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MOMENTS AND THE RANGE OF THE DERIVATIVE

Abstract

In this note we introduce three problems related to the topic of finite Hausdorff moments. Generally speaking, given the first $n+1$ ($n \in \mathbb{N} \cup \{0\}$) moments, α_0, α_1,..., α_n, of a real-valued continuously differentiable function f defined on $[0,1]$, what can be said about the size of the image of $\frac{df}{dx}$? We make the questions more precise and we give answers in the cases of three or fewer moments and in some cases for four moments. In the general situation of $n+1$ moments, we show that the range of the derivative should contain the convex hull of a set of n numbers calculated in terms of the Bernstein polynomials, $x^k(1-x)^{n+1-k}$, $k = 1, 2, ..., n$, which turn out to involve expressions just in terms of the given moments α_i, $i = 0, 1, 2, ...n$. In the end we make some conjectures about what may be true in terms of the sharpness of the interval range mentioned before.

1 Introduction

We are studying here a problem from real analysis which can be roughly stated in the following way:

> *Given a continuously differentiable function whose first n moments are prescribed, what can be said about the image of the derivative of this function?*

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One of the tools that we will use is the following classical so called first mean value theorem for integrals (see Section 30.9 in [1]).

Theorem 1. *Let h be a continuous function on $[a, b]$ and g a non-negative Riemann integrable function. Then there exists a value $c \in (a, b)$ such that*

$$\int_a^b h(x)g(x)dx = h(c)\int_a^b g(x)dx.$$

Moreover, if $h(x) \geq h(c)$ (or $h(x) \leq h(c)$) for all $x \in [a, b]$, then $h(x) = h(c)$ for every x point of continuity of g and $g(x) > 0$.

To introduce our hypothesis we let $n \in \mathbb{N} \cup \{0\}$ and let f be a continuously differentiable function which satisfies the following Hausdorff moment type interpolation conditions:

$$\int_0^1 x^k f(x)dx = \alpha_k, \quad k = 0, 1, 2, ..., n, \quad \alpha_k \in \mathbb{R}. \tag{1}$$

Let us observe that given arbitrary moments α_k the system (1) leads to a linear one if f is a polynomial function. The main matrix of the resulting system is a Hilbert matrix. This type of matrix is well know (see [2], for instance) and has a non-zero determinant.

Our investigation was motivated by a proposed problem in the College Mathematics Journal ([8]) which requires one to show that if $n = 2$ and $\alpha_k = k + 1$, there exist $c_1, c_2 \in [0, 1]$ such that $f'(c_1) = -24$ and $f'(c_2) = 60$. It turns out that this problem was inspired by a problem of C. Lupu (see [6]) which referred to only two moments, $\alpha_0 = \alpha_1 = 1$, and asked for a point c where $f'(c) = 6$. We wondered if these numbers were, in a certain sense which will be defined next, sharp. We will show that this is indeed the case in the next section (Theorem 2). Similar optimization questions, given the first n Hausdorff moments on $[0, 1]$ or $[-1, 1]$, are customary subjects in the literature (see [5], [7]) We are going to formulate the following very general questions that are our main interest in this paper.

Problem 1. *For a fixed n and α_k as before, what is the largest range $[A, B]$ such that $[A, B] \subseteq Range(f')$ for every f a continuously differentiable function on $[0, 1]$ satisfying (1)?*

Problem 2. *For a fixed n and α_k as before, what is the biggest number L such that for every f a continuously differentiable function on $[0, 1]$ satisfying (1) there exists some interval $[a, b]$ with $b - a = L$ that satisfies $[a, b] \subseteq Range(f')$?*

We observe that in order to prove that $[A,B]$ is the answer for Problem 1, it is necessary to show that $[A,B] \subseteq Range(f')$ for every f a continuously differentiable function on $[0,1]$ satisfying (1) and that for every $\epsilon > 0$ there exists f_l and f_r continuously differentiable functions on $[0,1]$ satisfying (1) and

$$Range(f_l') \subseteq (A-\epsilon, \infty) \text{ and } Range(f_r') \subseteq (-\infty, B+\epsilon). \tag{2}$$

It is clear that if A and B give the answer in Problem 1, then in trying to answer Problem 2 we must have $L \geq B - A$. If for every $\epsilon > 0$, one can find a function ($f_l = f_r$) that satisfies both conditions in (2), then the answer to Problem 2 is simply $L = B - A$.

Another related problem here is to characterize the case $L > B - A$ and calculate L in this case in terms of the α_k's. Perhaps Problem 2 may be easier if one restricts the class of functions in consideration to something more manageable like polynomials of a certain degree.

If we want to make the range of the derivative as small as possible, we just have to take moments that satisfy the necessary and sufficient condition for having a solution to the system that results from having a linear function, say $f(x) = u + vx$, $x \in [0,1]$, satisfying (1):

$$\frac{u}{k+1} + \frac{v}{k+2} = \alpha_k, \quad k = 0, 1, 2, ...n.$$

This is equivalent to

$$rank \begin{bmatrix} 1 & \frac{1}{2} & \alpha_0 \\ \frac{1}{2} & \frac{1}{3} & \alpha_1 \\ \frac{1}{3} & \frac{1}{4} & \alpha_2 \\ \dots & & \\ \frac{1}{n} & \frac{1}{n+1} & \alpha_{n-1} \\ \frac{1}{n+1} & \frac{1}{n+2} & \alpha_n \end{bmatrix} = 2.$$

On the other hand, if we want to make the range of f' as big as possible, it makes sense to restrict our moments to a finite range, say $[-1,1]$. We observe that the problem is homogeneous under dilations, so let us formulate a third problem.

Problem 3. *For a fixed n, what is the maximum of $B - A$ such that $[A,B] \subseteq Range(f')$ for every f a continuously differentiable function on $[0,1]$ satisfying (1), the maximum being taken over all possible moments $\alpha_k \in [-1,1]$?*

We will show in Section 2 that the answer to Problem 3 is 156 if $n = 2$, for the moments $\alpha_0 = 1$, $\alpha_1 = -1$ and $\alpha_2 = 1$. We observe that if the answer to

Problem 2 is zero, then the answer to Problem 3 is also zero. As suggested by one of the referees of our paper, one can ask similar questions about the range of f'' or higher derivatives, assuming these exist. We will make some remarks about these questions and see how the results for the first derivative could be applied for higher derivatives.

2 Small values of n

We have a few complete answers to Problem 1 for small values of n ($n \leq 3$). First, let us study what happens with $n = 0$. If we take $g(x) = 1 - x$ and $h = f'$ in Theorem 1, using integration by parts, we get

$$f'(c_1)\frac{1}{2} = \int_0^1 f'(x)(1-x)dx = -f(0) - \int_0^1 f(x)(-1)dx = \alpha_0 - f(0),\ c_1 \in (0,1),$$

or

$$f'(c_1) = 2(\alpha_0 - f(0)),\ c_1 \in (0,1).$$

If $f(0) = a$, then we can take $f(x) = a + (2\alpha_0 - 2a)x$ and observe that in case $n = 0$, there exists a function such that $\int_0^1 f(x)dx = \alpha_0$ and $Range(f') = \{2\alpha_0 - 2f(0)\}$. This gives us the following simple answers to Problem 1 and Problem 2.

Proposition 1. *For $n = 0$, there is no A and B that satisfy the requirements of Problem 1. The answer for Problem 2 ($n = 0$) is $L = 0$.*

Let us continue the analysis in the case $n = 1$. We can apply Theorem 1 to $g(x) = x(1 - x)$ and $h = f'$, $x \in [0, 1]$. Then, a similar calculation gives that for some $c_2 \in (0, 1)$,

$$f'(c_2)\frac{1}{6} = -\int_0^1 f(x)(1-2x)dx = 2\alpha_1 - \alpha_0, \Rightarrow f'(c_2) = 6(2\alpha_1 - \alpha_0).$$

If we apply Theorem 1 to $h = f'$ and $g(x) = (1 - x)^2$ instead,

$$\begin{aligned} f'(c_3)\frac{1}{3} &= \int_0^1 f'(x)(1-x)^2 dx = -f(0) - \int_0^1 f(x)(2x-2)dx \\ &= 2(\alpha_0 - \alpha_1) - f(0),\ c_3 \in (0,1), \end{aligned}$$

or

$$f'(c_3) = 6(\alpha_0 - \alpha_1) - 3f(0),\ \text{for some}\ \ c_3 \in (0,1).$$

So, if we take $a = 2(2\alpha_0 - 3\alpha_1)$ and $f(x) = a + mx$ where $m = 6(2\alpha_1 - \alpha_0) = 6(\alpha_0 - \alpha_1) - 3a$, we get a function which will give us what we need in this case, and therefore provide a similar answers to our problems.

Proposition 2. *For $n = 1$, we can take $A = B = 12\alpha_1 - 6\alpha_0$ to satisfy the requirements of Problem 1. The answer for Problem 2 ($n = 1$) is $L = 0$.*

The case $n = 2$ is getting a little more interesting; it is essentially non-trivial and at the same time pretty surprising. We have a definite answer to Problem 1 and Problem 3 and we show some inequality for L in Problem 2.

Theorem 2. *For $n = 2$, if $\Delta_0 := 6\alpha_2 - 6\alpha_1 + \alpha_0 > 0$, the values*

$$A := 12(4\alpha_1 - \alpha_0 - 3\alpha_2) \text{ and } B := 12(3\alpha_2 - 2\alpha_1)$$

satisfy the requirements of Problem 1 and if $\Delta_0 < 0$ then one needs to switch the values of A and B above in order to solve Problem 1. If $\Delta_0 = 0$, the values $A = B = 12(3\alpha_2 - 2\alpha_1)$ answer Problem 1 and $L = 0$ answers Problem 2.

PROOF. First, let us show that A and B are always in the range of the derivative. This is done as we have seen before by setting in the Theorem 1, $h = f'$, and $g(x) = x(1-x)^2 \geq 0$ ($x \in [0,1]$). Indeed, we have $\int_0^1 g(x)dx = \frac{1}{12}$ and

$$\int_0^1 f'(x)g(x)dx = f(x)g(x)|_0^1 - \int_0^1 f(x)(1 - 4x + 3x^2)dx = 4\alpha_1 - \alpha_0 - 3\alpha_2.$$

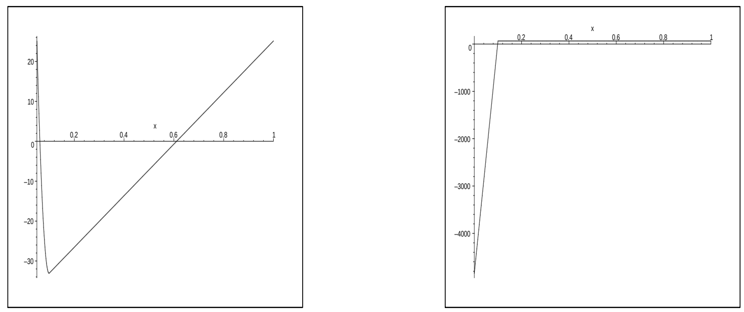
Hence, for some c_4 we must have $f'(c_4)\int_0^1 g(x)dx = 4\alpha_1 - \alpha_0 - 3\alpha_2$ which in turn gives $f'(c_4) = 12(4\alpha_1 - \alpha_0 - 3\alpha_2) = A$. Similarly, for $g(x) = x^2(1-x)$, ($x \in [0,1]$), one finds that $\int_0^1 g(x)dx = \frac{1}{12}$ still holds true and

$$\int_0^1 f'(x)g(x)dx = \int_0^1 f(x)(2x - 3x^2)dx = 3\alpha_2 - 2\alpha_1.$$

This insures that $B = 12(3\alpha_2 - 2\alpha_1)$ is also in the range of f'. Because f' is assumed to be continuous we get that the whole interval $[A, B]$ or $[B, A]$ is contained in the range of f'.

From here on, we are going to work under the first assumption ($\Delta_0 > 0$) which is equivalent to $A < B$ ($B - A = 12\Delta_0$). To show that A and B are sharp bounds we begin with B by constructing a spline function s_t for $t \in (0,1)$, defined by

$$s_t(x) = \begin{cases} a + bx + cx^2 \; for \; x \in [0,t] \\ \\ m + nx \;\; if \;\; x \in [t,1], \end{cases}$$



(a) $s_{\frac{1}{10}}$ (b) $s'_{\frac{1}{10}}$

Figure 1: Case $n = 2$ and $\alpha_k = k + 1$

where a, b, c, m and n are determined by the conditions $\int_0^1 s_t(x)dx = \alpha_0$, $\int_0^1 x s_t(x)dx = \alpha_1$, $\int_0^1 x^2 s_t(x)dx = \alpha_2$ and the restrictions necessary to insure that s_t is continuously differentiable at $x = t$.

In order to add an intuition element we included here the graphs of $s_{1/10}$ and its derivative for $\alpha_0 = 1$, $\alpha_1 = 1$ and $\alpha_2 = 2$.

It is easy to see that s_t is continuously differentiable at $x = t$ if and only if $m = a - ct^2$ and $n = b + 2ct$. This gives the new expression of s_t just in terms of a, b and c:

$$s_t(x) = \begin{cases} a + bx + cx^2 \ \ for\ x \in [0, t] \\ \\ (a - ct^2) + (b + 2ct)x \ \ \ if \ \ \ x \in [t, 1]. \end{cases} \tag{3}$$

One can check that the moment restrictions reduce to the following 3×3 relatively simple linear system of equations in a, b and c:

$$\begin{cases} a+\dfrac{b}{2}+\left(\dfrac{t^3}{3}-t^2+t\right)c=\alpha_0 \\[2ex] \dfrac{a}{2}+\dfrac{b}{3}+\left(\dfrac{t^4}{12}+\dfrac{2t}{3}-\dfrac{t^2}{2}\right)c=\alpha_1 \\[2ex] \dfrac{a}{3}+\dfrac{b}{4}+\left(\dfrac{t^5}{30}+\dfrac{t}{2}-\dfrac{t^2}{3}\right)c=\alpha_2. \end{cases} \tag{4}$$

It is clear that we have a unique solution for this system at least for infinitely many values of t since the main determinant of the system is a polynomial in t of degree at most 5. Let us observe that

$$\frac{d}{dx}(s_t)(x)=\begin{cases} b+2cx \ \ for \ x\in[0,t] \\[2ex] b+2ct \ \ \ if \ \ \ x\in[t,1]. \end{cases} \tag{5}$$

We observe that if $c>0$, the maximum of this function is $b+2ct$. With a little work one solves the system (4) and finds that

$$c=\frac{30\Delta_0}{t^3(6t^2-15t+10)},$$

$$b+2ct=\frac{12(6t^2\alpha_1-3t^2\alpha_0-20\alpha_1+30\alpha_2+5t\alpha_0-15t\alpha_2)}{6t^2-15t+10}.$$

It is clear from these expressions that under our hypothesis $c>0$ for every $t>0$ and that $B=12(3\alpha_2-2\alpha_1)=\lim\limits_{t\to 0}(b+2ct)$ which proves that B is sharp. In a similar way one can prove that A is sharp by taking a spline $\tilde{s}_t$ which is first a linear piece on $[0,t]$ and a quadratic piece on $[t,1]$. It turns out that the calculations are very much similar to the ones above with the only difference that this time we let t approach 1.

However, we are going to show that the lower bound A is sharp by using an invariance principle here by doing a "change of variable" so to speak and considering how the Problem 1 changes from f to g where $g(x)=f(1-x)$, $x\in[0,1]$. Let us denote by $\alpha_k(f)$ the k^{th} moment for the function f. One can see that

$$\alpha_0(g)=\alpha_0(f),\ \alpha_1(g)=\alpha_0(f)-\alpha_1(f)\ and\ \alpha_2(g)=\alpha_0(f)-2\alpha_1(f)+\alpha_2(f),$$

and of course, the relations are symmetric with respect to interchanging f and g , i.e.

$$\alpha_0(f)=\alpha_0(g),\ \alpha_1(f)=\alpha_0(g)-\alpha_1(g)\ and\ \alpha_2(f)=\alpha_0(g)-2\alpha_1(g)+\alpha_2(g).$$

Let us observe that the hypothesis that $\Delta_0(f)>0$ is in fact invariant under this change:

$$6\alpha_2(f)-6\alpha_1(f)+\alpha_0(f)=6\alpha_2(g)-6\alpha_1(g)+\alpha_0(g)>0.$$

By the first part of our proof, we see that

$$B(g)=12(3\alpha_2(g)-2\alpha_1(g))=12[3(\alpha_0(f)-2\alpha_1(f)+\alpha_2(f))-2(\alpha_0(f)-\alpha_1(f)),$$

or

$$B(g)=12(3\alpha_2(f)-4\alpha_1(f)+\alpha_0(f),$$

is a sharp bound for the range of g'. Since $g'(x)=-f'(1-x)$ we see that the range of g is just the range of f reflected into the origin and vice versa. Hence, $A(f)=-B(g)=12[4\alpha_1(f)-3\alpha_2(f)-\alpha_0(f)]$ is a sharp lower bound for f. The rest of the statements of the theorem follow from what we have shown so far. $\square$

Corollary 1. *In the case $n=2$, in respect to Problem 2, we have*

$$12|\Delta_0|\le L\le 32|\Delta_0|.$$

The maximum required in Problem 3 is 156.

PROOF. The first part is a simple consequence of the fact $B-A=12\Delta_0$ and the last part follows from the fact that $|\Delta_0|=|6\alpha_2-6\alpha_1+\alpha_0|\le 13$ if $\alpha_i\in[-1,1]$, $i=0,1,2$. To show the inequality $L\le 32|\Delta_0|$ we employ the same idea by constructing spline which is symmetric around 1/2:

$$\hat{s}_t(x)=\begin{cases} a+bx+c(x-2tx-t^2+t-1/4)\ for\ x\in[0,\frac{1}{2}-t] \\ \\ a+bx+cx^2\quad if\quad x\in[\frac{1}{2}-t,\frac{1}{2}+t] \\ \\ a+bx+c(x+2tx-t^2-t-1/4). \end{cases} \tag{6}$$

This spline is continuously differentiable on $[0,1]$ and depends on three parameters which if determined from the constraints given by the moments we get

$$c = \frac{120\Delta_0}{t(15-40t+48t^2)} > 0, \text{for all } t \in [0,1],$$

and which shows that the minimum and the maximum of the derivative of $\hat{s}_t$ is attained on the linear pieces. One can see that the difference between these two values is actually $4ct$ and so letting $t \to 0$ we get that

$$L \leq \lim_{t\to 0} 4ct = 32|\Delta_0|.$$

□

One can use the same techniques to show that for three moments, assuming the second derivative exists, the range of the second derivative should contain $30\delta_0$ and this is sharp because a polynomial of degree two exists solving the moments problem.

The case $n = 3$ is even more interesting and a lot more complicated. First of all we have at least three new possible values that we need to add to the range of f':

$$C = 20(4\alpha_3 - 3\alpha_2),\ D = 60(3\alpha_2 - 2\alpha_3 - \alpha_1),\ \text{and } E = 20(4\alpha_3 - 9\alpha_2 + 6\alpha_1 - \alpha_0) \tag{7}$$

obtained from Bernstein polynomials, $g_1(x) = x^3(1-x)$, $g_2(x) = x^2(1-x)^2$ and $g_3(x) = x(1-x)^3$ respectively.

Proposition 3. *Given A and B as defined in Theorem 2, we have the inclusion*

$$[min(A,B), max(A,B)] \subset [min(D,C,E), max(D,C,E)].$$

PROOF. Let us observe that $g_1(x) + g_2(x) = x^2(1-x)$ and $g_2(x) + g_3(x) = x(1-x)^2$. Differentiating and integrating against $f(x)$ we get the relations $\frac{1}{30}D + \frac{1}{20}E = \frac{1}{12}A$ or $A = \frac{2}{5}D + \frac{3}{5}E$ and similarly $B = \frac{3}{5}C + \frac{2}{5}D$. These two convex linear combinations are enough to conclude the desired statement. □

Of course, this proposition can be generalized to an arbitrary n. So we expect that the interval that answers Problem 1 contains the convex hull of the numbers constructed as usual, i.e.

$$D_k := -\frac{\int_0^1 f(x)\frac{d}{dx}(x^k(1-x)^{n+1-k})dx}{\int_0^1 x^k(1-x)^{n+1-k}dx}, k = 1, 2, ..., n, \tag{8}$$

given by the highest degree Bernstein basis polynomials possible.

We observe that if we define $\Delta_1 := 10\alpha_3 - 12\alpha_2 + 3\alpha_1$, then

$$C = E + 20\Delta_0 \text{ and } D = C - 20\Delta_1.$$

Hence, we observe that if we have $\Delta_0 > 0$ and $\Delta_1 < 0$ for instance, then $D > C > E$. Therefore, in light of Proposition 3, the candidates for the two values needed to answer Problem 1 are $\tilde{A} = E$ and $\tilde{B} = D$ under the given assumption. In fact, for various other situations we believe that the values A and B that answer Problem 1 are given for each case in the following table

No	Hypothesis	A	B
(i)	$\Delta_0 \geq 0, \Delta_1 \leq 0$	$20(4\alpha_3 - 9\alpha_2 + 6\alpha_1 - \alpha_0)$	$60(3\alpha_2 - 2\alpha_3 - \alpha_1)$
(ii)	$0 \leq \Delta_0 \leq \Delta_1$	$60(3\alpha_2 - 2\alpha_3 - \alpha_1)$	$20(4\alpha_3 - 3\alpha_2)$
(iii)	$0 \leq \Delta_1 \leq \Delta_0$	$20(4\alpha_3 - 9\alpha_2 + 6\alpha_1 - \alpha_0)$	$20(4\alpha_3 - 3\alpha_2)$
(iv)	$\Delta_0 \leq 0, \Delta_1 \geq 0$	$60(3\alpha_2 - 2\alpha_3 - \alpha_1)$	$20(4\alpha_3 - 9\alpha_2 + 6\alpha_1 - \alpha_0)$
(v)	$\Delta_1 \leq \Delta_0 \leq 0$	$20(4\alpha_3 - 3\alpha_2)$	$60(3\alpha_2 - 2\alpha_3 - \alpha_1)$
(vi)	$\Delta_0 \leq \Delta_1 \leq 0$	$20(4\alpha_3 - 3\alpha_2)$	$20(4\alpha_3 - 9\alpha_2 + 6\alpha_1 - \alpha_0)$

$$\text{where } \Delta_0 = 72 \begin{vmatrix} 1 & \frac{1}{2} & \alpha_0 \\ \frac{1}{2} & \frac{1}{3} & \alpha_1 \\ \frac{1}{3} & \frac{1}{4} & \alpha_2 \end{vmatrix} \text{ and } \Delta_1 = 720 \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & \alpha_1 \\ \frac{1}{3} & \frac{1}{4} & \alpha_2 \\ \frac{1}{4} & \frac{1}{5} & \alpha_3 \end{vmatrix}.$$

We have the following partial result along these lines.

Theorem 3. *For $n = 3$, the upper bound of (i) and the lower bound of (ii), in the table above, are correct. If $\Delta_0 = \Delta_1 = 0$ then $A = B = 20(4\alpha_3 - 3\alpha_2)$ and $L = 0$ solves Problem 1 and Problem 2.*

PROOF. First of all let us observe that the cases (iv), (v) and (vi) follow from (i), (ii) and (iii) respectively by simply changing f into $-f$. This simple transformation changes basically the order of A and B. It is easy to see that $\Delta_0 = \Delta_1 = 0$ implies the existence of a linear map that has the given moments and so $A = B$ and $L = 0$. Hence in what follows we will assume that $\Delta_0 \neq 0$ or $\Delta_1 \neq 0$.

Based on the invariance principle that we used in the proof of Theorem 2 we need to show the sharpness of only the upper bound in (i). Indeed we observe that if $g(x) = f(1 - x)$, $x \in [0, 1]$ then one can check that the hypothesis $\Delta_1(f) \leq 0$ changes into $\Delta_0(g) \leq \Delta_1(g)$. Also, the hypothesis $0 \leq \Delta_1 \leq \Delta_0$, is actually invariant under this change. One also needs to take into account that the bound D is invariant under this transformation but C and E interchange:

$$D(f) = D(g), \; C(f) = E(g), \text{ and } E(f) = C(g).$$

So, let us begin with case (i) and show that $B = 60(3\alpha_2 - 2\alpha_3 - \alpha_1)$ is sharp. For every $t \in (0, 1/2)$, consider a spline function $s_{1,t}$ which is quadratic on $[0,t]$, linear on $[t, 1-t]$ and another quadratic on $[1-t, 1]$. The constraints of having this spline a continuous and differentiable function give us a similar form for $s_{1,t}$ to the one constructed in the proof of Theorem 2 in equality (3), in terms of four free parameters a, b, c and d:

$$s_{1,t}(x) = \begin{cases} a + bx + cx^2 \ \ for \ x \in [0,t] \\ \\ (a - ct^2) + (b + 2ct)x \quad if \quad x \in [t, 1-t] \\ \\ d(1-t)^2 + a - ct^2 + [b + 2ct - 2d(1-t)]x + dx^2 \quad if \quad x \in [1-t, 1]. \end{cases}$$

The four parameters are then determined by imposing the four linear constraints given by the moments. The resulting system is

$$\begin{cases} a + \frac{b}{2} + (\frac{t^3}{3} - t^2 + t)c + \frac{t^3}{3}d = \alpha_0 \\ \\ \frac{a}{2} + \frac{b}{3} + \left(\frac{t^4}{12} + \frac{2t}{3} - \frac{t^2}{2}\right)c + \left(\frac{t^3}{3} - \frac{t^4}{4}\right)d = \alpha_1 \\ \\ \frac{a}{3} + \frac{b}{4} + \left(\frac{t}{2} - \frac{t^2}{3} + \frac{t^5}{30}\right)c + \left(\frac{t^3}{3} - \frac{t^4}{6} + \frac{t^5}{30}\right)d = \alpha_2 \\ \\ \frac{a}{4} + \frac{b}{5} + \left(\frac{2t}{5} - \frac{t^2}{4} + \frac{t^6}{60}\right)c + \left(\frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{10} - \frac{t^6}{60}\right)d = \alpha_3. \end{cases} \tag{9}$$

As we have observed before, the system has a unique solution for infinitely many values of $t \in (0, 1/2)$, since the main determinant of the system is a polynomial in t of degree at most 11. Because the derivative of $s_{1,t}$ is given by

$$s'_{1,t}(x) = \begin{cases} b(t) + 2c(t)x \ \ for \ x \in [0,t] \\ \\ b(t) + 2c(t)t \quad if \quad x \in [t, 1-t] \\ \\ b(t) + 2c(t)t - 2d(t)(1-t-x) \quad if \quad x \in [1-t, 1]. \end{cases}$$

One can use a symbolic calculator and check that

$$\lim_{t\to 0} b(t) + 2c(t)t = 60(3\alpha_2 - 2\alpha_3 - \alpha_1),$$

which is one necessary fact to prove the sharpness of B. Also, we need to check that for most of the values of t, $b(t) + 2c(t)t$ is a maximum of the derivative of $s_{1,t}$. For this end, it is enough to check that $c(t) > 0$ and $d(t) < 0$ for small values of t. Again, one can compute $\lim_{t\to 0} c(t)t^3 = \Delta_0 - \Delta_1 > 0$ under our assumption in case (i) (unless both numbers Δ_0, Δ_1 are zero). Also, the limit of $d(t)t^3$ as $t \to 0$ turns out to be equal to $3\Delta_1 \le 0$. If $\Delta_1 = 0$ we know that $\Delta_0 > 0$. In this case we have $\lim_{t\to 0} d(t)t^2 = -\frac{9}{4}\Delta_0 < 0$.

Using the duality via $g(x) = f(1-x)$, we see that $A = 60(3\alpha_2 - 2\alpha_3 - \alpha_1)$ is a sharp lower bound in the case (ii). □

In the case $n = 3$, assuming the table before Theorem 3 is correct, with respect to Problem 2, we have either $L \ge 20|\Delta_0|$, $L \ge 20|\Delta_1|$, or $L \ge 20|\Delta_0 - \Delta_1|$, depending upon the hypothesis in which the moments fall into as classified in Theorem 3. The maximum required in Problem 3 is 760 which is attained for $\alpha_0 = 1$, $\alpha_1 = -1$, $\alpha_2 = 1$, and $\alpha_3 = -1$. We wonder if alternating the signs of the moments and setting them $\alpha_k = (-1)^k$ will always give the maximum in Problem 3.

For higher derivatives we can show that $U := 120(3\alpha_1 - 12\alpha_2 + 10\alpha_3)$ and $V := 120(\alpha_0 - 9\alpha_1 + 18\alpha_2 - 10\alpha_3)$ are in the range of the second derivative. It does not seem to follow from our Theorem 2 applied to f' that these values are sharp, although the same idea of using a spline formed by a a cubic and a quadratic may work.

3 Higher values of n

We have noted the following statement after the proof of Proposition 3.

Theorem 4. *Given a continuously differentiable function satisfying the Hausdorff moments constraints (1) ($n \ge 2$), then the range of the derivative contains the interval $[A_n, B_n]$, where $A_n = \min\{D_k | k = 1, 2, ..., n\}$ and $B_n = \max\{D_k | k = 1, 2, ..., n\}$, with D_k given by (8). Moreover, for all $n \ge 2$ $[A_n, B_n] \subset [A_{n+1}, B_{n+1}]$.*

Proof. The first part follows with the same technique we have employed over and over here using Theorem 1. For the second part we are observing that the integrals which appear in the denominators of (8), are actually the well known values of the beta function, i.e. $B(k+1, n+2-k) = \int_0^1 x^k(1-x)^{n+1-k}dx$. Using the established formula for $B(\ ,\ )$, we see that

$$B(k+1,n+2-k) = \frac{\Gamma(k+1)\Gamma(n+2-k)}{\Gamma(n+3)} = \frac{k!(n+1-k)!}{(n+2)!} = \frac{1}{(n+2)\binom{n+1}{k}}.$$

This gives us a new expression of the $D_{k,n}$ which is basically in terms of the genuine Bernstein basis polynomials, i.e. $b_{\nu,n} = \binom{n}{\nu}x^\nu(1-x)^{n-\nu}$, $\nu = 0,1,...,n$:

$$D_{k,n} := -(n+2)\int_0^1 f(x)\frac{d}{dx}(b_{k,n+1})dx, \quad k=1,2,...,n. \tag{10}$$

It is easy to check that $x^k(1-x)^{n+1-k} + x^{k+1}(1-x)^{n-k} = x^k(1-x)^{n-k}$ which basically gives the convex combination formula

$$D_{k,n} = \frac{n+2-k}{n+3}D_{k,n+1} + \frac{k+1}{n+3}D_{k+1,n+1}, \quad k=1,2,...,n, \ n \geq 1.$$

These expressions imply the second claim of the theorem. $\square$

Let us observe that (11) implies the following form for $D_{k,n}$

$$D_{k,n} := (n+1)(n+2)\int_0^1 f(x)(b_{k,n} - b_{k-1,n})dx, \quad k=1,2,...,n. \tag{11}$$

which provides a simple way of computing $D_k's$ in terms of the moments α_0, α_1,..., α_n. In what follows we will describe yet another way of doing these computations, and for that purpose we generalize first the definitions of Δ_0 and Δ_1 in the following way

$$\Delta_k = \frac{(k+1)(k+2)^2(k+3)^2(k+4)}{2}\begin{vmatrix} \frac{1}{k+1} & \frac{1}{k+2} & \alpha_k \\ \frac{1}{k+2} & \frac{1}{k+3} & \alpha_{k+1} \\ \frac{1}{k+3} & \frac{1}{k+4} & \alpha_{k+2} \end{vmatrix}, k=0,1,2,....$$

or simply

$$\Delta_k = \frac{(k+3)(k+4)}{2}\alpha_{k+2} - (k+2)(k+3)\alpha_{k+1} + \frac{(k+1)(k+2)}{2}\alpha_k, \quad k \geq 0.$$

There are some relations between the $D_k's$ and $\Delta_k's$ in general which we will include in the next proposition.

Proposition 4. *For $k \geq 0$ and $n \geq 2$, we have in general*

$$\Delta_k = \frac{D_{k+2,k+2} - D_{k+1,k+1}}{2(k+3)}, k \geq 0. \tag{12}$$

$$D_{n,n} = 6(2\alpha_1 - \alpha_0) + 2\sum_{k=0}^{n-2}(k+3)\Delta_k. \tag{13}$$

Moreover, with the definitions of A_n and B_n from Theorem 4, $A_n = B_n$ if and only if $\Delta_i = 0$ for all $i = 0, 1, 2, ..., n-2$, if and only if there exists a linear function with moments α_k.

PROOF. Let us observe that we can simply write

$$x^k = x^{k+1} + x^k(1-x) = x^{k+1} + \frac{1}{k+1}b_{k,k+1} \Rightarrow$$
$$k\alpha_{k-1} = (k+1)\alpha_k - \frac{1}{(k+1)(k+2)}D_{k,k}.$$

Using this last formula we can calculate the expression of Δ_k:

$$c\Delta_k = \begin{vmatrix} \frac{1}{k+1} & \frac{1}{k+2} & \frac{k+2}{k+1}\alpha_{k+1} - \frac{D_{k+1,k+1}}{(k+1)(k+2)(k+3)} \\ \frac{1}{k+2} & \frac{1}{k+3} & \alpha_{k+1} \\ \frac{1}{k+3} & \frac{1}{k+4} & \frac{k+2}{k+3}\alpha_{k+1} + \frac{D_{k+2,k+2}}{(k+3)^2(k+4)} \end{vmatrix}$$
$$= \begin{vmatrix} \frac{1}{k+1} & \frac{1}{k+2} & -\frac{D_{k+1,k+1}}{(k+1)(k+2)(k+3)} \\ \frac{1}{k+2} & \frac{1}{k+3} & 0 \\ \frac{1}{k+3} & \frac{1}{k+4} & \frac{D_{k+2,k+2}}{(k+3)^2(k+4)} \end{vmatrix},$$

where $c = \frac{2}{(k+1)(k+2)^2(k+3)^2(k+4)}$. This last identity implies the formula (12). For the second part of our statement we observe that $\Delta_i = 0$ for all $i = 0, 1, 2, ...n-2$ if and only if there exists a linear function f with moments α_k. In this case the range of the derivative of f consists of only one point and therefore by Theorem 4 we must have $A_n = B_n$. For the converse, again using Theorem 4 we obtain that all $D_{i,j}$, $1 \leq i \leq j$, $1 \leq j \leq n$, have identical values and so by (12) we get $\Delta_k = 0$ for all $k = 0, 1, ..., n-2$.

Finally, let us observe that the equalities in (12) provide a telescopic sum for $D_{k,k}$ which allows one to arrive at formula (13). □

The convexity relations can be used to calculate all the $D_k's$ from the $D_{k,k}$ and so formulae (13) provide a way of computing all the $D_k's$ in terms of determinants Δ_i.

For the case $\alpha_k = k+1$, $k = 0, 1, 2, ..., n$, we calculate A_n and B_n in a more precise way. This generalizes the problem in [?].

Corollary 2. *Let $n \in \mathbb{N}$, $n \geq 2$, be fixed and f be a continuously differentiable satisfying (1) with $\alpha_k = k+1$, $k = 0, 1, 2, ..., n$. Then, the values of A_n and B_n as defined in Theorem 4 are*

$$A_n = -n(n+1)(n+2),\ B_n = (n+1)(n+2)(2n+1).$$

PROOF. Using the formula for Δ_k we get

$$\Delta_k = \frac{(k+3)^2(k+4)}{2} - (k+2)^2(k+3) + \frac{(k+1)^2(k+2)}{2} = 3k+7, \quad k \geq 0.$$

Then using formula (13) we obtain

$$D_{n,n} = 18 + 2\sum_{k=0}^{n-2}(k+3)(3k+7) = (n+1)(n+2)(2n+1).$$

Now we can use the convexity relations and compute $D_{n-1,n}$:

$$D_{n-1,n} = \frac{1}{2}\left((n+2)D_{n-1,n-1} - nD_{n,n}\right) = -n(n+1)(n+2).$$

Next, if one calculates $D_{n-2,n}$, some surprise appears:

$$D_{n-2,n} = \frac{1}{3}\left((n+2)D_{n-2,n-1} - (n-1)D_{n-1,n}\right) = 0.$$

Because of the convexity relation, it is easy to see that all the other $D_{k,n}$, $k \leq n-2$, are equal to zero. Therefore $A_n = -n(n+1)(n+2)$ and $B_n = (n+1)(n+2)(2n+1)$. □

Putting together what we did so far we now can say that for $\alpha_k = k+1$, the bounds above are sharp if $n = 2$ and the lower bound is sharp if $n = 3$.

Theorem 5. *For $n \geq 2$ fixed, with the definition of A_n and B_n as in Theorem 4, if $A_n < B_n$ it is not possible to have $L = B_n - A_n$ in Problem 2.*

PROOF. By way of contradiction let us assume that $L = B_n - A_n$. Hence, we can find a sequence of functions f_m, continuously differentiable, such that $Range(f'_m) \subset [A_n - \frac{1}{m}, B_n + \frac{1}{m}]$ and satisfying (1). Since f'_m can be considered in $L^2([0,1])$ we can find a subsequence of f'_m, say f'_{m_k}, weakly convergent to a function $f \in L^2([0,1])$. This implies that for every non-negative function $g \in L^2([0,1])$,

$$(A_n - \frac{1}{m})||g||_1 \leq \int_0^1 f'_m(x)g(x)dx \leq (B_n + \frac{1}{m})||g||_1,$$

where $||h||_1 = \int_0^1 h(x)dx$, $h \in L^1([0,1])$. Passing to the limit as $m_k \to \infty$, we get

$$A_n||g||_1 \leq \int_0^1 f(x)g(x)dx \leq B_n||g||_1,\ g \in L^2([0,1]), g \geq 0.$$

This implies that $A_n \leq f(x) \leq B_n$ for a.e. $x \in [0,1]$, by a standard measure theory argument. Since $B_n = D_k$ for some $k = 1, 2, ..., n$, and f_m satisfy (1) we can say that

$$B_n = -\frac{\int_0^1 f_m(x)\frac{d}{dx}[x^k(1-x)^{n+1-k}]dx}{||x^k(1-x)^{n+1-k}||_1} = \frac{\int_0^1 f'_m(x)x^k(1-x)^{n+1-k}dx}{||x^k(1-x)^{n+1-k}||_1}.$$

Letting $m_k \to \infty$ we obtain

$$B_n = \frac{\int_0^1 f(x)x^k(1-x)^{n+1-k}dx}{||x^k(1-x)^{n+1-k}||_1}.$$

Hence, re-writing this yields

$$\int_0^1 [B_n - f(x)]x^k(1-x)^{n+1-k}dx = 0,$$

which in turn implies, by what we have shown before about f, that $f(x) = B_n$ for a.e. $x \in [0,1]$. Similarly, we arrive at the conclusion $f(x) = A_n$ for a.e. $x \in [0,1]$. Since we assumed $A_n < B_n$ we clearly get a contradiction. Therefore, it remains that $L > B_n - A_n$. $\square$

This last theorem says that Problem 1 and Problem 2 are completely different in nature. We must admit that we do not have a definite answer to Problem 2, other than the trivial case $L = 0$, in any of the particular situations we have considered. We leave that to the interested reader.

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References

[1] R. G. Bartle, *The elements of real analysis*, John Wiley & Sons, Second Edition, 1976.

[2] Man-D. Choi, *Tricks or Treats with the Hilbert Matrix*, Amer. Math. Monthly, **5(90)** (1983), 301–312.

[3] G. Inglese, *Christoffel functions and finite moment problems*, Inverse Problems, **11** (1995), 949–960.

[4] G. Inglese, *A note about the discretization of finite moment problems*, Inverse Problems, **10** (1994) 401–414.

[5] G. Inglese, *A note about minimum relative entropy solutions of finite moment problems*, Numer. Funct. Anal. Optim. **16(9-10)** (1995), 1143–1153.

[6] C. Lupu, *Problem U37*, Mathematical Reflections, **6** (2006).

[7] G. Rodriguez and S. Seatzu, *On the solution of the finite moment problem.(English summary)* J. Math. Anal. Appl. **171(2)** (1992), 321–333.

[8] D. V. Thong, *The problem No 951*, College Math. J. (2011), 232–233.