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A DESCRIPTIVE DEFINITION OF THE KH-STIELTJES INTEGRAL†

Abstract

This paper gives a descriptive definition of Stieltjes integrals (on a compact interval of the real line) in the frame of Kurzweil-Henstock integration. Five conditions characterize the functions that are an indefinite integral with respect to some continuous function of generalized bounded variation.

1 Introduction

A descriptive definition of the Kurzweil-Henstock integral, involving differentiability almost everywhere together with some null condition, is known since a few years (cf. for instance [3]). A more complete fundamental theorem was given by W. B. Jurkat and R. W. Knizia for the multidimensional weak integral in [4] and [5], where these authors introduced a useful and natural outer measure associated to any (interval) function.

In a preceding paper [1], I gave such a fundamental theorem for the multidimensional integrals of J. Mawhin [6] and W. F. Pfeffer [8]. In the present one, I propose a similar theorem for the Kurzweil-Henstock-Stieltjes integral on a compact interval $[a, b] \subseteq \mathbb{R}$. Five equivalent conditions thus characterize the functions $F : [a, b] \to \mathbb{R}$ which are an indefinite integral of some function $f : [a, b] \to \mathbb{R}$ relatively to $U : [a, b] \to \mathbb{R}$, cf. Theorem 4.7 and Corollary 5.6. The function U is assumed to be continuous and VBG$^\circ$ (equivalently, VBG$_*$ in the sense of Saks [9]).

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Two difficulties arise in comparison with the non-Stieltjes case. First, the use of VBG° functions requires technical adjustments in many proofs (often along the same lines). Then, especially, a strong theorem on relative differentiation is needed, cf. Theorem 3.2 (and [2] for a more general version).

At the end of the paper, as an application of the fundamental theorem, a substitution theorem is given for the Kurzweil-Henstock integral, which uses a measurable and bounded function f. Such a theorem is well-known for the Lebesgue integral, but I have not found any reference for the KH-integral.

2 Preliminaries

Definition 2.1. A *system* S on a set $A \subseteq [a,b]$ is given by a finite family of intervals $a \le a_1 < b_1 \le \ldots \le a_r < b_r \le b$ together with a family of associated points $x_i \in [a_i, b_i] \cap A$. Now let $\delta : A \to \mathbb{R}_+$ be any *gauge* on the set A. One says that the system S is *δ-fine* if $[a_i, b_i] \subseteq \big(x_i - \delta(x_i), x_i + \delta(x_i)\big)$ for every $i = 1, \ldots, r$. We denote by $\mathcal{S}(A, \delta)$ the set of all δ-fine systems S on A.

Definition 2.2. A *division* of the interval $[a,b]$ is a system D on $[a,b]$ which satisfies $b_i = a_{i+1}$ for every $i = 0, \ldots, r$ (where $b_0 = a$ and $a_{r+1} = b$). Given two functions $f, U : [a,b] \to \mathbb{R}$ one can form the *Riemann-Stieltjes sum*

$$S(f, U, D) = \sum_{i=1}^{r} f(x_i)\big(U(b_i) - U(a_i)\big).$$

Then one says that the function f is *integrable relatively to the function U*, or shortly that f is *U-integrable*, if there exists a number $I \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to \mathbb{R}_+$ with the property

$$|S(f, U, D) - I| < \varepsilon \text{ for every } \delta\text{-fine division } D \text{ of } [a,b].$$

The *integral* $I \in \mathbb{R}$ is clearly unique, and denoted by $\int_a^b f \, \mathrm{d}U$. The following propositions 2.3 and 2.4 are well-known properties of the integral.

Proposition 2.3. *Let $f, U : [a,b] \to \mathbb{R}$ and $a < c < b$. Then f is integrable relatively to the function U on the interval $[a,b]$ if and only if both integrals $\int_a^c f \, \mathrm{d}U$ and $\int_c^b f \, \mathrm{d}U$ exist. And one has $\int_a^b f \, \mathrm{d}U = \int_a^c f \, \mathrm{d}U + \int_c^b f \, \mathrm{d}U$.*

Proposition 2.4. Saks-Henstock Lemma *Let $f : [a,b] \to \mathbb{R}$ be integrable relatively to the function $U : [a,b] \to \mathbb{R}$. We suppose given a gauge δ on the interval $[a,b]$ such that $|S(f, U, D) - \int_a^b f \, \mathrm{d}U| < \varepsilon$ for every δ-fine division D of $[a,b]$. Then for any δ-fine system S one has the following inequalities:*

1) $\big|\sum_{i=1}^{r} \big\{ f(x_i)\big(U(b_i) - U(a_i)\big) - \int_{a_i}^{b_i} f \; dU \big\}\big| \le \varepsilon,$

2) $\sum_{i=1}^r \bigl|f(x_i)\bigl(U(b_i)-U(a_i)\bigr)-\int_{a_i}^{b_i} f\ dU\bigr| \le 2\varepsilon$.

Definition 2.5. Let $F:[a,b]\to\mathbb{R}$ be any function. Given a system S on a set $A\subseteq[a,b]$ one forms the *variational sum* $W_F(S)=\sum_{i=1}^r |F(b_i)-F(a_i)|$. The F-*outer measure* of the subset A is the number

$$m_F(A)=\inf_\delta \sup\bigl\{W_F(S)\bigm/ S\in\mathcal{S}(A,\delta)\bigr\},$$

where δ runs over all gauges $A\to\mathbb{R}_+$. The following proposition shows that m_F is a metric outer measure (for the proof see Proposition 3.3 in [1]).

Proposition 2.6. *The functional m_F has the following properties:*

1) $m_F(A)\ge 0$ *for every* $A\subseteq[a,b]$, *and* $m_F(\emptyset)=0$,

2) $A\subseteq B$ *implies* $m_F(A)\le m_F(B)$,

3) $m_F\bigl(\bigcup_{n=1}^\infty A_n\bigr)\le\sum_{n=1}^\infty m_F(A_n)$ *for every sequence of sets* $A_n\subseteq[a,b]$,

4) $m_F(A\cup B)=m_F(A)+m_F(B)$ *provided A and B are contained in two disjoint open subsets of the interval* $[a,b]$.

Remark 2.7. As one could expect, in the special case where $F(x)=x$ the outer measure m_F is the Lebesgue outer measure, cf. Proposition 3.4 in [1].

Definition 2.8. Let $U:[a,b]\to\mathbb{R}$ be a fixed function. One says that a set $A\subseteq[a,b]$ is U-*null* if one can write $A=D\cup N$ with D at most denumerable and $m_U(N)=0$. As usual, a property is said to hold U-*almost everywhere* if the exceptional set is U-null.

Proposition 2.9. *For functions $f,U:[a,b]\to\mathbb{R}$ the following are equivalent:*

1) *f is U-integrable and* $\int_a^x f\ dU=0$ *for every* $x\in(a,b]$,

2) *the set* $E=\bigl\{x\in[a,b]\bigm/ f(x)\ne 0\bigr\}$ *satisfies* $m_U(E)=0$.

Proof. $(1\Rightarrow 2)$ We show that each set $E_n:=\bigl\{x\in[a,b]\bigm/ |f(x)|\ge\frac1n\bigr\}$ satisfies $m_U(E_n)=0$. Given $\varepsilon>0$ there exists a gauge $\delta:[a,b]\to\mathbb{R}_+$ such that $|S(f,U,D)|<\varepsilon$ for every δ-fine division D of $[a,b]$. Now let S be any δ-fine system on E_n. By Saks-Henstock Lemma one obtains

$$\tfrac1n W_U(S)=\sum_{i=1}^r \tfrac1n\bigl|U(b_i)-U(a_i)\bigr|\le\sum_{i=1}^r\bigl|f(x_i)\bigl(U(b_i)-U(a_i)\bigr)\bigr|\le 2\varepsilon,$$

and this proves that $m_U(E_n)\le 2n\varepsilon$. So the assertion follows.

(2 ⇒ 1) Let $E_n := \{x \in [a,b] \,/\, n-1 < |f(x)| \le n\}$. Then there exists for each $n \in \mathbb{N}$ a gauge $\delta_n : E_n \to \mathbb{R}_+$ such that $W_U(S) < \varepsilon \frac{1}{n} 2^{-n}$ for any system $S \in \mathcal{S}(E_n, \delta_n)$. Taking arbitrary $\delta(x)$ if $f(x) = 0$ and $\delta(x) = \delta_n(x)$ if $x \in E_n$, one gets a gauge $\delta : [a,b] \to \mathbb{R}_+$. For any δ-fine division D of $[a,b]$ one has

$$\big|S(f,U,D)\big| \le \sum_{n=1}^{\infty} \textstyle\sum_{x_i \in E_n} \big|f(x_i)\big(U(b_i) - U(a_i)\big)\big| \le \displaystyle\sum_{n=1}^{\infty} n\, W_U(S_n) < \varepsilon.$$

Therefore f is U-integrable on $[a,b]$ and $\int_a^b f \,\mathrm{d}U = 0$. □

3 Differentiation with Respect to VBG° Functions

Definition 3.1. Let $F, U : [a,b] \to \mathbb{R}$ be any functions. The *lower* and *upper derivatives* of F with respect to U,

$$\underline{\mathrm{D}}_U F(x) = \liminf_{y \to x} \frac{F(y) - F(x)}{U(y) - U(x)} \quad \text{and} \quad \overline{\mathrm{D}}_U F(x) = \limsup_{y \to x} \frac{F(y) - F(x)}{U(y) - U(x)},$$

are defined for all $x \in [a,b]$ such that $U(y) \ne U(x)$ in a neighborhood of x. The function F is *differentiable relatively to* U, or shortly *U-differentiable*, at x if $\underline{\mathrm{D}}F(x) = \overline{\mathrm{D}}F(x) \in \mathbb{R}$, this common value being denoted by $F'_U(x)$.

We shall use the following version of the Denjoy-Young-Saks theorem:

Theorem 3.2. *Let $U : [a,b] \to \mathbb{R}$ be any strictly increasing function. Then a function $F : [a,b] \to \mathbb{R}$ is U-differentiable at U-almost every point of the sets $\{x \in [a,b] \,/\, \underline{\mathrm{D}}_U F(x) > -\infty\}$ and $\{x \in [a,b] \,/\, \overline{\mathrm{D}}_U F(x) < \infty\}$.*

Proof. This is a particular case of Théorème 7 in [2]. □

Definition 3.3. One says that a function $F : [a,b] \to \mathbb{R}$ is *of bounded variation* on a set $E \subseteq [a,b]$, or *VB°* on E, if one has $m_F(E) < \infty$. One says that the function F is *of generalized bounded variation*, or *VBG°*, if there exists a decomposition $[a,b] = \bigcup_{n=1}^{\infty} E_n$ (not necessarily disjoint) such that F is of bounded variation on each subset E_n.

Remark 3.4. Since a function $F : [a,b] \to \mathbb{R}$ is continuous at x if and only if $m_F(\{x\}) = 0$, it follows that the set of discontinuities of a VBG° function is at most denumerable.

Lemma 3.5. *If a function $F : [a,b] \to \mathbb{R}$ is of bounded variation on a subset $E \subseteq [a,b]$, then there exist a strictly increasing function $H : [a,b] \to \mathbb{R}$ and a gauge $\delta : E \to \mathbb{R}_+$ such that*

$$x \in E \text{ and } |y - x| < \delta(x) \text{ imply } \big|F(y) - F(x)\big| \le \big|H(y) - H(x)\big|.$$

Proof. There exists a gauge $\delta : E \to \mathbb{R}_+$ such that $W_F(S) < m_F(E) + 1$ for every δ-fine system S on E. Then the function

$$H(x) := x + \sup\big\{W_F(S) \big/ S \in \mathcal{S}(E,\delta) \text{ and } S \subseteq [a,x]\big\}$$

satisfies the desired condition (easy verification). □

Lemma 3.6. *A function $F : [a,b] \to \mathbb{R}$ is of generalized bounded variation if and only if there exists a strictly increasing function $H : [a,b] \to \mathbb{R}$ such that*

$$|D|_H F(x) := \limsup_{y \to x} \left| \frac{F(y) - F(x)}{H(y) - H(x)} \right| < \infty \text{ for every } x \in [a,b].$$

Proof. ($\Rightarrow$) By definition one has $[a,b] = \bigcup_{n=1}^{\infty} E_n$ with $m_F(E_n) < \infty$ for every $n \in \mathbb{N}$. Considering for each integer n a function $H_n : [a,b] \to \mathbb{R}$ and a gauge $\delta_n : E_n \to \mathbb{R}_+$ as in the preceding lemma, one defines the function

$$H(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{H_n(x) - H_n(a)}{H_n(b) - H_n(a)}.$$

For $x \in E_n$ one remarks that $|D|_H F(x) \le 2^n \big(H_n(b) - H_n(a)\big)$.

($\Leftarrow$) For each set $E_n := \big\{x \in [a,b] \big/ |D|_H F(x) < n\big\}$ one easily proves the inequality $m_F(E_n) \le n\big(H(b) - H(a)\big)$. □

Remark 3.7. According to a theorem of Ward (cf. [9] page 236) it follows that a function $F : [a,b] \to \mathbb{R}$ is VBG$^\circ$ if and only if it is bounded and VBG$_*$ in the sense of Saks.

Lemma 3.8. *Let $H : [a,b] \to \mathbb{R}$ be a strictly increasing function and let A be a subset of $[a,b]$ with $m_H(A) = 0$. If the function $F : [a,b] \to \mathbb{R}$ satisfies $|D|_H F(x) < \infty$ for every $x \in A$, then one has $m_F(A) = 0$.*

Proof. We show that $m_F(A_n) = 0$, where $A_n := \big\{x \in A \big/ |D|_H F(x) < n\big\}$. Given $\varepsilon > 0$ there exists a gauge $\delta : A_n \to \mathbb{R}_+$ such that $W_H(S) < \varepsilon$ for every system $S \in \mathcal{S}(A_n, \delta)$. We may assume that $x \in A_n$ and $|y - x| < \delta(x)$ imply $|F(y) - F(x)| < n\,|H(y) - H(x)|$. Then $W_F(S) < n\,\varepsilon$ for every $S \in \mathcal{S}(A_n, \delta)$, and this proves that $m_F(A_n) \le n\,\varepsilon$. □

Lemma 3.9. *Let $H : [a,b] \to \mathbb{R}$ be a strictly increasing function. If a function $F : [a,b] \to \mathbb{R}$ satisfies $F'_H(x) = 0$ for every $x \in A$, then $m_F(A) = 0$.*

Proof. Easy verification (cf. Lemme 5 in [2]). □

Proposition 3.10. *Let $F, U : [a, b] \to \mathbb{R}$ be two VBG° functions. Then F is U-differentiable at U-almost every point of $[a, b]$.*

PROOF. Let $H_F, H_U : [a, b] \to \mathbb{R}$ be strictly increasing functions as in 3.6 and consider the function $H(x) := H_F(x) + H_U(x)$. By Theorem 3.2 the interval $[a, b]$ can be decomposed into the disjoint union of

1) a set E_1 where F and U are H-differentiable, and

2) a H-null set E_2.

By 3.9 the set $E_0 = \{x \in E_1 \,/\, U'_H(x) = 0\}$ is U-null, and by 3.8 the set E_2 is U-null. Now if $x \in E_1 \setminus E_0$, then one has $F'_U(x) = F'_H(x) \cdot U'_H(x)^{-1}$. □

4 The Fundamental Theorem

Throughout this section $U : [a, b] \to \mathbb{R}$ is a fixed continuous VBG° function.

Definition 4.1. A function $F : [a, b] \to \mathbb{R}$ is called *U-Lipschitzian* on a set $E \subseteq [a, b]$, or LZ_U on E, if there exists $C > 0$ such that $m_F(A) \leq C \cdot m_U(A)$ for every subset $A \subseteq E$. The function F is called *generalized U-Lipschitzian*, or LZG_U, if there exists some decomposition $[a, b] = \bigcup_{n=1}^{\infty} E_n$ such that F is U-Lipschitzian on each subset E_n.

Similarly, a function $F : [a, b] \to \mathbb{R}$ is called *U-absolutely continuous* on a set E, or AC_U on E, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $A \subseteq E$ and $m_U(A) < \delta$ imply $m_F(A) < \varepsilon$. And it is called *generalized U-absolutely continuous*, or ACG_U, if there exists some decomposition $[a, b] = \bigcup_{n=1}^{\infty} E_n$ such that F is U-absolutely continuous on each subset E_n.

Finally, one says that a function $F : [a, b] \to \mathbb{R}$ is *U-variationally normal*, or shortly *U-normal*, if $m_U(A) = 0$ implies $m_F(A) = 0$.

Lemma 4.2. *If the function U is of bounded variation on the set $E \subseteq [a, b]$, then the function $V(x) = m_U\big(E \cap [a, x]\big)$ is continuous.*

PROOF. Since U is continuous one has $m_U\big(E \cap [c, d]\big) = m_U\big(E \cap (c, d)\big)$ for every subinterval $[c, d] \subseteq [a, b]$. Now let x_n be a strictly increasing sequence with $x_0 = a$ and $\lim x_n = x$. We show that $V(x_n)$ converges to $V(x)$. Using the subadditivity of m_U, cf. Proposition 2.6, one obtains

$$V(x) \leq \sum_{n=1}^{\infty} m_U\big(E \cap [x_{n-1}, x_n)\big) = \sum_{n=1}^{\infty} m_U\big(E \cap (x_{n-1}, x_n)\big).$$

And using Proposition 2.6 once again one concludes that

$$\sum_{n=1}^{s} m_U\big(E \cap (x_{n-1}, x_n)\big) = m_U\big(E \cap \textstyle\bigcup_{n=1}^{s} (x_{n-1}, x_n)\big) = V(x_s)$$

for every $s \in \mathbb{N}$. Thus $V(x) \leq \lim V(x_s) \leq V(x)$ and the assertion is proved. The continuity on the right side of x is proved similarly, by considering the function $V(b) - V(x) = m_U\big(E \cap (x, b]\big)$. □

Lemma 4.3. *Any LZG_U function is ACG_U, and any ACG_U function is VBG° and U-variationally normal.*

Proof. We show that if $F : [a, b] \to \mathbb{R}$ is AC_U and the function U is VB° on a set $E \subseteq [a, b]$, then F is VB° on E (the other affirmations are evident). We consider the function $V(x)$ of the preceding lemma. By definition there exists $\delta > 0$ such that $A \subseteq E$ and $m_U(A) < \delta$ imply $m_F(A) < 1$. And by continuity of the function V we can choose a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that $V(x_i) - V(x_{i-1}) = m_U\big(E \cap [x_{i-1}, x_i]\big) < \delta$ for every $i = 1, \ldots, n$. Thus we obtain $m_F(E) < n$, and the assertion is proved. □

Proposition 4.4. *Let $f : [a, b] \to \mathbb{R}$ be integrable relatively to U. Then the indefinite integral $F(x) = \int_a^x f \, \mathrm{d}U$ is LZG_U.*

Proof. We show that if the function U is VB° on the set $E \subseteq [a, b]$, then F is LZ_U on each subset $E_n := \big\{x \in E \big/ |f(x)| \leq n\big\}$. So let $A \subseteq E_n$ be a fixed subset. Given $\varepsilon > 0$ there exist two gauges δ_1 on $[a, b]$ and δ_2 on A such that

1) $\big|S(f, U, D) - \int_a^b f \, \mathrm{d}U\big| < \varepsilon$ for every δ_1-fine division D of $[a, b]$,

2) $W_U(S) < m_U(A) + \varepsilon$ for every system $S \in \mathcal{S}(A, \delta_2)$.

We consider the gauge $\delta : A \to \mathbb{R}_+$ defined by $\delta(x) = \min\big(\delta_1(x), \delta_2(x)\big)$. Now let S be any δ-fine system on A. By Saks-Henstock Lemma we have

$$W_F(S) = \sum_{i=1}^r \big|F(b_i) - F(a_i)\big| \leq \sum_{i=1}^r \big|f(x_i)\big(U(b_i) - U(a_i)\big)\big| + \sum_{i=1}^r \big|F(b_i) - F(a_i) - f(x_i)\big(U(b_i) - U(a_i)\big)\big| \leq n\big(m_U(A) + \varepsilon\big) + 2\varepsilon.$$

Thus we obtain $m_F(A) \leq n \cdot m_U(A) + (n + 2)\,\varepsilon$, and since ε is arbitrary this proves that F is U-Lipschitzian on the set E_n. □

For the next proposition it is useful to introduce some notations. Given a function F on $[a, b]$ we put $E_F = \big\{x \in [a, b] \big/ F \text{ is not } U\text{-differentiable at } x\big\}$, and we define the derivative $D_U F : [a, b] \to \mathbb{R}$ by $D_U F(x) = F'_U(x)$ if $x \notin E_F$ and $D_U F(x) = 0$ if $x \in E_F$.

Proposition 4.5. *Let $F : [a, b] \to \mathbb{R}$ be a function such that $m_F(E_F) = 0$. Then the derivative $D_U F$ is integrable relatively to U. Furthermore, one has $\int_a^x D_U f \, \mathrm{d}U = F(x) - F(a)$ for every $x \in (a, b]$.*

PROOF. We show that $\int_a^b D_U f \, \mathrm{d}U = F(b) - F(a)$. Let $[a,b] = \bigcup_{n=1}^{\infty} E_n$ be a disjoint decomposition such that $m_U(E_n) < \infty$ for every $n \in \mathbb{N}$. There exists for each n a gauge $\delta_n' \colon E_n \to \mathbb{R}_+$ such that $W_U(S_n) < m_U(E_n) + 1$ for every system $S_n \in \mathcal{S}(E_n, \delta_n')$. Define $\varepsilon_n > 0$ by $2^n \varepsilon_n \big(m_U(E_n) + 1\big) = \varepsilon$. For each $x \in E_n \setminus E_F$ there exists $\delta_n(x) > 0$ such that $|y - x| < \delta_n(x)$ implies

$$\big|F(y) - F(x) - F_U'(x)\big(U(y) - U(x)\big)\big| \leq \varepsilon_n \big|U(y) - U(x)\big|.$$

One may assume that $\delta_n(x) \leq \delta_n'(x)$. And by hypothesis there exists a gauge $\delta : E_F \to \mathbb{R}_+$ such that $W_F(S) < \varepsilon$ for every system $S \in \mathcal{S}(E_F, \delta)$. One thus gets a gauge $\delta : [a,b] \to \mathbb{R}_+$. Now let D be any δ-fine division of the interval $[a,b]$. Then one has the following inequality:

$$\begin{aligned}
&\big|S(D_U F, U, D) - F(b) + F(a)\big| \leq \textstyle\sum_{x_i \in E_F} \big|F(b_i) - F(a_i)\big| + \\
&\sum_{n=1}^{\infty} \textstyle\sum_{x_i \in E_n \setminus E_F} \big|F_U'(x_i)\big(U(b_i) - U(x_i)\big) - \big(F(b_i) - F(x_i)\big)\big| + \\
&\sum_{n=1}^{\infty} \textstyle\sum_{x_i \in E_n \setminus E_F} \big|F_U'(x_i)\big(U(x_i) - U(a_i)\big) - \big(F(x_i) - F(a_i)\big)\big| < \\
&\varepsilon + \sum_{n=1}^{\infty} \varepsilon_n W_U({S_n}^+) + \sum_{n=1}^{\infty} \varepsilon_n W_U({S_n}^-) \leq \varepsilon + 2 \sum_{n=1}^{\infty} 2^{-n} \varepsilon = 3\varepsilon,
\end{aligned}$$

and this proves that $D_U F$ is integrable with respect to U. □

Corollary 4.6. *Let $F : [a,b] \to \mathbb{R}$ be a continuous function. If there exists a denumerable set $D \subseteq [a,b]$ such that F is U-differentiable on $[a,b] \setminus D$, then $F(x) = F(a) + \int_a^x D_U F \, \mathrm{d}U$ for every $x \in (a,b]$.*

PROOF. This is immediate since $m_F(D) = 0$, cf. Remark 3.4. □

Theorem 4.7. *For a function $F : [a,b] \to \mathbb{R}$ the following are equivalent:*

1) *F is an indefinite integral relatively to U,*
2) *F is LZG_U,*
3) *F is ACG_U,*
4) *F is VBG° and U-normal,*
5) *F is U-differentiable U-almost everywhere and U-normal.*

PROOF. This follows from Propositions 4.4, 4.3, 3.10 and 4.5 (another equivalent condition will be given in Corollary 5.6). □

Corollary 4.8. *Let $f : [a,b] \to \mathbb{R}$ be a U-integrable function and let $F(x) = \int_a^x f \, \mathrm{d}U$ be its indefinite integral. Then $F_U'(x) = f(x)$ U-almost everywhere.*

PROOF. By 2.9 the set $\big\{x \in [a,b] \big/ f(x) \neq D_U F(x)\big\}$ is U-null. □

5 The Lusin Condition (N)

Let $F : [a, b] \to \mathbb{R}$ be fixed. We want to compare the two following conditions (where m denotes the Lebesgue outer measure):

1) $m(A) = 0$ implies $m_F(A) = 0$ (see Definition 4.1), and

2) $m(A) = 0$ implies $m\big(F(A)\big) = 0$, i.e. the Lusin condition (N).

Lemma 5.1. *For any set $A \subseteq [a, b]$ with $m_F(A) = 0$ one has $m\big(F(A)\big) = 0$.*

PROOF. Given $\varepsilon > 0$ there exists a gauge $\delta : A \to \mathbb{R}_+$ such that $W_F(S) < \varepsilon$ for every system $S \in \mathcal{S}(A, \delta)$. By the so-called Covering Lemma (McLeod [7] page 143) there exist two (possibly finite) sequences of non-overlapping intervals $I_n = [a_n, b_n]$ and of points $x_n \in I_n \cap A$ such that

$$I_n \subseteq \big(x_n - \delta(x_n), x_n + \delta(x_n)\big) \text{ for every } n, \text{ and } A \subseteq \textstyle\bigcup_n I_n.$$

For each n we define $m_n = \inf(F, I_n)$ and $M_n = \sup(F, I_n)$, and we choose a point $y_n \in I_n$ with $M_n - m_n \leq 3\,|F(y_n) - F(x_n)|$. For every finite sum one has $\sum_{n=1}^{r}(M_n - m_n) \leq 3\,W_F(S_r) < 3\varepsilon$. Therefore $\sum_n (M_n - m_n) \leq 3\varepsilon$, and this shows that $m\big(F(A)\big) \leq 3\varepsilon$ since $F(A) \subseteq \bigcup_n [m_n, M_n]$. □

Lemma 5.2. *Let $C_F = \big\{x \in [a, b] \,/\, y \leq x \leq z$ implies $F(y) \leq F(x) \leq F(z)\big\}$. If the function F is continuous on a subset $A \subseteq C_F$ satisfying $m\big(F(A)\big) = 0$, then one has $m_F(A) = 0$.*

PROOF. Given $\varepsilon > 0$ there exists by hypothesis a gauge $\eta : F(A) \to \mathbb{R}_+$ such that $W_{\mathrm{id}}(T) < \varepsilon$ for every system $T \in \mathcal{S}\big(F(A), \eta\big)$, cf. Remark 2.7 (one may also work with the usual definition of sets of measure zero). By continuity of F there exists a gauge $\delta : A \to \mathbb{R}_+$ such that $x \in A$ and $|y - x| < \delta(x)$ imply $\big|F(y) - F(x)\big| < \eta\big(F(x)\big)$. If S is any δ-fine system on A, then one has

$$W_F(S) = \sum_{i=1}^{r} \big(F(b_i) - F(x_i) + F(x_i) - F(a_i)\big) = W_{\mathrm{id}}(T_1) + W_{\mathrm{id}}(T_2)$$

(use that $x_i \in C_F$ for every $1 \leq i \leq r$), and therefore $m_F(A) \leq 2\varepsilon$. □

Proposition 5.3. *For a subset $A \subseteq [a, b]$ the following are equivalent:*

1) $m_F(A) = 0$,

2) F is continuous on A, $m\big(F(A)\big) = 0$ and $m_F(A) < \infty$.

PROOF. ($2 \Rightarrow 1$) Since F is of bounded variation on A there exists by 3.5 a strictly increasing function $H : [a,b] \to \mathbb{R}$ and a gauge $\delta : A \to \mathbb{R}_+$ such that $x \in A$ and $|y-x| < \delta(x)$ imply $\big|F(y) - F(x)\big| \le \big|H(y) - H(x)\big|$. We remark that this implies $m_F(N) = 0$ for every subset $N \subseteq A$ satisfying $m_H(N) = 0$. Since by Theorem 3.2 the set $E = \big\{x \in A \big/ F \text{ is not } H\text{-differentiable at } x\big\}$ is H-null we deduce that $m_F(E) \le m_F(N) + m_F(D) = 0$.

By Lemma 3.9 the set $A_0 = \big\{x \in A \big/ F'_H(x) = 0\big\}$ satisfies $m_F(A) = 0$. So it remains to consider the sets $A_\pm = \big\{x \in A \big/ \pm F'_H(x) > 0\big\}$. Obviously, one has $A_+ \subseteq \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \big\{x \in A \big/ x \in [y,z] \subseteq (x - \tfrac{1}{n}, x + \tfrac{1}{n}) \Rightarrow F(y) \le F(x) \le F(z)\big\}.$$

By the preceding lemma one obtains $m_F(A_n) = 0$ for every $n \in \mathbb{N}$ (divide the interval $[a,b]$ into finitely many small intervals). Therefore $m_F(A_+) = 0$, and similarly $m_F(A_-) = 0$, which proves the proposition. □

Question 5.4. *The example of Saks ([9] p. 224) shows that the hypothesis $m_F(A) < \infty$ cannot be released. But could one put in place of it the weaker assumption that F is differentiable almost everywhere? Or in other words, is there any function satisfying the Lusin condition (N) that is continuous and differentiable almost everywhere when not VBG°?*

Definition 5.5. Let $U : [a,b] \to \mathbb{R}$ be a continuous VBG° function as in the preceding section. One says that a function $F : [a,b] \to \mathbb{R}$ satisfies the *Lusin condition U-(N)* if $m\big(U(A)\big) = 0$ implies $m\big(F(A)\big) = 0$.

Corollary 5.6. *For a function $F : [a,b] \to \mathbb{R}$ the following are equivalent;*

1) *F is an indefinite integral with respect to U,*

6) *F is continuous, VBG° and it satisfies the Lusin condition U-(N).*

PROOF. Using Proposition 5.3 one obtains $m\big(U(A)\big) = 0$ iff $m_U(A) = 0$, and similarly $m\big(F(A)\big) = 0$ iff $m_F(A) = 0$. □

As another corollary of Proposition 5.3 we give the following substitution theorem for the Kurzweil-Henstock integral (which might be proved also by a more direct method):

Corollary 5.7. *Let $U : [a,b] \to \mathbb{R}$ be continuous and VBG°, and consider the interval $[c,d] = U\big([a,b]\big)$. If the function $f : [c,d] \to \mathbb{R}$ is measurable and bounded, then $f \circ U$ is integrable relatively to U and $\int_a^b f \circ U \, \mathrm{d}U = \int_{U(a)}^{U(b)} f$.*

PROOF. Let $F(y) = \int_c^y f$ be the indefinite integral of f. Clearly, the function F is Lipschitzian, and this implies that $F \circ U$ is LZG_U. By the fundamental theorem 4.7 we obtain $\int_{U(a)}^{U(b)} f = F\big(U(b)\big) - F\big(U(a)\big) = \int_a^b D_U(F \circ U)\,\mathrm{d}U$. So we are led to consider the following sets:

1) $A = \big\{x \in [a,b] \big/ F \circ U \text{ is not } U\text{-differentiable at } x\big\}$,

2) $B = \big\{x \notin A \big/ (F \circ U)'_U(x) \neq f\big(U(x)\big)\big\}$,

3) $C = \big\{y \in [c,d] \big/ F \text{ is not differentiable at } y \text{ or } F'(y) \neq f(y)\big\}$,

One has $m_U(A) = 0$ by Theorem 3.10 (use that U is continuous). And since $U(B) \subseteq C$ is of measure zero one gets $m_U(B) = 0$ by Proposition 5.3. Hence the set $E = \big\{x \in [a,b] \big/ D_U(F \circ U)\,(x) \neq f\big(U(x)\big)\big\}$ satisfies $m_U(E) = 0$, and the assertion follows from Proposition 2.9. □

In particular, if U is an indefinite integral, i.e. $U(x) = U(a) + \int_a^x g$, then $(f \circ U) \cdot g$ is integrable and $\int_a^b (f \circ U) \cdot g = \int_{U(a)}^{U(b)} f$ (left as an exercise).

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