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VARIATION-DIMINISHING WAVELETS AND WAVELET TRANSFORMS

Abstract

Using Schoenberg's theory of variation-diminishing integral operators of convolution type variation diminishing wavelets and wavelets of specific changes in sign are constructed. An inversion formula involving derivatives of the wavelet transform is established. Wavelets generated by Tanno's form of convolution kernels and H-functions are also investigated. Results are illustrated by means of examples and figures.

1 Introduction

A function $\phi(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with the property

$$\int_{-\infty}^{\infty} \phi(t)\, dt = 0 \tag{1.1}$$

is called a wavelet (small wave), because these conditions imply that $\phi(t)$ changes sign and $\phi(t)$ vanishes for large values of $|t|$. Sometimes it is assumed that the Fourier transform $\hat{\phi}(w)$ of $\phi(t)$ satisfies the so called "admissibilty condition" [1, p.60]:

$$C_\phi := \int_{-\infty}^{\infty} \frac{|\hat{\phi}(w)|^2}{|w|}\, dw < \infty, \tag{1.2}$$

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the existence of this integral requires that $\hat{\phi}(0)=0$; which yields (1.1). Construction of new wavelets possessing specified properties is one of the interesting problems in wavelet analysis. It is well known that if ψ is a wavelet and ϕ is a bounded integrable function on $\mathbb{R}$ then the convolution function $\psi * \phi$ is a wavelet [2, Theorem 6.2.1, p. 369]. The new wavelet possesses differentiability and other properties even though ψ may not be differentiable. Moreover, wavelet transform of ϕ is also a convolution of ϕ with dilated ψ. Hence the theory of convolution transforms [6] seems to be applicable for development of theory of wavelets and wavelet transforms [1, 11], a brief account of which is given below.

Let $G(t) \in L^1(\mathbb{R})$ and let

$$f(x) := (G * \phi)(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)\, dt, \tag{1.3}$$

where $\phi(t)$ is bounded and continuous. The kernel $G(t)$ is said to be *variation diminishing* if the number of changes of sign of $f(x)$ for $-\infty < x < \infty$ never exceeds the number of changes of sign of $\phi(t)$ for $-\infty < t < \infty$; i.e.,

$$V[G * \phi] \leq V[\phi], \tag{1.4}$$

where $V[\phi]$ denotes number of changes of sign of ϕ on $\mathbb{R}$.

It has been shown by Schoenberg [12] that $G(t)$ is variation diminishing if and only if (after multiplication by a suitable constant)

$$\hat{G}(w) := \int_{-\infty}^{\infty} G(t)e^{-iwt}\, dt = [e^{cw^2+ibw} \prod_{k=1}^{\infty} (1 - \frac{iw}{a_k}) e^{iw/a_k}]^{-1}, \tag{1.5}$$

where the a_k's are real, $\sum_{k=1}^{\infty} a_k^{-2} < \infty$, b is real and c is real and non-negative. From (1.5) it follows that

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwt} [e^{cw^2+ibw} \prod_{k=1}^{\infty} (1 - \frac{iw}{a_k}) e^{iw/a_k}]^{-1}\, dw \tag{1.6}$$

and

$$G^{(r)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (iw)^r e^{iwt} [e^{cw^2+ibw} \prod_{k=1}^{\infty} (1 - \frac{iw}{a_k}) e^{iw/a_k}]^{-1}\, dw, \quad r \in \mathbb{Z}_+. \tag{1.7}$$

Examples of $G(t)$ [6, p. 4] are

$$G(t) = \frac{1}{2} e^{-|t|}, \quad e^{-e^t} e^t, \quad (2\pi)^{-1} \operatorname{sech}(t/2), \quad g(t), \tag{1.8}$$

where

$$g(t) = \begin{cases} e^t & -\infty < t < 0 \\ 0 & 0 \leq t < \infty. \end{cases}$$

Notice that in (1.6) the infinite product may be reduced to finite number of factors by assuming that after a certain point $k = N, a_k = \infty \ \forall k = N+1, \cdots, \infty$. The kernel $G(t)$ is said to be of degree N.

If $G(t)$ is of degree N, then from [6, p. 91],

$$G^{(r)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (iw)^r e^{iwt} [e^{cw^2 + ibw} \prod_{k=1}^{N} (1 - \frac{iw}{a_k}) e^{iw/a_k}]^{-1} \, dw \tag{1.9}$$

has exactly r changes of sign for $r = 0, 1, \cdots, N-1$.

Next, we state certain properties of Pólya class E of entire functions [6, p. 42] to be used in the sequel.

Definition 1.1. *An entire function $E(s)$ belongs to class E, if and only if it is of the form*

$$E(s) = e^{-cs^2 + bs} \prod_{k=1}^{\infty} (1 - \frac{s}{a_k}) e^{s/a_k}, \tag{1.10}$$

where $c \geq 0, b, a_k (k = 1, 2, \cdots)$ are real and

$$\sum_k a_k^{-2} < \infty. \tag{1.11}$$

Examples of functions in the class E [6, p. 42] are 1, $(1-s)$, e^s, e^{-s^2}, $\sin s/s$, $1/\Gamma(1-s)$.

Product of any two elements of class E is also an element of class E.

We shall derive an inversion formula for the wavelet transform using derivatives of the transform. For this purpose we shall exploit the following inversion theorem for the convolution transform valid for $c = 0$ [6, Theorem 7.1, p. 57]. The case $c \neq 0$ may similarly be treated using other theorems of Hirschman and Widder [6], Tanno [13, 14] and Ditzian et al [3, 4, 5].

Theorem 1.1. *Let $c = 0$ and $E(s) \in E$ be given by (1.10) and let $G(t)$ be given by (1.6). Assume that $f(x)$ is given by (1.3), where ϕ is bounded and continuous. Set*

$$P_n(s) = e^{(b - \varepsilon_n)s} \prod_{k=1}^{n} (1 - \frac{s}{a_k}) e^{s/a_k}, \quad \varepsilon_n = o(1), \ n \to \infty.$$

Then

$$E(D)f(x) = \lim_{n \to \infty} P_n(D) f(x) = \phi(x), \qquad -\infty < x < \infty.$$

In this paper using the theory of convolution transform developed by Schoenberg [12] and Hirschman and Widder [6] we construct variation diminishing wavelets, wavelets with specific changes of sign and study their changes of trend. Using (1.9) we construct some new wavelets. Also we show that certain classical wavelets (or their Fourier transforms) can be derived from (1.9) as special cases. For wavelet transform involving such a wavelet we obtain an inversion formula in operational form. Tanno's kernel, a generalization of Hirchman-Widder kernel, is exploited to derive several other new wavelets in closed form. Wavelets generated by H-function [9] are also investigated.

2 Variation Diminishing Wavelets

In this section we exploit admissibility condition (1.2) for construction of variation diminishing wavelets.

Theorem 2.1. *Let ϕ be a wavelet. Let G be a variation diminishing kernel defined by (1.6), where $c \geq 0, b, a_k (k = 1, 2, \cdots)$ are real and (1.11) holds. Then $\phi * G$ is a variation diminishing wavelet.*

Proof. By (1.5) we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |(\phi * G)\hat{}(w)|^2 \frac{dw}{|w|} &= \int_{-\infty}^{\infty} |\hat{\phi}(w)|^2 \frac{|\hat{G}(w)|^2}{|w|}\, dw \\
&\leq \sup_w |e^{iw^2 + ibw} \prod_{k=1}^{\infty} (1 - \frac{iw}{a_k}) e^{iw/a_k}|^{-2} \int_{-\infty}^{\infty} \frac{|\hat{\phi}(w)|^2}{|w|}\, dw \\
&\leq \sup_w \frac{1}{\prod_k (1 + \frac{w^2}{a_k^2})} \cdot C_\phi \leq C_\phi < \infty.
\end{aligned}$$

Moreover, it is variation diminishing by the aforesaid result of Schoenberg [12]. □

Example 2.1. Let $G(t) = \frac{1}{2} e^{-|t|}$ be the variation diminishing kernel and $\theta(t)$ be the Haar wavelet [11, p. 10] defined by

$$\theta(t) := \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

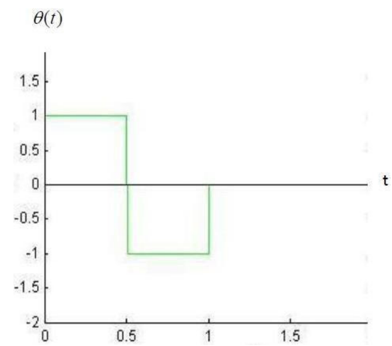

Figure 1: Haar wavelet $\theta(t)$

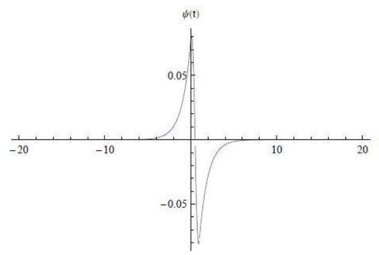

Figure 2: Wavelet $\psi(t)$

Then

$$\begin{aligned}\psi(t) &:= \int_{-\infty}^{\infty} G(t-x)\theta(x)\,dx \\ &= \begin{cases} \frac{1}{2}(1-e^{-\frac{1}{2}})^2 e^t & t \le 0 \\ \frac{1}{2}(2-2e^{t-\frac{1}{2}}+e^{t-1}-e^{-t}) & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(2e^{-t+\frac{1}{2}}-e^{-t}+e^{t-1}-2) & \frac{1}{2} \le t \le 1 \\ -\frac{1}{2}(e^{\frac{1}{2}}-1)^2 e^{-t} & t \ge 1. \end{cases}\end{aligned} \tag{2.2}$$

From figures 1 and 2, number of change of signs of $\theta(t)$ and $\psi(t)$ is 1.

Example 2.2. Let the variation diminishing kernel be

$$G(t) = \begin{cases} e^t & -\infty < t < 0 \\ 0 & 0 \le t < \infty \end{cases}$$

and

$$\theta(t) = (1 - t^2)e^{-\frac{1}{2}t^2} = -\frac{d^2}{dt^2}(e^{-\frac{1}{2}t^2})$$

be the Mexican hat wavelet. Then

$$\begin{aligned}\psi(t) &= (G * \theta)(t) = -\int_{-\infty}^{0} e^x \frac{d^2}{dx^2} e^{-\frac{1}{2}(t-x)^2}\, dx \\ &= \int_{0}^{\infty} e^{-y} \frac{d^2}{dy^2} e^{-\frac{1}{2}(t+y)^2}\, dy \\ &= (1-t)e^{-\frac{1}{2}t^2} - (\pi/2)^{\frac{1}{2}} e^{t+\frac{1}{2}} Erfc\left(\frac{t+1}{\sqrt{t}}\right).\end{aligned}$$

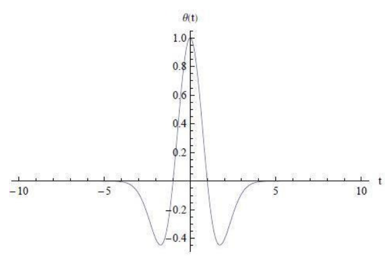

Figure 3: Maxican hat wavelet $\theta(t)$

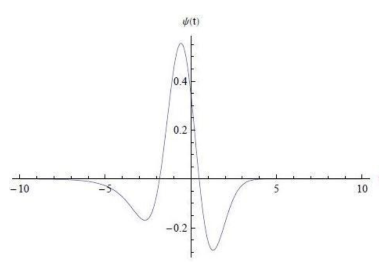

Figure 4: Wavelet $\psi(t)$

From figures 3 and 4 it is obvious that $\psi(t)$ has the same number of changes of sign as $\theta(t)$; in both cases it is 2.

3 Wavelets with specific changes of sign

We show that $\psi(t) := G^{(r)}(t)$, $r = 1, 2, \cdots$, where $G^{(r)}(t)$ may be given by (1.7) or (1.9) according as $G(t)$ is of infinite degree, or of finite degree, is a wavelet.

Theorem 3.1. *Let $c \geq 0$ and $G^{(r)}(t)$, $r = 1, 2, \cdots$,N-1, be defined by (1.9). Then $G^{(r)}(t)$ is a wavelet which has exactly r changes of sign. The points associated with changes of sign of $G^{(r)}(t)$, $r = 1, 2, \cdots$,N-3, are simple zeros of $G^{(r)}(t)$. In case $G^{(r)}(t)$ is defined by (1.7), then $G^{(r)}(t)$ has atmost r changes of sign.*

PROOF. Let $G(t)$ be of degree N. Then from (1.9) we have

$$\mathscr{F}[G^{(r)}(t)](w) = (iw)^r [e^{cw^2+ibw} \prod_{k=1}^{N} (1 - \frac{iw}{a_k}) e^{iw/a_k}]^{-1}.$$

Hence

$$\begin{aligned} C_\psi &= \int_{-\infty}^{\infty} |\mathscr{F}[G^{(r)}(t)](w)|^2 \frac{dw}{|w|} \\ &= \int_{-\infty}^{\infty} |w|^{2r-1} |e^{2cw^2} \prod_{k=1}^{N} (1 + \frac{w^2}{a_k^2})|^{-1}\, dw \\ &\leq \int_{-\infty}^{\infty} \frac{|w|^{2r-1} e^{-2cw^2}}{|1 + \frac{w^2}{a_k^2}|}\, dw < \infty \end{aligned}$$

for all $r > 0$ and $c > 0$. Therefore, for $r = 1, 2, \cdots$, $\psi(t) = G^{(r)}(t)$ satisfies admissibility condition (1.2); it represents a wavelet.

Next, assume that $c = 0$. Then

$$\begin{aligned} C_\psi &= \int_{-\infty}^{\infty} |w|^{2r-1} \left| \prod_{k=1}^{N} (1 + \frac{w^2}{a_k^2}) \right|^{-1} dw \\ &\leq \int_{-1}^{1} |w|^{2r-1}\, dw + C \int_{|w| \geq 1} |w|^{2r-2N-1}\, dw < \infty \end{aligned}$$

for $1 \leq r \leq N$ and certain constant $C > 0$. Thus, if $G(t)$ is of degree N, then $G^{(r)}(t)$ given by (1.9) with $c = 0$, is also a wavelet which has exactly r changes of sign $\forall\, r, 1 \leq r \leq N - 1$ [6, p. 91].The proof of the second part follows from [6, Theorem 5.3, p. 93].

Moreover, if $G(t)$ is of infinite degree then the corresponding C_ψ can also be shown to be bounded, and $G^{(r)}(t)$ is a wavelet having atmost $r, 1 \le r < \infty$, changes of sign [6, p. 92]. □

Example 3.1. As in [6, p. 35], let us choose $a_k = k$, $k = 1, 2, \cdots, n$. Then

$$E(s) = (1 - \frac{s}{1})(1 - \frac{s}{2}) \cdots (1 - \frac{s}{n})$$

and

$$G(t) = \begin{cases} ne^t(1-e^t)^{n-1} & -\infty < t < 0 \\ 0 & 0 < t < \infty \end{cases}$$

is of finite degree. Then

$$\begin{aligned} |G^{(r)}(t)| &= \left| \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{s^r e^{st}}{(1-\frac{s}{1})(1-\frac{s}{2})\cdots(1-\frac{s}{n})}\, ds \right| \\ &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{e^{iyt}(iy)^r}{(1-iy)(1-\frac{iy}{2})\cdots(1-\frac{iy}{n})} \right| dy \\ &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|y|^r}{[(1+y^2)(1+\frac{y^2}{2^2})\cdots(1+\frac{y^2}{n^2})]^{\frac{1}{2}}}\, dy \\ &< \infty, \quad 1 \le r \le n-2. \end{aligned}$$

Thus $G^{(r)}(t)$ exists $\forall r$, $1 \le r \le n-2$.

Now, consider the special case $n = 4$. Then the wavelet

$$\psi(t) := G^{(2)}(t) = \begin{cases} 4D^2[e^t(1-e^t)^3] & -\infty < t < 0 \\ 0 & 0 \le t < \infty \end{cases}$$

has exactly 2 changes of sign; see Fig. 5.

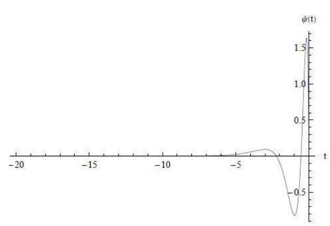

Figure 5: Wavelet $\psi(t) = G^{(2)}(t)$

Example 3.2. The graph of the fifth derivative of the variation diminishing kernel $(2\pi)^{-1}\operatorname{sech}(t/2)$ is given in Fig. 6. It is clear that the number of changes of sign in the wavelet

$$\psi(t) = (d/dt)^5[(2\pi)^{-1}\operatorname{sech}(t/2)]$$

is 3, as asserted in the theorem.

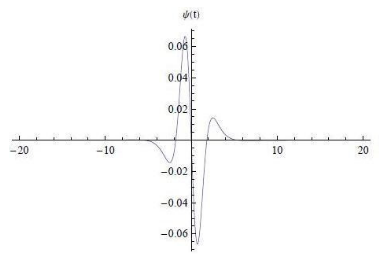

Figure 6: Wavelet $\psi(t) = D^{(5)}\left[\frac{1}{2\pi}\operatorname{sech}(t/2)\right]$

Example 3.3. The graph of

$$\psi(t) = (d/dt)^5(e^{-e^t}e^t)$$

is given in Fig. 7. The number of changes of sign in this case is 4, which is less than 5, the order of the derivative of $G^{(5)}(t) = D^{(5)}(e^{-e^t}e^t)$.

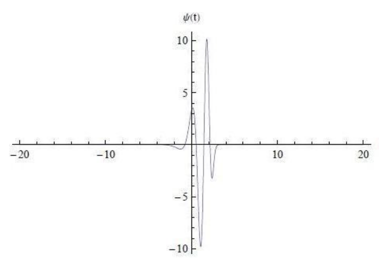

Figure 7: Wavelet $\psi(t) = D^{(5)}\left[\exp(-e^t)e^t\right]$

Certain well-known wavelets, and in some cases their Fourier transforms, can be shown to be special cases of (1.9).

Example 3.4. (Mexican hat wavelet). In (1.9) set $c = 1/2$, $b = 0$ and $a_k = \infty$ for all $k = 1, 2, \cdots$. Then

$$G^{(2)}(t) = -(2\pi)^{-1} \int_{-\infty}^{\infty} e^{iwt} w^2 e^{-\frac{1}{2}w^2}\, dw$$
$$= -(2\pi)^{-1/2}(1-t^2)e^{-\frac{1}{2}t^2};$$

so that $-(2\pi)^{1/2}G^{(2)}(t) = (1-t^2)e^{-\frac{1}{2}t^2}$, the Mexican hat wavelet [11, p. 11].

Example 3.5. (Morlet wavelet). If we choose $c = 1/2$,$b = -t_0$ and $a_k = \infty$ for all $k = 1, 2, \cdots$, then (1.6) gives

$$G(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{iwt} e^{-\frac{1}{2}w^2 - it_0 w}\, dw$$
$$= (8\pi)^{-1/2} e^{-\frac{1}{2}(t-t_0)^2};$$

so that $4\pi G(t) = (2\pi)^{1/2} e^{-\frac{1}{2}(t-t_0)^2}$, which is the Fourier transform of Morlet wavelet [7, p. 30].

Example 3.6. (Cauchy wavelet). If we choose c=0,$b = -r-1$ and $a_k = 1$ for $k = 1, ..., r+1$, $a_k = \infty$ for $k = r+2, r+3, \cdots$, then (1.5) gives

$$\hat{G}(w) = [e^{-i(r+1)w}((1-iw)e^{iw})^{r+1}]^{-1}$$
$$= (1-iw)^{-(1+r)}$$
$$= \frac{2\pi}{\Gamma(r+1)}\psi(w),$$

where $\psi(t) = (2\pi)^{-1}\Gamma(r+1)(1-it)^{-(r+1)}$ for r=1,2,..., is called a Cauchy wavelet[7, p. 29].

4 Changes of trend

Following [6, p. 93] we study intersection property of wavelets.

Let

$$E_1(s) \;:=\; e^{-c's^2+b's} \prod_{k=1}^{\infty}(1-\frac{s}{a'_k})e^{s/a'_k}$$
$$E_2(s) \;:=\; e^{-c''s^2+b''s} \prod_{k=1}^{\infty}(1-\frac{s}{a''_k})e^{s/a''_k},$$

where c', c'', b', b'', a'_k, a''_k are real and $c' \geq 0$, $c'' \geq 0$, $\sum_k (a'_k)^{-2} < \infty$, $\sum_k (a''_k)^{-2} < \infty$ and let

$$H_1(t) = (2\pi)^{-1} \int_{-i\infty}^{i\infty} (E_1(s))^{-1} e^{st}\, ds, \quad H_2(t) = (2\pi)^{-1} \int_{-i\infty}^{i\infty} (E_2(s))^{-1} e^{st}\, ds.$$

If $E(s)$ is given by (1.10), then the relations $E(s) = E_1(s)E_2(s)$ and $G(t) = (H_1 * H_2)(t)$ are equivalent [6, p. 94]. Clearly,

$$\begin{aligned} G^{(r)}(x) &= \int_{-\infty}^{\infty} H_1^{(r)}(x-t) H_2(t)\, dt \\ &= \int_{-\infty}^{\infty} H_1(x-t) H_2^{(r)}(t) dt, \qquad r = 1, 2, \cdots . \end{aligned}$$

Theorem 4.1. *For any a, $-\infty < a < \infty$, $G^{(r)}(x) - aH_1^{(r)}(x)$ is a wavelet which has at most $r+2$ changes of trend.*

PROOF. Let

$$\begin{aligned} \Delta(t) &= 0 \qquad\qquad |t| \geq 1 \\ &= 1 - |t|, \qquad |t| < 1. \end{aligned}$$

Then [6, p. 94]

$$H_1^{(r)}(x) = \lim_{h \to 0+} \int_{-\infty}^{\infty} H_1^{(r)}(x-t)\{h^{-1}\Delta(t/h)\}\, dt.$$

Hence

$$\begin{aligned} G^{(r)}(x) - aH_1^{(r)}(x) &= \lim_{h\to 0+} \{ \int_{-\infty}^{\infty} H_1^{(r)}(x-t) H_2(t)\, dt \\ &\qquad -a \int_{-\infty}^{\infty} H_1^{(r)}(x-t)\{h^{-1}\Delta(t/h)\}\, dt\} \\ &= \lim_{h\to 0+} \int_{-\infty}^{\infty} H_1^{(r)}(x-t)\{H_2(t) - ah^{-1}\Delta(t/h)\}\, dt. \end{aligned}$$

From Theorem 3.1, we know that $H_1^{(r)}(x)$ has at most r changes of sign, and for sufficiently small h, $H_2(t) - ah^{-1}\Delta(t/h)$ has at most two changes of sign [6, p.94]. Therefore, $G^{(r)}(x) - aH_1^{(r)}(x)$ has at most $r+2$ changes of trend. □

5 Inversion of the wavelet transform

In this section we assume that the wavelet is a derivative of $G(t)$ given by (1.7) and obtain an inversion formula for the wavelet transform [8, p. 63]:

$$W_r(x,a) := a^{-\rho}\int_{-\infty}^{\infty}\phi(t)G^{(r)}(\frac{t-x}{a})\,dt \qquad \rho>0, a>0, r=1,2,\cdots \quad (5.1)$$

in terms of derivatives of the transform. From [6, pp. 55, 108-110] we know that for each r = 1, 2,..., $G^{(r)}(t) \in L^1(\mathbb{R})$; therefore the above integral exists for any bounded continuous function ϕ.

Theorem 5.1. *Let ϕ be bounded and continuous on $\mathbb{R}$. Let $E(s) \in E$ be given by (1.10) and $G(t)$ be given by (1.6) with $c = 0$, b, $a_k \in \mathbb{R}, \sum_{k=1}^{\infty} a_k^{-2} < \infty$. Assume that for each $r = 1, 2, \cdots$, the wavelet transform $W_r(x,a)$ is defined by (5.1). Then*

$$\phi(x) = E(aD)a^{\rho-1-r}(D^{-1})^r F_r(x,a),$$

where $D = \frac{d}{dx}$, $D^{-1} = \int_{-\infty}^{x}\cdots$ and $F_r(-x,a) = W_r(\phi(-t))(x,a)$.

Proof. Let us set

$$F_r(x,a) := a^{-\rho}\int_{-\infty}^{\infty}\phi(t)G^{(r)}(\frac{x-t}{a})\,dt. \qquad (5.2)$$

Then by change of variables and using Fubini's theorem we can write

$$\begin{aligned} D_x^{-1}[F_r(\xi a,a)](x) &= \int_{-\infty}^{x} F_r(\xi a,a)\,d\xi \\ &= a^{-\rho+1}\int_{-\infty}^{\infty}\phi(au)du\int_{-\infty}^{x} G^{(r)}(\xi-u)\,d\xi \\ &= a^{-\rho+1}\int_{-\infty}^{\infty}\phi(au)du\int_{-\infty}^{x-u} G^{(r)}(v)dv \\ &= a^{-\rho+1}\int_{-\infty}^{\infty}\phi(au)G^{(r-1)}(x-u)du. \end{aligned}$$

Similarly, we can show that

$$D_x^{-r}[F_r(\xi a,a)](x) = a^{-\rho+1}\int_{-\infty}^{\infty}\phi(au)G(x-u)\,du.$$

Now, applying Theorem 1.1 we get

$$E(D_x)a^{\rho-1}(D_x^{-1})^r F_r(xa,a) = \phi(ax).$$

This by a change of variables yields

$$\phi(z) = E(aD)a^{\rho-1-r}[(D_z)^{-1}]^r F_r(z,a).$$

$\square$

6 Tanno's kernel and wavelets

Tanno [13, 14], Ditzian and Jakimovski [4, 5] studied the following form of convolution transform. Assume that

$$F(s) = e^{bs}\{\prod_{k=1}^{\infty}(1 - sa_k^{-1})e^{s/a_k}/(1 - sc_k^{-1})e^{s/c_k}\} \tag{6.1}$$

where b, $\{a_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ are real numbers, $0 \le (a_k/c_k) < 1$ and $\sum_{k=1}^{\infty} a_k^{-2} < \infty$.

We may choose $c_k = \infty$ when $a_k > 0$ or $c_k = -\infty$ when $a_k < 0$ by which we shall mean $(1 - sc_k^{-1})e^{s/c_k} = 1$.

Assume further that

$$\begin{aligned} \alpha_1 &= \max\{a_k, -\infty | a_k < 0\}, & \alpha_2 &= \min\{a_k, \infty | a_k > 0\}, \\ \gamma_1 &= \max\{c_k, -\infty | c_k < 0\} \quad \text{and} & \gamma_2 &= \min\{c_k, \infty | c_k > 0\}. \end{aligned}$$

Then from [4, Theorem 2.1, p. 171],

$$|F(i\tau)|^{-1} \le |F(0)|^{-1} \text{ for } (\gamma_1 + \alpha_1)/2 \le 0 \le (\gamma_2 + \alpha_2)/2. \tag{6.2}$$

Next, for a sequence of real numbers $\{b_k\}$, let $N(\{b_k\}, x)$ denote the number of elements of the sequence $\{b_k\}$ between 0 and x (may be equal to $|x|$). For a pair of sequences $\{a_k\}$ and $\{c_k\}$ of real numbers define

$$\begin{aligned} N_+ &\equiv N_+(\{a_k\},\{c_k\}) = \liminf_{x\to\infty}(N(\{a_k\},x) - N(\{c_k\},x)), \\ N_- &\equiv N_-(\{a_k\},\{c_k\}) = \liminf_{x\to-\infty}(N(\{a_k\},x) - N(\{c_k\},x)) \\ &\text{and} \\ N &\equiv N(\{a_k\},\{c_k\}) \equiv N_+ + N_-. \end{aligned}$$

Then from [4, Theorem 2.2, p. 171],

$$|F(i\tau)|^{-1} = O(|\tau|^{-n}), \qquad |\tau| \to \infty \tag{6.3}$$

for all integers n satisfying $n \le N$.

The following theorem due to Ditzian and Jakimovski [5, p. 181] will be used in the sequel.

Theorem 6.1. *If for $F(s)$ given by (6.1) with $b=0$, $N \geq 2$, then there exists a density function $G(t)$ satisfying*

$$\frac{1}{F(s)} = \int_{-\infty}^{\infty} e^{-st} G(t)\, dt \qquad \alpha_1 < \Re s < \alpha_2 \tag{6.4}$$

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{F(s)}\, ds \tag{6.5}$$

$$G^{(n)}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{s^n e^{st}}{F(s)}\, ds \qquad n \leq N-2. \tag{6.6}$$

Theorem 6.2. *Let $\psi(x)$ be a wavelet and let $(F(it))^{-1}$ be the Fourier transform of some function $G(x) \in L^1(\mathbb{R})$. Then for $\gamma_1+\alpha_1 \leq 0 \leq \gamma_2+\alpha_2$, $G*\psi$ is a wavelet.*

Proof. We have

$$\begin{aligned} C_\psi &= \int_{-\infty}^{\infty} \frac{|\hat{G}(w)\hat{\psi}(w)|^2}{|w|} = \int_{-\infty}^{\infty} |[F(iw)]^{-1}\hat{\psi}(w)|^2\, \frac{dw}{|w|} \\ &\leq |F(0)|^{-2}\int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|}\, dw \text{ for } \gamma_1+\alpha_1 \leq 0 \leq \gamma_2+\alpha_2, \\ &< \infty, \end{aligned}$$

on using (6.2). □

Theorem 6.3. *Let $F(s)$ be given by (6.1) and let $N \geq 3$. Then $G^{(q)}(x)$ defined by (6.6) is a wavelet for all $q, 1 \leq q \leq N-2$.*

Proof. From (6.6) for $q \leq N-2$ we have

$$\begin{aligned} C_\psi &= \int_{-\infty}^{\infty} |(G^{(q)}\hat{)}(w)|^2 \frac{dw}{|w|} \\ &= \int_{-\infty}^{\infty} |w^q \hat{G}(iw)|^2 \frac{dw}{|w|} \\ &= \int_{-\infty}^{\infty} |w|^{2q-1}|F(iw)|^{-2}\, dw \\ &\leq \int_{-1}^{1} |w|^{2q-1}|F(iw)|^{-2}\, dw + \int_{|w|\geq 1} |w|^{2q-1}|F(iw)|^{-2}\, dw \\ &\leq \int_{-1}^{1} |w|^{2q-1}|\prod_{k=1}^{\infty}(1-iw/a_k)e^{iw/a_k}/(1-iwc_k^{-1})e^{iw/c_k}|^{-2}\, dw \end{aligned}$$

$$+C\int_{|w|\geq 1}|w|^{2q-1}|w|^{-2n}\,dw \qquad \text{by (6.3), for } C>0$$

$$\leq \int_{-1}^{1}|w|^{2q-1}\prod_{k=1}^{\infty}\frac{|1+w^2/c_k^2|}{|1+w^2/a_k^2|}\,dw + C\int_{|w\geq 1}|w|^{2q-2n-1}\,dw.$$

Since $|a_k/c_k|<1$, and (6.3) holds $\forall n<N$, we choose $n>q$. Then the integrals are convergent for $1\leq q\leq N-2$. □

Example 6.1. From [14, p. 181] we know that if $G(t)$ is defined by

$$G(t)=\begin{cases}\frac{1}{\Gamma(\nu+1)}[1-e^{-t}]^{\nu}e^{t} & t\geq 0\\ 0 & t<0,\end{cases}$$

then

$$[F(s)]^{-1}=\frac{\Gamma(s+1)}{\Gamma(s+\nu+2)}, \qquad s=\sigma+i\tau,\sigma>-1,\nu>1;$$

so that

$$|F(\sigma+i\tau)|^{-1}=O(|\tau|^{-\nu-1}), \qquad |\tau|\to\infty.$$

Then for $\nu=9$, $\psi(t):=G^{(5)}(t)$ is a wavelet shown in Fig. 8.

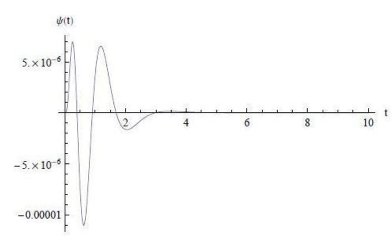

Figure 8: Wavelet $\psi(t)=G^{(5)}(t)$

7 Wavelets generated by H-function

Fox's H-function is a general special function defined by Mellin-Barnes type integral:

$$\begin{aligned} H_{p,q}^{m,n}(z) &:= H_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}(a_1,\alpha_1), & \cdots, & (a_p,\alpha_p)\\ (b_1,\beta_1), & \cdots, & (b_q,\beta_q)\end{matrix}\right)\\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{H}_{p,q}^{m,n}(s)z^{-s}ds, \qquad \gamma\in\mathbb{R}, z\in\mathbb{C} \qquad (7.1)\end{aligned}$$

where

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, s=\gamma+iy, \quad (7.2)$$

$a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+ = (0,\infty)$ $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, q$.

It is assumed that

$$\alpha_i(b_j+l) \neq \beta_j(a_i-k-1), \quad i=1,\cdots,n; j=1,\cdots,m; k,l=0,1,2,\cdots, \quad (7.3)$$

so that poles of Gamma functions $\Gamma(b_j + \beta_j s)$ and $\Gamma(1 - a_i - \alpha_i s)$ do not coincide. For details of assumptions and properties of the H-function and existence of the above contour integral we may refer to [9, pp. 1-4].

Let

$$a^* := \sum_{i=1}^{n}\alpha_i - \sum_{i=n+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j \quad (7.4)$$

$$\mu := \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{1}{2}(p-q) \quad (7.5)$$

$$\xi := \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j + \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i \quad (7.6)$$

$$c^* := m+n-\frac{1}{2}(p+q). \quad (7.7)$$

Then from [9, Lemma 1.2, p. 4],

$$|\mathcal{H}_{p,q}^{m,n}(iy)| \approx C|y|^{\Re(\mu)}\exp[-\frac{\pi}{2}(|y|a^* + \Im(\xi)sgn(y))], \quad |y|\to\infty, \quad (7.8)$$

where

$$C = (2\pi)^{c^*} e^{-c^*-\Re(\mu)} \prod_{i=1}^{p} \alpha_i^{\frac{1}{2}-\Re(a_i)} \prod_{j=1}^{q} \beta_j^{\Re(b_j)-\frac{1}{2}}. \quad (7.9)$$

Since for $\gamma = 0$, using change of variable $z = e^t$, (7.1) can be reduced to the inverse Fourier transform

$$H_{p,q}^{m,n}(e^t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{H}_{p,q}^{m,n}(-iy)e^{iyt}\,dy, \quad (7.10)$$

from [9, Theorem 2.2, p. 43] we get

$$\mathscr{F}[H_{p,q}^{m,n}(e^t)](y) = \mathcal{H}_{p,q}^{m,n}(-iy), \quad (7.11)$$

for

$$-\min_{1\le i\le m}\Re(b_j/\beta_j) < 0 < \min_{1\le i\le n}\Re(1-a_i)/\alpha_i \text{ when } a^*>0 \tag{7.12}$$

and additionally

$$\Re(\mu) < -1 \text{ when } a^* = 0. \tag{7.13}$$

Notice that in (7.2) Gamma functions can be replaced by product of sequences, and (7.1) can be shown to be a special case of Tanno's kernel (6.5).

Theorem 7.1. *Assume that conditions (7.12) – (7.13) hold. Then* $\psi(t) := (d/dt)^r H_{q,q}^{m,n}(e^t)$ *is a wavelet* $\forall r \in \mathbb{Z}_+$ *if* $a^*>0$*; and for* $1 \le r < -\frac{1}{2}\Re(\mu)$ *if* $a^* = 0$.

PROOF. We have

$$\begin{aligned} C_\psi &:= \int_{-\infty}^{\infty} |\mathscr{F}[D^r H_{p,q}^{m,n}(e^t)](y)|^2 \frac{dy}{|y|} \\ &\le \int_{-1}^{1} |y|^{2r-1}|\mathcal{H}_{p,q}^{m,n}(iy)|^2\,dy + \int_{|y|\ge 1} |y|^{2r-1}|\mathcal{H}_{p,q}^{m,n} iy|^2\,dy. \end{aligned} \tag{7.14}$$

Using properties of Gamma functions and the asymptotic behaviour (7.8) it can be shown that the two integrals in (7.14) are finite under the assumptions of the theorem. □

Example 7.1. Let us choose the following special case of (7.1) given in [9, p. 63].

$$H_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} (1-a,1) & \\ (0,1), & (1-c,1) \end{matrix}\right) = \frac{\Gamma(a)}{\Gamma(c)}\ {}_1F_1(a;c;-z).$$

Then $a^* = 1 > 0$ and therefore by above theorem,

$$\psi(t) := (d/dt)^r\ {}_1F_1(a;c;-e^t)$$

is a wavelet for all $r = 1, 2, \cdots$; see Fig. 9.

Example 7.2. From [9, p. 64] we know that

$$H_{0,2}^{1,0}\left(\frac{z^2}{4} \,\middle|\, \overline{\left(\frac{-a+\eta}{2},1\right),\left(\frac{-(a+\eta)}{2},1\right)}\right) = \left(\frac{z}{2}\right)^{-a} J_\eta(z).$$

In this case $a^* = 0$ and $\mu = -a-1$.

Therefore, by the above theorem, for $0 < \Re(a), \Re(-a+\eta) < 0$,

$$\psi(t) := (d/dt)^r[e^{-ta/2}J_\eta(2e^{t/2})]$$

is a wavelet when $1 \le r < \frac{1}{2}\Re(a+1)$; see Fig. 10.

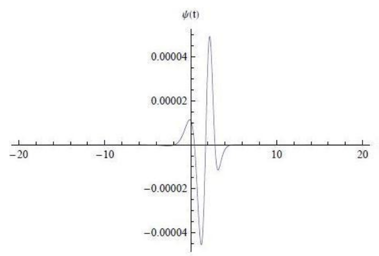

Figure 9: Wavelet $\psi(t) = D^{(5)}{}_1F_1(9/2; 10; -e^t)$

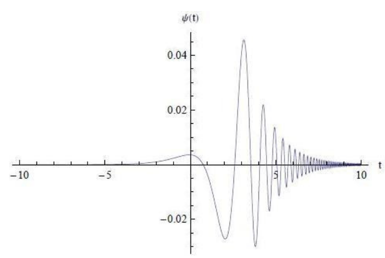

Figure 10: Wavelet $\psi(t) = D^{(2)}\left[e^{-3t/2}J_5(2e^{t/2})\right]$

Remark 1. Previous analysis can be applied to study wavelet transform involving Tanno's form of convolution kernels [13, 14]; these have also been investigated by Ditzian et al [4, 5]. Moreover results can also be extended to generalized functions following the techniques of Zemanian [15], Pandey and Zemanian [10] and Ditzian [3].

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