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ON THE APPROXIMATELY CONTINUOUS INTEGRALS OF BURKILL AND KUBOTA

Abstract

The exact relations between the approximately continuous Perron and Denjoy integrals of Burkill [1] and Kubota [5, 6] are re-established by rectifying the faulty proofs of Kubota, and the related questions of Gordon [3] are resolved completely.

In [2, p. 269] Gordon asked: *Is there an approximately continuous integral that includes both the general Denjoy integral, $\mathcal{D}$-integral [10], and the approximately continuous Perron integral, AP-integral [1], of Burkill*? This is a pertinent question since Tolstoff [17, p. 658] gave a function which is $\mathcal{D}$-integrable but not AP-integrable. But this question was resolved long ago by the author in the affirmative (see [15, p. 352], [12, 13]), by introducing the (T_aP)- and (T_aD)- integrals where T_a is the approximate limit process.

A still earlier solution is the approximately continuous Denjoy integral, AD-integral [5], and its equivalent the AP^*-integral [6], of Kubota. But in [3] Gordon asked the same question again, referring to a flaw in Kubota's proof [5, Theorem 2] that the AD-integral includes the AP-integral, and pointing out also certain flaws both in the indirect attempt of Lee [7] and in the direct attempt of Lin [8] to rectify Kubota's proof.

In this note we assume that the reader is familiar with the notions of VB, AC, VBG, Lusin's condition (N), and approximate continuity and derivative [10]. Also, we refer the reader to [15, p. 337] for the precise definitions of the following concepts: AC above, AC below, ACG above, ACG below, ACG, (ACG) above, (ACG) below, (ACG), (VBG) and (PAC). We mention that, a function F is ACG [resp. VBG] on $[a,b]$ if $[a,b]$ is the union of a sequence of sets $\{E_n\}_{n=1}^{\infty}$ such that F is AC [resp. VB] on each E_n; if further each E_n can be taken to be closed, then F is said to be (ACG) [resp. (VBG)] on $[a,b]$. Note that F is not required to be continuous on $[a,b]$. For (PAC) we shall use the following equivalent definition:

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Definition (13, p. 296)**.** A function $F : [a, b] \to \mathbb{R}$ is (PAC) on $[a, b]$ if for every $\epsilon > 0$ there exist an increasing sequence of sets $\{E_n\}$ with union $[a, b]$ and a sequence of positive numbers $\{r_n\}$, such that for every n and every finite family $\{(a_i, b_i)\}$ of pairwise disjoint open intervals with endpoints in E_n and with $\sum_i (b_i - a_i) < r_n$, we have $\sum_i |F(b_i) - F(a_i)| < \epsilon$.

Now, we find that Kubota used in fact the following type of fallacious arguments in three of his papers [4, 5, 6]:

- If $\{M_n\}$ is a sequence of functions each of which is (ACG) below on $[a, b]$, then there exists a sequence of closed sets $\{E_k\}$ with union $[a, b]$ such that each M_n is AC below on every E_k.

In the absence of other conditions, this is certainly not a valid argument (see [3, p. 837]). We will show, however, that Kubota's results [5, Theorem 2] and [6, Theorem 3.6] are correct in the context of these two papers. This resolves, in particular, the specific question 2 of Gordon [3, p. 838] in the affirmative. We make no attempt to rectify the proof of [4, Theorem 4.1], as it appears that Kubota abandoned [4] in favour of [5].

The related specific question 1 of Gordon [3, p. 838] is in essence the following: *If a function F satisfies the Lusin's condition (N) and is approximately continuous and VBG on $[a, b]$, then must F be (ACG) on $[a, b]$?* The answer to this is an emphatic **NO**. Dwelling on this point, long ago the author constructed a function [15, Example 3.1, p. 342] which is approximately continuous and (PAC), but not even ACG below or ACG above on $[a, b]$. It is to be noted that, by [15, Theorem 3.6], a function is (PAC) on $[a, b]$ iff it satisfies the condition (N) and is (VBG) on $[a, b]$.

In this connection Gordon obtained a set of sufficient conditions [3, Theorem 4] for a function to be Baire* 1 on $[a, b]$. But this theorem is only a very special case of a more extensive result of the author [16, Theorem 2.1, p. 14]. It should be mentioned here that, Sargent [11, p. 117] calls a function F *continuous in the generalized sense*, (CG), on $[a, b]$ if $[a, b]$ is the union of a sequence of closed sets $\{E_n\}$ such that $F_{|E_n}$ is continuous for each n, and O'Malley [9] calls such a function Baire* 1.

As a solution to his opening question Gordon offered the AK_N-integral [3, p. 834], using the concept of VBG_N functions. But there appears to be a serious oversight in his proof of the uniqueness of this integral, as it is not at all obvious that the difference of two VBG_N functions always satisfies the Lusin's condition (N). The difficulty lies in the use of the condition VBG rather than (VBG). But if the condition VBG_N is replaced by $(VBG)_N$, then the resulting AK_N-integral reduces to the (T_aP) and (T_aD)- integrals [12, 13, 15].

We now prove the two results of Kubota. The derivative and the upper and lower derivatives, in the approximate sense, of a function F will be denoted by F'_{ap}, $\overline{AD}F$ and $\underline{AD}F$, respectively. The Lebesgue measure of a set E will be denoted by $|E|$. We consider functions

$$f : [a,b] \to [-\infty, +\infty] \quad \text{and} \quad M, m : [a,b] \to (-\infty, +\infty).$$

Burkill's AP-integral [1, § 3] can be defined as follows.

If $-\infty \neq \underline{AD}M(x) \geq f(x)$ for each x in $[a,b]$, $M(a) = 0$, and M is approximately continuous on $[a,b]$, then the function M is called an *AP-major function* of f on $[a,b]$.

If $\infty \neq \overline{AD}m(x) \leq f(x)$ for each x in $[a,b]$, $m(a) = 0$, and m is approximately continuous on $[a,b]$, then the function m is called an *AP-minor function* of f on $[a,b]$.

The function f is said to be *AP-integrable* on $[a,b]$ if f has both AP-major functions M and AP-minor functions m and $\inf\{M(b)\} = \sup\{m(b)\}$, and then this common finite value is defined to be the definite AP-integral of f on $[a,b]$, denoted by $(AP)\int_a^b f$.

We remark that Burkill assumed f to be measurable and finite almost everywhere. But these can be proved for AP-integrable f.

Kubota [5, § 3] defines the function f to be *AD-integrable* on $[a,b]$ if there is a function F which is approximately continuous and (ACG) on $[a,b]$ and is such that $F'_{ap}(x) = f(x)$ *a.e.* on $[a,b]$, and then $F(b) - F(a)$ is called the definite AD-integral of f on $[a,b]$, denoted $(AD)\int_a^b f$.

Theorem 1. *The AD-integral includes the AP-integral.*

Proof. Let f be AP-integrable on $[a,b]$. Then [1, § 4]

$$F(x) = (AP)\int_a^x f\,, \quad F(a) = 0\,, \quad a \leq x \leq b\,,$$

is well defined, F is approximately continuous on $[a,b]$, and $F'_{ap} = f$ *a.e.* on $[a,b]$. So the proof will be complete once we can show that F is (ACG) on $[a,b]$. To this end, by [15, Theorem 3.5, p. 340] it is enough to show that F is both (PAC) and (CG) on $[a,b]$.

To show that F is (PAC) on $[a,b]$ we use the method of proof of [14, Theorem 5.4, p. 39]. Given $\epsilon > 0$, select an AP-major function M and an AP-minor function m of f on $[a,b]$ such that

$$H(b) < \epsilon \quad \text{where} \quad H = M - m\,.$$

For each positive integer n and each point x in $[a,b]$, put

$$A_n^x = \left\{ y \in [a,b] \,:\, \frac{M(y)-M(x)}{y-x} \le -n \ \text{ or } \ \frac{m(y)-m(x)}{y-x} \ge n \right\}.$$

Then let E_n denote the set of points x in $[a,b]$ such that

$$|A_n^x \cap [u,v]| < \frac{1}{2}(v-u) \ \text{ if } \ x \in [u,v] \ \text{ and } \ v-u < \frac{1}{n}\,. \tag{1}$$

Since $A_{n+1}^x \subseteq A_n^x$, $\underline{AD}M(x) > -\infty$ and $\overline{AD}m(x) < \infty$ for all n, x, clearly $\{E_n\}$ is an increasing sequence of sets with union $[a,b]$.

Now, if $u, v \in E_n$ and $0 < v-u < 1/n$, then (1) implies that there are points $y \in (u,v) \setminus (A_n^u \cup A_n^v)$, and then we have

$$M(y)-M(u) > -n(y-u)\,, \quad m(y)-m(u) < n(y-u)\,,$$

$$M(v)-M(y) > -n(v-y)\,, \quad m(v)-m(y) < n(v-y)\,.$$

Since $M-F$ and $F-m$ are nondecreasing, we get

$$\begin{aligned} F(v)-F(u) \le M(v)-M(u) &= H(v)-H(u)+m(v)-m(u) \\ &< H(v)-H(u)+n(v-u) \end{aligned}$$

and

$$\begin{aligned} F(u)-F(v) \le m(u)-m(v) &= H(v)-H(u)+M(u)-M(v) \\ &< H(v)-H(u)+n(v-u)\,. \end{aligned}$$

Hence

$$|F(v)-F(u)| < H(v)-H(u)+n(v-u)\,.$$

Since H is nondecreasing on $[a,b]$, it follows that for each n and for every finite family of nonoverlapping intervals $\{[u_i,v_i]\}$ with endpoints in E_n and with $\sum_i (v_i-u_i) < \epsilon/n$, we have

$$\sum_i |F(v_i)-F(u_i)| < H(b)-H(a)+\epsilon = H(b)+\epsilon < 2\epsilon\,.$$

Hence F is (PAC) on $[a,b]$.

Finally, since M is approximately continuous and $\underline{AD}M > -\infty$ on $[a,b]$, as a special case of [16, Theorem 2.1, p. 14] M is (CG) on $[a,b]$. Also, $M-F$ is continuous on $[a,b]$ since it is nondecreasing and approximately continuous on $[a,b]$. Hence $F = M-(M-F)$ is (CG) on $[a,b]$. This completes the proof. □

Remark. Since F is (ACG) on $[a,b]$, there is a sequence of closed sets $\{B_n\}$ with union $[a,b]$ such that F is AC on each B_n. Then for all AP-major functions M and all AP-minor functions m of f on $[a,b]$, since $M-F$ and $F-m$ are nondecreasing on $[a,b]$, obviously each M is AC below and each m is AC above on every B_n. Thus the assertion of Kubota in his proof of [5, Theorem 2] is true, though not in his way.

Kubota's AP^*-integral [6, § 3] is defined as follows.

The function M is called an AP^*-*upper function* of f on $[a,b]$ if $M(a)=0$, M is approximately continuous and (ACG) below on $[a,b]$, and $M'_{ap}(x)\geq f(x)$ *a.e.* on $[a,b]$.

The function m is called an AP^*-*lower function* of f on $[a,b]$ if $m(a)=0$, m is approximately continuous and (ACG) above on $[a,b]$, and $m'_{ap}(x)\leq f(x)$ *a.e.* on $[a,b]$.

The function f is said to be AP^*-*integrable* on $[a,b]$ if f has both AP^*-upper functions M and AP^*-lower functions m on $[a,b]$ and $\inf\{M(b)\}=\sup\{m(b)\}$, and then this common finite value is defined to be the definite AP^*-integral of f on $[a,b]$, denoted $(AP^*)\int_a^b f$.

Theorem 2. *The AD-integral is equivalent to the AP^*-integral.*

PROOF. This was proved by Kubota [6, Theorem 3.6]. But, as discussed above, there is a flaw in his proof that the AD-integral includes the AP^*-integral. So we will prove only this part.

Let f be AP^*-integrable on $[a,b]$. Then [6, § 3]

$$F(x)=(AP^*)\int_a^x f\,,\quad F(a)=0\,,\quad a\leq x\leq b\,,$$

is well-defined, F is approximately continuous on $[a,b]$, and $F'_{ap}=f$ *a.e.* on $[a,b]$. So it remains only to show that F is (ACG) on $[a,b]$, that is that F is both (PAC) and (CG) on $[a,b]$.

Given $\epsilon>0$, select an AP^*-upper function M and an AP^*-lower function m of f on $[a,b]$ such that

$$H(b)<\epsilon \quad\text{where}\quad H=M-m\,.$$

Since M is (ACG) below and m is (ACG) above on $[a,b]$, we can find a sequence of closed sets $\{E_n\}$ with union $[a,b]$ such that, M is AC below and m is AC above on each E_n. Then for each n there is a $\delta_n>0$ such that, for every finite family of nonoverlapping intervals $\{[a_p,b_p]\}$ with endpoints in E_n and with $\sum_p(b_p-a_p)<\delta_n$, we have

$$\sum_p(M(b_p)-M(a_p))>-\frac{\epsilon}{2^n}\quad\text{and}\quad\sum_p(m(b_p)-m(a_p))<\frac{\epsilon}{2^n}\,.$$

Now, by [15, Lemma 2.1, p. 337], there is an increasing sequence of closed sets $\{F_n\}$ with union $[a,b]$ such that

$$F_n = \cup_{k=1}^{n} F_{kn}\,, \quad F_{kn} \subseteq E_k\,, \quad \operatorname{dist}(F_{in}, F_{jn}) \geq \frac{1}{n} \ \text{ for } i \neq j\,.$$

Consider any n and any finite family of nonoverlapping intervals $\{[a_p, b_p]\}$ with endpoints in F_n and with

$$\sum_p (b_p - a_p) < \min\left\{\frac{1}{n}, \delta_1, \ldots, \delta_n\right\}.$$

Since $\operatorname{dist}(F_{in}, F_{jn}) \geq 1/n$ for $i \neq j$, so for each p both a_p and b_p must belong to precisely one of the sets F_{kn}, $k = 1, 2, \ldots, n$. Then, since $F_{kn} \subseteq E_k$, we clearly have

$$\sum (M(b_p) - M(a_p)) = \sum_{k=1}^{n} \sum_{a_p \in F_{kn}} (M(b_p) - M(a_p)) > \sum_{k=1}^{n} \frac{-\epsilon}{2^k} > -\epsilon\,,$$

$$\sum (m(b_p) - m(a_p)) = \sum_{k=1}^{n} \sum_{a_p \in F_{kn}} (m(b_p) - m(a_p)) < \sum_{k=1}^{n} \frac{\epsilon}{2^k} < \epsilon\,.$$

Since $M - F$, $F - m$ and H are nondecreasing, we get

$$\begin{aligned}\sum (F(b_p) - F(a_p)) &\leq \sum (M(b_p) - M(a_p)) \\ &= \sum (H(b_p) - H(a_p)) + \sum (m(b_p) - m(a_p)) \\ &< H(b) - H(a) + \epsilon = H(b) + \epsilon < 2\epsilon\,,\end{aligned}$$

$$\begin{aligned}\sum (F(a_p) - F(b_p)) &\leq \sum (m(a_p) - m(b_p)) \\ &= \sum (H(b_p) - H(a_p)) + \sum (M(a_p) - M(b_p)) \\ &< H(b) - H(a) + \epsilon = H(b) + \epsilon < 2\epsilon\,.\end{aligned}$$

Thus $|\sum (F(b_p) - F(a_p))| < 2\epsilon$. Hence, clearly, $\sum |F(b_p) - F(a_p)| < 4\epsilon$. Hence, as before, by definition F is (PAC) on $[a,b]$.

Finally, both $M - F$ and $F - m$ are continuous on $[a,b]$ since they are nondecreasing and approximately continuous on $[a,b]$. Since, further, M is AC below and m is AC above on each E_n, it follows readily from $F = M - (M - F) = (F - m) + m$ that $F_{|E_n}$ is continuous for each n. Hence F is (CG) on $[a,b]$, which completes the proof. □

References

[1] J. C. Burkill, *The approximately continuous Perron integral*, Math. Zeit. **34** (1931), 270–278.

[2] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Mathematics, AMS **4**, Providence 1994.

[3] R. A. Gordon, *Some comments on an approximately continuous Khintchine integral*, Real Analysis Exchange **20** (1994-95), no. 2, 831–841.

[4] Y. Kubota, *On the approximately continuous Denjoy integrals*, Tôhoku Math. J. **15** (1963), 253–264.

[5] Y. Kubota, *An integral of Denjoy type*, Proc. Japan Acad. **40** (1964), 713–717.

[6] Y. Kubota, *An integral of Denjoy type. II*, Proc. Japan Acad. **42** (1966), no. 7, 737–742.

[7] C. M. Lee, *An analogue of the theorem of Hake-Alexandroff-Looman*, Fund. Math. **C** (1978), 69–74.

[8] Y. Lin, *A note on that Kubota's AD-integral is more general than Burkill's AP-integral*, J. Mathematical Study **27** (1994), 116–120.

[9] R. J. O'Malley, *Baire* 1, Darboux functions*, Proc. Amer. Math. Soc. **60** (1976), 187–192.

[10] S. Saks, *Theory of the integral*, 2nd. rev. ed., vol. PWN, Monografie Matematyczne, Warsaw, 1937.

[11] W. L. C. Sargent, *Some properties of $\mathcal{C}_\lambda$-continuous functions*, J. London Math. Soc. **26** (1951), 116–121.

[12] D. N. Sarkhel, *A wide Perron integral*, Bull. Austral. Math. Soc. **34** (1986), 233–251.

[13] D. N. Sarkhel, *A wide constructive integral*, Math. Japonica **32** (1987), 295–309.

[14] D. N. Sarkhel and A. K. De, *The proximally continuous integrals*, J. Austral. Math. Soc. (Series A) **31** (1981), 26–45.

[15] D. N. Sarkhel and A. B. Kar, *(PVB) functions and integration*, J. Austral. Math. Soc. (Series A) **36** (1984), 335–353.

[16] D. N. Sarkhel and P. K. Seth, *On some generalized approximative relative derivatives*, Rendi. Circ. Mat. Palermo (2) **35** (1986), 5–21.

[17] G. P. Tolstoff, *Sur l'intégrale de Perron*, Mat. Sb. **5** (1939), no. 47, 647–660.

The author wishes to thank the referee for the following additional references concerning the AP^*- and AD-integrals.

[18] J. Ridder, *Über approximativ stetige Denjoy-Integrale*, Fund. Math. **21** (1933), 1–10.

[19] J. Ridder, *Über die gegenseitigen Beziehungen vereschiedener approximativ stetigen Denjoy-Perron Integrale*, Fund. Math. **22** (1934), 136–162.

The AP^*-integral was first introduced by Ridder in [18], (he calls it the D_4-integral in Definition 3, p. 5). It also appears in [19] (Definition 8, p. 149).

The AD-integral is in fact the β-Ridder integral introduced in [19] (Definition 7, p. 148). In fact in [18, p. 6] Ridder asserts that the AP^*-integral is equivalent to the AD- integral (with his notations of course) and makes the same faulty proof as Kubota.

The following related paper appeared after the acceptance of the present paper.

[20] C. M. Lee, *Kubota's AD-integral is more general than Burkill's AP-integral*, Real Analysis Exchange **22** (1996-97), no. 1, 433–436.