F_n^{'}\}_n$ converges in measure to g on P, it follows that it also converges in measure to g_o on $[a, b]$. By Corollary 3, g_o is Lebesgue integrable on $[a, b]$ and $\{F_n\}_n$ converges uniformly to G on $[a, b]$, where $G(x) = F(a) + (\mathcal{L}) \int_a^x g_o(t)dt$. Since $\{F_n\}_n$ converges to F on $[a, b]$ it follows that $F = G$ on $[a, b]$. Hence $F^{'}(x) = G^{'}(x) = g_o(x)$ *a.e.* on $[a, b]$. Therefore $F^{'}(x) = g(x)$ *a.e.* on P. □

Remark 3. In Lemma 5, the condition "$\{F_n\}_n$ converges pointwise to $F : P \to \mathbb{R}$ on P" is essential. Indeed, let $P = [0, 1/3] \cup [2/3, 1]$ and let $F_n : P \to \mathbb{R}$,

$$F_n(x) = \begin{cases} 0 & \text{if } x \in [0, 1/3] \\ n & \text{if } x \in [2/3, 1] \end{cases}$$

For $\epsilon > 0$ and $\delta_\epsilon < 1/3$ we see easily that $\{F_n\}_n$ is UAC on P, but $\{\tilde{F}_n\}_n$ is not UAC on $[0, 1]$.

Corollary 4. *Let P be a closed subset of $[a, b]$. Let $F, F_n : [a, b] \to \mathbb{R}$, $n = 1, 2, \ldots$.*

(I) Suppose that $\{F_n\}_n$ is UAC on P and converges pointwise to F on P. We have:

(i) $F \in AC$ on P. Consequently F and F_n are approximately derivable a.e. on P;

(ii) If $\{(F_n)_{ap}^{'}\}_n$ converges in measure to an almost everywhere finite function g on P, then $F_{ap}^{'}(x) = g(x)$ a.e. on P.

(II) Suppose that $\{F_n\}_n$ is UAC^* on P and converges pointwise to F on $[a, b]$. We have:

(i) $F \in AC^*$ on P. Consequently F and F_n are derivable a.e. on P.

(ii) If $\{F_n^{'}\}_n$ converges in measure to an almost everywhere finite function g on P, then $F^{'}(x) = g(x)$ a.e. on P.

PROOF. (I) (i) This follows by Lemma 4, (i).
(ii) We may suppose without loss of generality that $a, b \in P$ and that the

set of all intervals contiguous to P is infinite. Let $\{(c_k, d_k)\}_k$ be the intervals contiguous to P. Let $\tilde{F}, \tilde{F}_n : [a,b] \to \mathbb{R}$ be defined by

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if } x \in P \\ linear & \text{on each } [c_k, d_k] \end{cases} \text{ and } \tilde{F}_n(x) = \begin{cases} F_n(x) & \text{if } x \in P \\ linear & \text{on each } [c_k, d_k] \end{cases}$$

Clearly $\tilde{F}_n$ converges pointwise to $\tilde{F}$ on $[a,b]$. By Lemma 5 it follows that $\{\tilde{F}_n\}_n$ is UAC on $[a,b]$ and $\tilde{F} \in AC$ on $[a,b]$. Clearly $\tilde{F}_n'(x) = (F_n)_{ap}'(x)$ and $\tilde{F}'(x) = F_{ap}'(x)$ *a.e.* on P. By hypothesis $\{\tilde{F}_n'\}_n$ converges in measure to an almost everywhere finite function g on P, so, by Lemma 5, $\tilde{F}'(x) = g(x)$ *a.e.* on P. Hence $F_{ap}'(x) = g(x)$ *a.e.* on P.

(II) (i) follows by Lemma 4, (ii); (ii) follows by (II) (i) and (I) (ii). □

Remark 4. The condition "$\{F_n\}_n$ is UAC on P" in Corollary 4, (I) is essential (see Example 2). The condition "$\{F_n\}_n$ is UAC^* on P" in Corollary 4, (II) is essential. It cannot be replaced by "$\{f_n\}_n$ is UAC on P" (see Example 1).

Theorem 1. *Let $P \subseteq [a,b]$ and let $F, F_n : [a,b] \to \mathbb{R}$, $n = 1, 2, \ldots$, be measurable functions such that $\{F_n\}_n$ converges pointwise to F on $[a,b]$.*

(i) Suppose that F_n is approximately derivable a.e. on $[a,b]$, $\{F_n\}_n$ is $UACG$ on P, and $\{(F_n)_{ap}'\}_n$ converges in measure to a measurable function g, finite a.e. on $[a,b]$. Then F is approximately derivable a.e. on P and $F_{ap}'(x) = g(x)$ a.e. on P.

(ii) Suppose that F_n is derivable a.e. on $[a,b]$, $\{F_n\}_n$ is UAC^*G on P, and $\{F_n'\}_n$ converges in measure to a measurable function g, finite a.e. on $[a,b]$. Then F is derivable a.e. on P and $F'(x) = g(x)$ a.e. on P.

PROOF. (i) We may suppose without loss of generality that $\{F_n\}_n$ is UAC on P. By Corollary 1 there exists $G_n : \overline{P} \to \mathbb{R}$ such that $\{G_n\}_n$ is UAC on $\overline{P}$ and $(G_n)_{/P} = F_n$ for each n. Let $P_n = \{x \in \overline{P} : F_n(x) = G_n(x)\}$. Then each P_n is a Lebesgue measurable set which contains P. Let $Q = \cap_{n=1}^{\infty} P_n$. Then Q is a Lebesgue measurable subset of $[a,b]$ which contains P. It follows that Q can be written as the union of an ascending sequence of closed sets $\{Q_i\}_i$ and a null set Z. For each i, $\{F_n\}_n$ is UAC on Q_i. By hypothesis and Corollary 4, (I), it follows that F is approximately derivable *a.e.* on Q_i and $F_{ap}'(x) = g(x)$ *a.e.* on Q_i, for each i. Hence $F_{ap}'(x) = g(x)$ *a.e.* on P.

(ii) We may suppose without loss of generality that $\{F_n\}_n$ is UAC^* on P. By Lemma 4, (ii) it follows that $F \in AC^*$ on P. Therefore F is derivable *a.e.* on P. Now the proof follows by (i). □

Remark 5. The condition "$\{F_n\}_n$ is $UACG$ on P" in Theorem 1, (i) is essential (see Example 2). The condition "$\{F_n\}_n$ is UAC^*G on P" in Theorem 1, (ii) is also essential. It cannot be replaced by "$\{F_n\}_n$ is UAC on P" (see Example 1).

Remark 6. In Corollary 3, Lemma 5, Corollary 4 and Theorem 1 the condition "converges in measure" may be replaced by "converges *a.e.*" (see for example Lebesgue's theorem of [8], p. 95).

3 Applications of the Main Theorem to Some Integrals, More General Than $\mathcal{D}$ and $\mathcal{D}^*$

Definition 5. Let $\mathcal{M}([a,b]) = \{F : [a,b] \to \mathbb{R} : F$ is a Lebesgue measurable function on $[a,b]\}$. Let L_1, L_2, L_3 and L_4 be linear subspaces of $\mathcal{M}([a,b])$ with the following properties:

1) If $F \in ACG \cap L_1$ on $[a,b]$ and $F'_{ap} = 0$ *a.e.* on $[a,b]$, then F is a constant function on $[a,b]$.

2) If $F \in [ACG] \cap L_2$ on $[a,b]$ and $F'_{ap} = 0$ *a.e.* on $[a,b]$, then F is a constant function on $[a,b]$.

3) If $F \in AC^*G \cap L_3$ on $[a,b]$ and $F' = 0$ *a.e.* on $[a,b]$, then F is a constant function on $[a,b]$.

4) If $F \in [AC^*G] \cap L_2$ on $[a,b]$ and $F' = 0$ *a.e.* on $[a,b]$, then F is a constant function on $[a,b]$.

Remark 7. Clearly there are more subspaces of type L_2 than of type L_1, and there are more subspaces of type L_4 than of type L_3.

Definition 6. Let $f : [a,b] \to \overline{\mathbb{R}}$

- f is said to be $L_1\mathcal{D}$ (respectively $[L_2\mathcal{D}]$) integrable on $[a,b]$, if there exists $F : [a,b] \to \mathbb{R}$ such that $F \in ACG \cap L_1$ (respectively $F \in [ACG] \cap L_2$) on $[a,b]$, and $F'_{ap}(x) = f(x)$ *a.e.* on $[a,b]$.
- f is said to be $L_3\mathcal{D}^*$ (respectively $[L_4\mathcal{D}^*]$) integrable on $[a,b]$, if there exists $F : [a,b] \to \mathbb{R}$ such that $F \in AC^*G \cap L_3$ (respectively $F \in [AC^*G] \cap L_4$) on $[a,b]$, and $F'(x) = f(x)$ *a.e.* on $[a,b]$.

We shall say that the function F is an indefinite $L_1\mathcal{D}$ (respectively $[L_2\mathcal{D}]$, $L_3\mathcal{D}^*$, $[L_4\mathcal{D}^*]$) integral of $f(x)$. Its increment $F(b) - F(a)$ is called the definite $L_1\mathcal{D}$ (respectively $[L_2\mathcal{D}]$, $L_3\mathcal{D}^*$, $[L_4\mathcal{D}^*]$) integral of $f(x)$, and we denote it by $L_1\mathcal{D}\int_a^b f(t)dt$ (respectively $[L_2\mathcal{D}]\int_a^b f(t)dt$, $L_3\mathcal{D}^*\int_a^b f(t)dt$, $[L_4\mathcal{D}^*]\int_a^b f(t)dt$).

Remark 8.

- If $L_1 = L_2 = L_3 = L_4 = \mathcal{C}$, then $\mathcal{CD} = [\mathcal{CD}] = \mathcal{D}$ (the wide Denjoy integral), and $\mathcal{CD}^* = [\mathcal{CD}^*] = \mathcal{D}^*$ (the Denjoy* integral).
- If $L_1 = L_2 = L_3 = L_4 = \mathcal{C}_{ap}$, then $[\mathcal{C}_{ap}\mathcal{D}]$ is the β-Ridder integral (see Definition 7 of [9], p. 148), which is also called the AD-integral of Kubota (see [5], p. 715).
- We have

$$AC^*G \cap \mathcal{C}_{ap} \subset VB^*G \cap \mathcal{C}_{ap} \cap (N) = [VB^*G] \cap \mathcal{C}_{ap} \cap [\mathcal{C}G] \cap (N) =$$

$$= [VB^*G] \cap [ACG] \cap \mathcal{C}_{ap} \subset [AC^*G] \cap \mathcal{C}_{ap} \quad \text{on} \quad [a,b] \, .$$

  For the first equality see Theorem 2.10.3, (vi) of [4] and use the fact that a $\mathcal{C}_{ap}$ function is a Darboux function on an interval. The second equality follows by the Banach-Zarecki Theorem ([11], p. 227). The last inclusion follows by Theorem 2.12.1, (ii) of [4]. Therefore

$$AC^*G \cap \mathcal{C}_{ap} = [AC^*G] \cap \mathcal{C}_{ap} \, ,$$

  so

$$\mathcal{C}_{ap}\mathcal{D}^* = [\mathcal{C}_{ap}\mathcal{D}^*] = \alpha - \text{Ridder integral}$$

  (for the α-Ridder integral see Definition 2 of [9], p. 138).
- The LDG integrals, introduced by C. M. Lee [6] are $[L_2\mathcal{D}]$-type integrals.
- (*Question*) Does the $\mathcal{C}_{ap}\mathcal{D}$ integral strictly extend the $[\mathcal{C}_{ap}\mathcal{D}]$ integral?

Theorem 2. *Let $\{f_n\}_n \subset L_1\mathcal{D}$ (respectively $[L_2\mathcal{D}]$) on $[a,b]$ such that*

$$\lim_{n\to\infty} f_n \to f \, , \ \textit{a.e.} \quad \textit{on} \quad [a,b] \, .$$

For each positive integer n, let F_n be the indefinite $L_1\mathcal{D}$ (respectively $[L_2\mathcal{D}]$) integral of f_n. Suppose that $\{F_n\}_n$ converges pointwise to F on $[a,b]$, $F \in L_1$ (respectively L_2). If $\{F_n\}_n \in UACG$ (respectively $[UACG]$) on $[a,b]$, then $f \in L_1\mathcal{D}$ (respectively $[L_2\mathcal{D}]$) on $[a,b]$ and

$$\lim_{n\to\infty} L_1\mathcal{D}\int_a^b f_n(t)dt = L_1\mathcal{D}\int_a^b f(t)dt$$

(respectively

$$\left.\lim_{n\to\infty} [L_2\mathcal{D}]\int_a^b f_n(t)dt = [L_2\mathcal{D}]\int_a^b f(t)dt\right) .$$

PROOF. See Lemma 4, (i) and Theorem 1, (i). □

Theorem 3. *Let* $\{f_n\}_n \subset L_3\mathcal{D}^*$ *(respectively* $[L_4\mathcal{D}^*]$*) on* $[a,b]$ *such that*

$$\lim_{n\to\infty} f_n \to f\,,\ a.e. \quad on \quad [a,b]\,.$$

For each n*, let* F_n *be the indefinite* $L_2\mathcal{D}^*$ *(respectively* $[L_4\mathcal{D}^*]$*) integral of* f_n*. Suppose that* $\{F_n\}_n$ *converges pointwise to* F *on* $[a,b]$*,* $F \in L_3$ *(respectively* L_4*). If* $\{F_n\}_n \in UAC^*G$ *(respectively* $[UAC^*G]$*) on* $[a,b]$*, then* $f \in L_3\mathcal{D}^*$ *(respectively* $[L_4\mathcal{D}^*]$*) on* $[a,b]$ *and*

$$\lim_{n\to\infty} L_3\mathcal{D}^* \int_a^b f_n(t)dt = L_3\mathcal{D}^* \int_a^b f(t)dt$$

(respectively

$$\left. \lim_{n\to\infty} [L_4\mathcal{D}^*] \int_a^b f_n(t)dt = [L_4\mathcal{D}^*] \int_a^b f(t)dt \right)\,.$$

PROOF. See Lemma 4, (ii) and Theorem 1, (ii). □

Remark 9. Suppose that L_1, L_2, L_3 and L_4 are closed under uniform convergence. Then the condition "$\{F_n\}_n$ converges pointwise to F on $[a,b]$, $F \in L_1$ (respectively L_2)" in Theorem 2 may be replaced with the condition "$\{F_n\}_n$ converges uniformly to F on $[a,b]$". Similarly the condition "$\{F_n\}_n$ converges pointwise to F on $[a,b]$, $F \in L_3$ (respectively L_4)" in Theorem 3 may be replaced with the condition "$\{F_n\}_n$ converges uniformly to F on $[a,b]$".

Note that Theorem 2 contains Theorem 47, a) of [3] and Theorem 3 contains Theorem 47, b) of [3] (in fact Theorem 47, b) is identical with L. P. Yee's Theorem 7.6 of [7]). Theorem 3 also contains L. P. Yee's Corollary 7.7 of [7].

4 Sequences of Approximately Derivable Functions on an Interval

We recall the following classical theorems.

Theorem A. *([10], p. 140). Let* $\{f_n\}_n$ *be a sequence of differentiable functions on* $[a,b]$*, such that* $\{f_n(x_o)\}_n$ *converges for some point* x_o *on* $[a,b]$*. If* $\{f_n^{'}\}_n$ *converges uniformly on* $[a,b]$ *to* g*,, then* $\{f_n\}_n$ *converges uniformly on* $[a,b]$ *to a function* f*, and* $f^{'}(x) = g(x)$ *on* $[a,b]$*.*

Remark 10. If in Theorem A, the condition "$\{f_n^{'}\}_n$ converges uniformly on $[a,b]$ to g" is replaced by "$\{f_n^{'}\}_n$ converges pointwise on $[a,b]$ to g", then, even if $\{f_n\}_n$ converges uniformly to f on $[a,b]$, it may happen that $f^{'}(x)$ does not exist (finite or infinite) on a perfect set of positive measure as close as we want to $b-a$. It follows that $f^{'}(x) \neq g(x)$ on a set of positive measure (see Example 1).

Theorem B. *([2], p. 44). Let $\{f_n\}_n$ be a sequence of approximately differentiable functions on $[a,b]$, such that $\{f_n(x_o)\}_n$ converges for some point x_o on $[a,b]$. If $\{(f_n)_{ap}^{'}\}_n$ converges uniformly on $[a,b]$ to g, then $\{f_n\}_n$ converges uniformly on $[a,b]$ to a function f, and $f_{ap}^{'}(x) = g(x)$ on $[a,b]$.*

PROOF. We follow the proof of [2], p. 44. Since

$$(f_n)_{ap}^{'} \longrightarrow g\ [unif] \text{ on } [a,b].$$

it follows that there exists a positive integer n_1 such that

$$|(f_n)_{ap}^{'}(x) - (f_{n_1})_{ap}^{'}(x)| < 1\,, \quad (\forall)\, n \geq n_1\,.$$

By Tolstoff's Theorem ([1], p. 175) it follows that $f_n - f_{n_1}$ is a Lipschitz function, and by the Khintchine–Mišik Theorem ([12], p. 139 or [1], Theorem 2.4, p. 155) we have

$$(f_n)_{ap}^{'}(x) - (f_{n_1})_{ap}^{'}(x) = (f_n - f_{n_1})^{'}(x) \quad \text{on} \quad [a,b], \quad (\forall)\, n \geq n_1\,.$$

Hence

$$(f_n - f_{n_1})^{'} \longrightarrow g - (f_{n_1})_{ap}^{'}\ [unif] \text{ on } [a,b].$$

By Theorem A

$$f_n - f_{n_1} \longrightarrow f - f_{n_1}\ [unif] \text{ on } [a,b] \text{ for some } f$$

and

$$(f - f_{n_1})^{'}(x) = g(x) - (f_{n_1})_{ap}^{'}(x) \text{ on } [a,b]\,.$$

Therefore

$$(f_n)_{ap}^{'}(x) = (f_{n_1})_{ap}^{'}(x) + (f - f_{n_1})^{'}(x) = g(x) \text{ on } [a,b]\,.$$

□

Remark 11. If in Theorem A the condition "$\{f_n^{'}\}_n$ converges uniformly on $[a,b]$ to g" is replaced by "$\{f_n^{'}\}_n$ converges pointwise on $[a,b]$ to g", then, even if $\{f_n\}_n$ converges uniformly to f on $[a,b]$, it may happen that $f^{'}$ exists and is continuous on $[a,b]$, but $f' \neq g$ on a perfect set of positive measure as close as we want to $b-a$ (see Example 2).

If in Theorem B the condition "$\{(f_n)'_{ap}\}_n$ converges uniformly on $[a,b]$ to g" is replaced by "$\{(f_n)'_{ap}\}_n$ converges pointwise on $[a,b]$ to g", then, even if $\{f_n\}_n$ converges uniformly to f on $[a,b]$, it may happen that f'_{ap} exists and is continuous on $[a,b]$, but $f'_{ap} \neq g$ on a perfect set of positive measure as close as we want to $b-a$ (see Example 2).

5 Examples

Example 1. First we construct a Cantor type perfect set, contained in $[0,1]$. Let $\beta \in (0,1]$ and let $\{\beta_n\}_n$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} 2^{n-1}\beta_n = \beta$. We extract from $[0,1]$ the open interval $G_1 = (a_1, b_1)$, centered in $1/2$ with length β_1.

Let

$$P_1 = [0,1] \setminus G_1 .$$

Clearly P_1 consists of two disjoint closed intervals, each of length

$$\frac{1-\beta_1}{2} .$$

From each of the two intervals of P_1 we extract from the left to the right the centered open intervals (a_2, b_2) and (a_3, b_3), with length β_2. Let

$$G_2 = G_1 \cup (a_2, b_2) \cup (a_3, b_3) \quad \text{and} \quad P_2 = [0,1] \setminus G_2 .$$

Clearly P_2 consists of 2^2 nonoverlapping closed intervals, each of length

$$\frac{1-(\beta_1 + 2\beta_2)}{2^2} .$$

Suppose we have already defined the sets G_{n-1} and P_{n-1}, $n \geq 2$. Then P_{n-1} consists of 2^{n-1} nonoverlapping closed intervals, each of length

$$\frac{1-(\beta_1 + 2\beta_2 + \cdots + 2^{n-1}\beta_{n-1})}{2^{n-1}} .$$

From each interval of P_{n-1} we extract from the left to the right the centered open intervals

$$(a_{2^{n-1}}, b_{2^{n-1}}), (a_{2^{n-1}+1}, b_{2^{n-1}+1}), \ldots, (a_{2^n-1}, b_{2^n-1})$$

with length β_n. Let

$$G_n = G_{n-1} \cup \left(\cup_{i=2^{n-1}}^{2^n-1} (a_i, b_i) \right) \quad \text{and} \quad P_n = [0,1] \setminus G_n \, .$$

Then P_n consists of 2^n nonoverlapping closed intervals, each of length

$$\frac{1 - (\beta_1 + 2\beta_2 + \cdots + 2^n \beta_n)}{2^n} \, .$$

Let

$$G = \cup_{n=1}^{\infty} G_n \quad \text{and} \quad P = \cap_{n=1}^{\infty} P_n \, .$$

Then $m(G) = \beta$ and $m(P) = 1 - \beta$.

Let $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & \text{if } x \in P \\ \frac{1}{4n} \left(1 + \cos\left(\frac{2\pi}{(b_i - a_i)} (x - a_i) - \pi \right) \right) & \text{if } x \in (a_i, b_i), \\ & i = 2^{n-1}, 2^{n-1}+1, \ldots, 2^n - 1 \\ & n = 1, 2, \ldots \end{cases}$$

Let $f_n : [0,1] \to \mathbb{R}$ (for f_3 see Figure 1), $f_n(x) = \begin{cases} f(x) & \text{if } x \in G_n \\ 0 & \text{if } x \in P_n. \end{cases}$

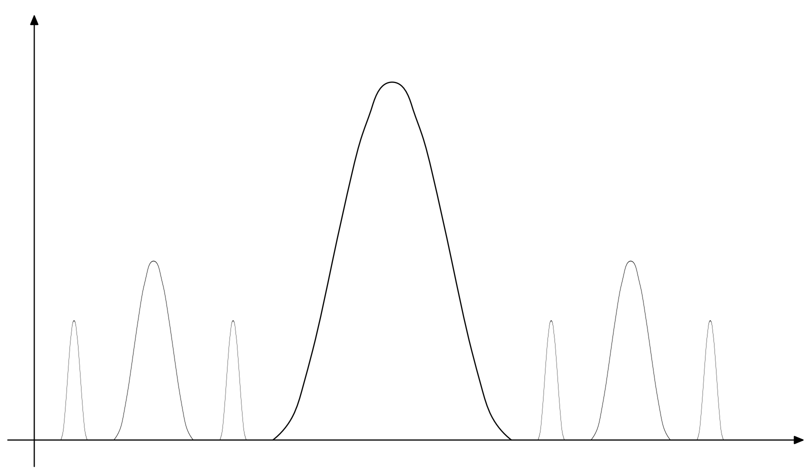

Figure 1: The graph of f_3 in Example 1

Then we have

1) $f_n^{'}(x) = \begin{cases} 0 & \text{if } x \in P_n \\ f^{'}(x) & \text{if } x \in G_n,\ n = 1, 2, \ldots . \end{cases}$ hence $\{f_n\}_n \in C^1([0,1])$;

2) $f_n \longrightarrow f\ [unif]$ on $[0,1]$;

3) Let $g : [0,1] \to \mathbb{R}$, $g(x) = \begin{cases} 0 & x \in P \\ f^{'}(x) & x \in G\,. \end{cases}$
Then $f_n^{'}(x) \to g(x)$, $(\forall)\, x \in [0,1]$.

4) $f^{'}(x)$ does not exist (finite or infinite) if $x \in P$, but $f_{ap}^{'} = g$ *a.e.* on $[0,1]$.

5) $\{f_n\}_n$ is UAC on P, but $\{f_n\}_n$ is not UAC^* (and neither UAC^*G) on P (see Corollary 4, (II) and Theorem 1, (ii)).

Example 2. We consider all the notations of Example 1. Let $\{\alpha_n\}$ be a strictly increasing sequence of positive numbers, converging to 1. From each (a_i, b_i), $i = 1, 2, \ldots, 2^n - 1$, we extract the centered closed interval $[c_i^n, d_i^n]$ of length $\alpha_n(b_i - a_i)$. Let

$$K_n = \cup_{i=1}^{2^n-1}[c_i^n, d_i^n]\,.$$

Then $m(K_n) = \alpha_n \cdot m(G_n)$ and $G = \cup_{n=1}^{\infty} K_n$. Let $f_n : [0,1] \to \mathbb{R}$, $n = 1, 2, \ldots$. First we define f_n on $P_n \cup K_n$ by

$$f_n(x) = \begin{cases} \alpha & \text{if } x \in [\alpha, \beta] \\ x - \frac{m(P_n)}{2^{n+1}} & \text{if } x \in K_n, \end{cases}$$

where $[\alpha, \beta]$ is any of the 2^n closed intervals of P_n. Clearly f_n is increasing on $P_n \cup K_n$. On each $[a_i, c_i^n]$, $i = 1, 2, \ldots, 2^n - 1$, we define f_n such that f_n is strictly increasing, f_n has a continuous derivative on $[a_i, c_i^n]$, $f_n^{'}(a_i) = 0$ and $f_n^{'}(c_i^n) = 1$. On each $[d_i^n, b_i]$, $i = 1, 2, \ldots, 2^n - 1$, we define f_n such that f_n is strictly increasing, f_n has a continuous derivative on $[d_i^n, b_i]$, $f^{'}(d_i^n) = 1$ and $f_n^{'}(b_i) = 0$ (for f_3 and f see Figure 2).

Then we have

1) $f_n \in C^1([0,1])$;

2) $f_n \longrightarrow f\ [unif]$ on $[0,1]$, where $f : [0,1] \to [0,1]$, $f(x) = x$, $f \in C^1[0,1]$;

3) $f_n^{'}(x) = 0$ on P_n. Hence $f_n^{'}(x) = 0$ on P.

4) $\lim_{n\to\infty} f_n^{'}(x) = 1$ for each $x \in G$. (Indeed, for $x \in G = \cup_{n=1}^{\infty} K_n$, there exists a positive integer m such that $x \in \text{int}(K_n)$, $(\forall)\, n \ge m$, because $K_1 \subset K_2 \subset \ldots \subset K_m \subset \ldots \subset K_n \subset \ldots$; it follows that $f_n^{'}(x) = 1$, $(\forall)\, n \ge m$, so $\lim_{n\to\infty} f_n^{'}(x) = 1$).

5) $\lim_{n\to\infty} f_n^{'}(x) = g(x)$, $x \in [0,1]$, where $g : [0,1] \to [0,1]$,

$$g(x) = \begin{cases} 0 & \text{if } x \in P \\ 1 & \text{if } x \in G \end{cases}$$

6) $\{f_n\}_n$ is UAC (or $UACG$) neither on $[0,1]$ nor on P (see Corollary 4, (I) and Theorem 1, (i)).

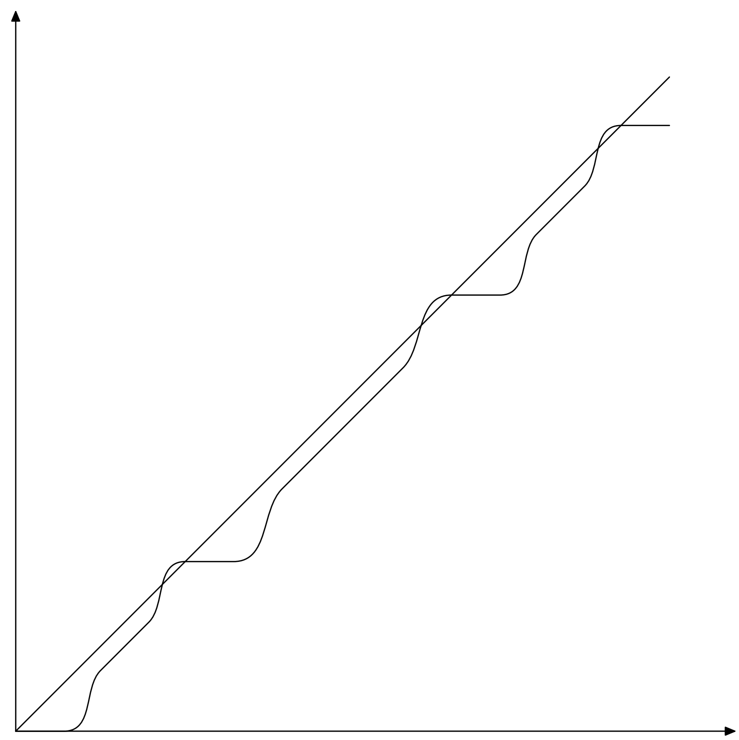

Figure 2: The graph of f_3 and f in Example 2

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