RESEARCH

Real Analysis Exchange
Vol. 36(2), 2010/2011, pp. 435–448

Caterina La Russa, Department of Mathematics, University of Palermo, via Archirafi 34, 90123 Palermo, Italy. email: larussa@math.unipa.it

HENSTOCK-TYPE INTEGRAL FOR VECTOR VALUED FUNCTIONS IN A COMPACT METRIC SPACE

Abstract

We define a Henstock-type integral for vector valued functions defined in a probability metric compact Radon space, using a suitable family $\mathcal{B}$ of measurable sets which play the role of "*intervals*". When $\mathcal{B}$ is the family of all subintervals of $[0, 1]$ we obtain the classical Henstock–Kurzweil integral on the real line, whereas if $\mathcal{B}$ is the family of all subintervals of $[0, 1]^2$, or that of all subintervals of $[0, 1]^2$ with a fixed regularity, we obtain the classical Henstock integral on the plane with respect to the Kurzweil base or the Kempisty base respectively.

1 Introduction

The theory of integration introduced by Lebesgue in 1902 is a powerful tool which, perhaps because of its abstract character, does not have the intuitive appeal of the Riemann integral. Moreover, as Lebesgue himself observed in his thesis [10], his integral does not integrate all unbounded derivatives and so it does not provide a solution for the problem of recovering a function from its derivative. Besides the Lebesgue theory does not cover nonabsolutely convergent integrals. In 1957 Kurzweil [9] and, independently, in 1963 Henstock [8] gave a new definition of integral, which is more general than that of Lebesgue. Its construction uses Riemann sums associated to interval partitions which are pointwise fine instead of uniformly fine (as in case of the classical Riemann integral). The Henstock-Kurzweil integral has the power of Lebesgue's one and

Mathematical Reviews subject classification: Primary: 28B20; Secondary: 26A39, 28B05, 46G10, 54C60
Key words: Henstock integral, McShane integral, partitions, gauge
Received by the editors November 29, 2010
Communicated by: Udayan B. Darji

includes it. Moreover it integrates all derivatives. The interval partitions used in the Riemann sums are closely connected with the topology of the real line and in an easy way it is possible to generalize the definition of the Henstock integral to real (or vector) valued functions defined in $\mathbb{R}^2$ (see [13]).

When the functions are defined in a more general setting different from the euclidean one, the situation becomes more complicated. The biggest difficulties are to find a suitable family of measurable sets which play the role of "*intervals*" in the construction of the Riemann sums and to prove the existence of partitions that are fine with respect to a fixed gauge.

In 2000 N. W. Leng and L. P. Yee, defined a Henstock-type integral on a metric measure space ([11]). When the space is the plane, their generalized intervals are the family of polygons with vertical or horizontal edges or the family of simply connected domains in the plane with piecewise circular edges (see Remark 1).

The aim of the present paper is to provide a definition of the classical Henstock integral in the case of vector valued functions defined in a probability metric compact Radon space.

To this end in Section 3 we define "*intervals*" the sets of a family $\mathcal{B}$ satisfying suitable properties. Then we consider a method of integration (the $\mathcal{B}_H$-integral) which involves finite Henstock partitions by non overlapping sets of $\mathcal{B}$. If $\mathcal{B}$ is the family of all subintervals of $[0,1]$, the corresponding $\mathcal{B}_H$-integral is the classical Henstock–Kurzweil integral on the real line; whereas if $\mathcal{B}$ is the family of all subintervals of $[0,1]^2$ or of all subintervals of $[0,1]^2$ with a fixed regularity α, where $0 < \alpha < 1$, we obtain the Henstock integral on the plane with respect to the Kurzweil base or the Kempisty base, respectively.
Besides, using the Axiom of Choice, we construct, in any probability metric compact Radon space, a family of sets satisfying the properties of the "*intervals*" (see Theorem 1).

In Section 4, using McShane partitions of "*intervals*" instead of Henstock partitions, we obtain a kind of McShane integral that is equivalent to the generalized McShane integral introduced by Fremlin in [3, Theorem 2].

2 Preliminaries

Throughout this paper (Ω, d, Σ, μ) is a probability metric compact Radon space i.e.:

(i) (Ω, d) is a metric space;

(ii) (Ω, Σ, μ) is a probability complete space;

(iii) $\mathcal{T} \subset \Sigma$, where $\mathcal{T}$ is the topology induced by the metric d on X;

(iv) the measure μ is regular, i.e.

$$\mu(E) = \sup\{\mu(F) : F \subseteq E \;\; F \text{ closed}\} = \inf\{\mu(G) : E \subseteq G \;\; G \in \mathcal{T}\}$$

for every $E \in \Sigma$;

(v) μ is τ-*additive*, i.e. if $\mathcal{G} \subseteq \mathcal{T}$ is non-empty and upwards directed by inclusion, then

$$\mu(\bigcup_{G\in\mathcal{G}} G) = \sup\{\mu(G) : G \in \mathcal{G}\}.$$

For $A \subset \Omega$, A^0, $\overline{A}$, A^c and ∂A are the interior part, the closure, the complement and the boundary of A, respectively. For $A, B \subset \Omega$, we denote by $A \triangle B$ the symmetric difference of A and B.
If $A \subset \Omega$, $diam(A) := \sup\{d(x, y) : x, y \in A\}$ is the *diameter* of A. Each set $B_r(w) = \{z \in \Omega : d(w, z) < r\}$, where $w \in \Omega$ and $r > 0$, is called an open ball. Denote by $\mathcal{T}_1$ the family of all open balls. Throughout this paper, moreover we assume that the measure μ satisfies the following property:

$$(A) \qquad \mu(S) > 0 \text{ and } \mu(S) = \mu(\overline{S}) \text{ for all } S \in \mathcal{T}_1.$$

Denote by Σ_∂ the family of all $A \in \Sigma$ such that $\mu(A) > 0$ and $\mu(\partial A) = 0$.
We say that two measurable sets A and B are *non overlapping* if $A^0 \cap B^0 = \emptyset$ and $\mu(\partial A \cap \partial B) = 0$.

Given a non-empty family $\mathcal{A} \subseteq \Sigma$, a finite collection $\mathcal{P} = \{(A_n, \omega_n) : n = 1, \dots, p\}$ of pairwise non overlapping sets $A_n \in \mathcal{A}$ and points $\omega_n \in \overline{A_n}$ is said to be a *Henstock* $\mathcal{A}$-*partition* (briefly $\mathcal{A}_H$-*partition*). If we assume only that $\omega_n \in \Omega$, for $1 \leq n \leq p$, then $\mathcal{P}$ is said to be a *McShane* $\mathcal{A}$-*partition* (briefly $\mathcal{A}_{Mc}$-*partition*). If $A \in \mathcal{A}$ and $\mu(A \triangle (\bigcup_{n=1}^p A_n)) = 0$, we say that $\mathcal{P}$ is an $\mathcal{A}_H$-*partition* (resp. $\mathcal{A}_{Mc}$-*partition*) *of* A.
Denote by $\mathcal{A}^U$ the family of all finite unions of non overlapping sets of $\mathcal{A}$.

Each function $\Delta : \Omega \to \mathcal{T}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$ is called *gauge*. Let Δ be a gauge and let $E \subset \Omega$. An $\mathcal{A}_H$-partition ($\mathcal{A}_{Mc}$-partition) $\mathcal{P} = \{(A_n, \omega_n) : n = 1, \dots, p\}$ is said to be:

(a) Δ-*fine*, if $A_n \subset \Delta(\omega_n)$ for each $1 \leq n \leq p$;

(b) *tagged in* E, if $\omega_n \in E$ for each $1 \leq n \leq p$.

For simplicity we write $(\mathcal{A}_H, \Delta)$-*partition* (resp. $(\mathcal{A}_{Mc}, \Delta)$-*partition*), for an $\mathcal{A}_H$-partition (resp. $\mathcal{A}_{Mc}$-partition) that is Δ-fine.

Given $f : \Omega \to Y$, where Y is any Banach space, and a partition $\mathcal{P} = \{(A_n, \omega_n) : n = 1, \dots, p\}$, we set $\sigma(f, \mathcal{P}) := \sum_{n=1}^p f(\omega_n)\mu(A_n)$.

3 The family $\mathcal{B}$ of "intervals"

One of the most important problems in a theory of gauge integrals is the existence of partitions, fine with respect to a fixed gauge. So in our framework, first of all, it is essential to define a suitable family of measurable sets which play the role of "*intervals*" in the construction of Riemann sums.

Definition 1. We say that $\mathcal{B} \subset \Sigma$ is a family of *intervals* in (Ω, d, Σ, μ) if it satisfies the following properties:

(j) $\mathcal{B} \subseteq \Sigma_\partial$;

(jj) $\Omega \in \mathcal{B}$;

(jjj) for each $B \in \mathcal{B}$, there exist in $\mathcal{B}$ non overlapping subsets $B_1, ..., B_k$ of $\overline{B}$ such that

$$\mu(B \setminus \bigcup_{i=1}^{k} B_i) = 0 \quad and \quad diam(B) > c \cdot diam(B_i) \tag{1}$$

for every $i = 1, ..., k$, where $c > 1$ is a fixed constant;

(jv) if $B \in \mathcal{B}$, for each $C \subseteq B$ and $C \in \mathcal{B}^U$, then $B \setminus C$ belongs to $\mathcal{B}^U$ unless a set of zero measure.

Example 1. Let $\Omega = [0, 1]$ be endowed with the Lebesgue measure and the Euclidean topology and let $\mathcal{B}$ be the family of all subintervals of Ω. Then $\mathcal{B}$ satisfies properties (j)–(jv).

Example 2. Let $\Omega = [0, 1]^2$ be endowed with the Lebesgue measure and the Euclidean topology and let $\mathcal{B}$ be the Kurzweil base (i.e. the family of all subintervals of Ω). Then $\mathcal{B}$ satisfies properties (j)–(jv).

Example 3. Let $\Omega = [0, 1]^2$ be endowed with the Lebesgue measure and the Euclidean topology and let $\mathcal{B}$ be the Kempisty base (i.e. the family of all the subintervals of $[0, 1]^2$ whose regularity[1] is greater than a fixed α, for $0 < \alpha < 1$). Then $\mathcal{B}$ satisfies properties (j)–(jv).

In [11], N. W. Leng and L. P. Yee consider a metric measure space (Y, d, μ) that satisfies condition (A) and such that:

[1] Recall that if $I = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ the *regularity* of I is the number $r(I) = \min_{i=1,2}(b_i - a_i)/\max_{i=1,2}(b_i - a_i)$.

(B) for every measurable set $W \subset Y$ and every $\epsilon > 0$, there exist an open set U and a closed set Z such that $Z \subset W \subset U$ and $\mu(U \setminus Z) < \epsilon$.

Note that each probability, metric, compact Radon space (Ω, d, Σ, μ), satisfies condition (B).

In their framework, N. W. Leng and L. P. Yee define the following families of sets:

$$\mathcal{H}_1 = \left\{ \overline{B_1} \setminus \overline{B_2} : B_1, B_2 \in \mathcal{T}_1 \text{ and } B_1 \not\subseteq B_2,\ B_2 \not\subseteq B_1 \right\},$$

$$\mathcal{H}_2 = \left\{ \bigcap_{i \in \Lambda} X_i \neq \emptyset : X_i \in \mathcal{H}_1 \text{ and } \Lambda \text{ is a finite index set} \right\}.$$

They call the members of $\mathcal{H}_2$ *generalized intervals* and a finite union of mutually disjoint generalized intervals *elementary set.* Then using the elementary sets, they are able to define a Henstock-type integral on Y.

Remark 1. If $Y = \mathbb{R}^2$ with the metric $d_1(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, a generalized interval of $\mathcal{H}_2$ looks like a polygon with vertical or horizontal edges, and each edge is not necessarily included. Instead, if we consider $Y = \mathbb{R}^2$ with the metric $d_2(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{\frac{1}{2}}$, a generalized interval is a simply connected domain in the plane with piecewise circular edges and each edge is not necessarily included. In any case we obtain a base, different from both the Kurzweil and the Kempisty base.

In [11] the following result is shown:

Lemma 1. ([11], Theorem p. 36) *Given a gauge Δ on a generalized interval $I \in \mathcal{H}_2$, there exists a Δ-fine division*[2] *of I.*

Lemma 2. *Given a gauge Δ on (Ω, d, Σ, μ), there exists a Δ-fine division of Ω.*

PROOF. Taking in account that Ω is compact, the proof follows as in Theorem on page 36 of [11], after suitable changes. $\square$

Lemma 3. *In the space (Ω, d, Σ, μ) the family $\mathcal{H}_2 \bigcup \Omega$ satisfies the properties (j)–(jjj).*

PROOF. Properties (j) and (jj) hold. In order to show that also the property (jjj) is satisfied, fix $B \in \mathcal{H}_2 \bigcup \Omega$ and a constant $c > 1$. It is enough to consider

[2]A *division* of I is a finite collection $\{(I_n, x_n) : n = 1, \dots, p\}$ of pairwise mutually disjoint subintervals $I_n \subset I$ and points $x_n \in I_n$ such that $\bigcup_{n=1}^{p} I_n = I$.

a gauge Δ such that $\Delta(w) := B_r(w)$ for each $w \in B$, with $r = \frac{diam(B)}{4c}$ and to apply Lemma 1 or Lemma 2. $\square$

Now, using the Axiom of Choice, we are going to construct, in any probability metric compact Radon space, a family $\mathcal{B}$ of intervals.

Proposition 1. *Let $\mathcal{A}$ be a family of subsets of Ω satisfying properties (j)–(jjj). Then there exists a subfamily $\mathcal{B} \subseteq \mathcal{A}$ which satisfies also the property (jv).*

Proof. We construct the subfamily $\mathcal{B}$ by induction.
First step: Let $B_1^{k_1} = \Omega$, where $k_1 = 1$.
Second step: Since $\Omega \in \mathcal{A}$, by the property (jjj), (and the Axiom of Choice) we can choose in $\mathcal{A}$, non overlapping subsets of Ω, $B_2^1, ..., B_2^{k_2}$ such that $\mu(\Omega \setminus \bigcup_{m=1}^{k_2} B_2^m) = 0$ and $diam(\Omega) > c \cdot diam(B_2^m)$, for every $m = 1, ..., k_2$, and where c is a fixed constant greater then 1.
Third step: For each B_2^m, with $m = 1, 2, ...k_2$, by the property (jjj) we can choose in $\mathcal{A}$ a finite number of sets satisfying the condition (1) of property (jjj), with B replaced by B_2^m with $m = 1, 2, ...k_2$. Let call by $B_3^1, ..., B_3^{k_3} \in \mathcal{A}$ all the sets obtained in this step.
Then call by $B_n^1, ..., B_n^{k_n} \in \mathcal{A}$, with $k_n \in \mathbb{N}$, the sets obtained in the nth step. The subfamily $\mathcal{B} = \{B_n^m\}_{m,n} \subset \mathcal{A}$ with $n \in \mathbb{N}$ and $m = 1, 2, ..., k_n$ satisfies also the property (jv). $\square$

Theorem 1. *In any space (Ω, d, Σ, μ), there exists a family of intervals.*

Proof. It follows by Proposition 1 and by Lemma 3. $\square$

Proposition 2. *Let $\mathcal{B}$ be a family of intervals in (Ω, d, Σ, μ). For each $B \in \mathcal{B}$ and for each gauge Δ there exists a $(\mathcal{B}_H, \Delta)$-partition of B.*

Proof. Let $B \in \mathcal{B}$ and assume by contradiction that there is not such a partition of B. By property (jjj) there exists in $\mathcal{B}$ a set $B^{(1)} \subset \overline{B}$ with $diam(B) > c \cdot diam(B^{(1)})$ and such that there is not a $(\mathcal{B}_H, \Delta)$-partition of $B^{(1)}$. Proceeding by induction, we can construct a sequence $\{B^{(k)}\}_k$ of $\mathcal{B}$ sets such that $\overline{B^{(k)}} \supset \overline{B^{(k+1)}}$, $diam(B^{(k)}) > c \cdot diam(B^{(k+1)})$ and there is not a $(\mathcal{B}_H, \Delta)$-partition of $B^{(k)}$ for each $k \in \mathbb{N}$.

As Ω is compact, $\bigcap_{k=1}^{\infty} \overline{B^{(k)}} \neq \emptyset$. Let $\omega_0 \in \bigcap_{k=1}^{\infty} \overline{B^{(k)}}$. By construction $\lim_{k\to\infty} diam(B^{(k)}) = 0$. So there exists an index k_0 such that $B^{(k_0)} \subset \Delta(\omega_0)$ and the pair $(B^{(k_0)}, \omega_0)$ is a $(\mathcal{B}_H, \Delta)$-partition of $B^{(k_0)}$ and this is a contradiction. $\square$

Proposition 3. *Let $\mathcal{B}$ a family of intervals in (Ω, d, Σ, μ). Let Δ be a gauge and let $\mathcal{P}$ be a $(\mathcal{B}_H, \Delta)$-partition of $B \in \mathcal{B}$. Then the partition $\mathcal{P}$ can be extended to a $(\mathcal{B}_H, \Delta)$-partition of Ω.*

PROOF. It follows at once from property (jv) and Proposition 2. $\square$

Definition 2. A family $\mathcal{F} \subset \Sigma$ is said to be *weakly fine* on Ω if for each $w \in \Omega$ and for each $G \in \mathcal{T}$ with $w \in G$, there exists $F \in \mathcal{F}$ such that $\mu(F) > 0$, $\omega \in \overline{F}$ and $F \subseteq G$.

Proposition 4. *Let $\mathcal{B}$ be a family of intervals in (Ω, d, Σ, μ). Then $\mathcal{B}$ is weakly fine on Ω.*

PROOF. Consider $w \in \Omega$ and $G \in \mathcal{T}$ with $w \in G$. By properties (jj) and (jjj) there exist in $\mathcal{B}$ non overlapping sets $B_1, ..., B_k$ such that $\mu(\Omega \setminus \bigcup_{i=1}^k B_i) = 0$ and $diam(\Omega) > c \cdot diam(B_i)$, for every $i = 1, ..., k$ and $c > 1$ fixed constant. Note that by condition (A), each non empty set of $\mathcal{T}$ has positive measure. Then $\mu(\Omega \setminus \overline{\bigcup_{i=1}^k B_i}) = 0$ implies that $\Omega \setminus \overline{\bigcup_{i=1}^k B_i} = \emptyset$. So $w \in \overline{B_i}$ for some $i = 1, ..., k$.

Call $I^{(1)}$ one of such intervals. Then $w \in \overline{I^{(1)}}$ and $diam(\Omega) > c \cdot diam(I^{(1)})$. By property (jjj) there exist in $\mathcal{B}$ non overlapping subsets $B_1^1, ..., B_1^{k_1}$ of $\overline{I^{(1)}}$ such that $\mu(I^{(1)} \setminus \bigcup_{j=1}^{k_1} B_1^j) = 0$ and $diam(I^{(1)}) > c \cdot diam(B_1^j)$, for every $j = 1, ..., k_1$.
Call $I^{(2)}$ one of such intervals. Then $w \in \overline{I^{(2)}}$ and $diam(I^{(1)}) > c \cdot diam(I^{(2)})$. Now again by property (jjj) there exist in $\mathcal{B}$ non overlapping subsets $B_2^1, ..., B_2^{k_2}$ of $\overline{I^{(2)}}$ such that $\mu\left(I^{(2)} \setminus \bigcup_{n=1}^{k_2} B_2^n\right) = 0$ and $diam(I^{(2)}) > c \cdot diam(B_2^n)$, for every $n = 1, ..., k_2$.
Call $I^{(3)}$ one of such intervals. Then $w \in \overline{I^{(3)}}$ and $diam(I^{(2)}) > c \cdot diam(I^{(3)})$. Proceeding by induction, we construct a sequence $\{I^{(m)}\}_{m \in \mathbb{N}}$ of $\mathcal{B}$ sets with decreasing diameter such that $\overline{I^{(m)}} \supset \overline{I^{(m+1)}}$ and $w \in \overline{I^{(m)}}$ for all $m \in \mathbb{N}$. Then taking in account that G is open and $w \in G$, there exists an index m_0 such that $\omega \in \overline{I^{(m_0)}}$ and $I^{(m_0)} \subset G$. $\square$

In the following we will use the property:

Definition 3. A family $\mathcal{F} \subseteq \Sigma$ *separates points off closed sets* if given $\varepsilon > 0$, $\omega \in \Omega$ and an open set O of positive measure and containing ω, there exists $F \in \mathcal{F}$ such that $\omega \in \overline{F}$, $F \subseteq O$ and $\mu(O \setminus F) < \varepsilon$.

Note that the previous definition is slightly weaker then the definition given in [1]. In fact there is also the condition that $\omega \in \overline{F}$.

Proposition 5. *Let $\mathcal{B}$ be a family of intervals in (Ω, d, Σ, μ). Then the family $\mathcal{B}^U$ separates points off closed sets.*

Proof. Let $\omega_0 \in \Omega$ and $O \in \mathcal{T}$ be given, with $\omega_0 \in O$, and let $\varepsilon > 0$ be fixed. Since μ is regular, let F be a compact $F \subset O$ such that $\mu(O \setminus F) < \varepsilon$. Now define a gauge Δ in Ω in the following way: $\Delta(\omega) \subset O$ if $\omega \in O$, $\Delta(\omega) \subset \Omega \setminus F$, if $\omega \in O^c$. Since $\mathcal{B}$ is weakly fine on Ω (Proposition 4), then let $B_0 \subset \Delta(\omega_0)$ be a $\mathcal{B}$ set such that $\omega_0 \in \overline{B_0}$. Applying Proposition 3 to the partition $\mathcal{P} = \{(B_0, \omega_0)\}$ and to the gauge Δ, $\mathcal{P}$ can be extended to a $(\mathcal{B}_H, \Delta)$-partition $\mathcal{Q}$ of Ω. Set $U = \bigcup B$ where the union is extended to all the sets $B \in \mathcal{B}$ such that $(B, \omega) \in \mathcal{Q}$ and $B \subset O$. Therefore U is the required set. $\square$

4 Henstock and McShane $\mathcal{B}$-integrals

Let $\mathcal{B}$ be a fixed family of intervals in (Ω, d, Σ, μ) and let $(X, \|\cdot\|)$ be a Banach space.

Definition 4. We say that a function $f \colon \Omega \to X$ is *$\mathcal{B}_H$-integrable ($\mathcal{B}_{Mc}$-integrable) on* Ω if there exists $w \in X$ satisfying the following property:
for each $\varepsilon > 0$ there exists a gauge $\Delta \colon \Omega \to \mathcal{T}$ such that

$$\|\sigma(f, \mathcal{P}) - w\| < \varepsilon, \tag{2}$$

for every $(\mathcal{B}_H, \Delta)$-partition ($(\mathcal{B}_{Mc}, \Delta)$-partition) $\mathcal{P}$ of Ω. We set

$$w = (\mathcal{B}_H) \int_\Omega f \, d\mu \qquad \left(w = (\mathcal{B}_{Mc}) \int_\Omega f \, d\mu \right).$$

Given a measurable set $E \subset \Omega$ we say that f is $\mathcal{B}_H$-integrable ($\mathcal{B}_{Mc}$-integrable) on E if the function $f\chi_E$ is $\mathcal{B}_H$-integrable ($\mathcal{B}_{Mc}$-integrable) on Ω, where as usual, χ_E is the characteristic function of the set E.
We set $w(E) = (\mathcal{B}_H) \int_\Omega f\chi_E \, d\mu$ $\big(w(E) = (\mathcal{B}_{Mc}) \int_\Omega f\chi_E \, d\mu\big)$.

Remark 2. If $\Omega = [0, 1]$ is endowed with the Lebesgue measure and the Euclidean topology, $X = \mathbb{R}$ and $\mathcal{B}$ is the family of all subintervals of Ω, then the $\mathcal{B}_H$-integral is the classical Henstock–Kurzweil integral on the real line. If $\Omega = [0, 1]^2$ is endowed with the Lebesgue measure and the Euclidean topology and $\mathcal{B}$ is the Kurzweil base or the Kempisty base, then the $\mathcal{B}_H$-integral is the Henstock integral on the plane with respect to the Kurzweil base or the Kempisty base (see [13]).

Proposition 6. *A function $f \colon \Omega \to X$ is $\mathcal{B}_H$-integrable on $B \in \mathcal{B}$ if and only if the following Cauchy condition holds:*

for each $\varepsilon > 0$ there exist a gauge Δ such that

$$\|\sigma(f, \mathcal{P}) - \sigma(f, \mathcal{Q})\| < \varepsilon, \tag{3}$$

for each couple $\mathcal{P}$, $\mathcal{Q}$ of $(\mathcal{B}_H, \Delta)$-partitions of B.

PROOF. The proof follows as in Proposition 2 of [1] after suitable changes. □

The above property guarantees that

Proposition 7. *Let $f: \Omega \to X$ be a $\mathcal{B}_H$-integrable function on Ω. Then the function $f\chi_B$ is $\mathcal{B}_H$-integrable on Ω for every set $B \in \mathcal{B}$.*

PROOF. The proof follows from Proposition 3 and Proposition 6. □

Note that the $\mathcal{B}_H$-integral is uniquely determined, closed under addition and scalar multiplication. Moreover also the Henstock Lemma version for vector valued function holds (see [14]).

In [3] D.H. Fremlin studies, in a σ-finite outer regular quasi-Radon space, a method of integration for vector-valued functions which is a generalization of the McShane process of integration [12]. This method involves infinite McShane partitions by disjoint families of measurable sets of finite measure. However, in the compact case, the method may use finite McShane partitions with disjoint measurable sets (see [3, Proposition E1]).

Definition 5. We say that a function $f: \Omega \to X$ is *Fremlin-integrable* on Ω ([3, Proposition E1]) if there exists $w \in X$ satisfying the following property: for each $\varepsilon > 0$ there exists a gauge Δ such that

$$\|\sigma(f, \mathcal{P}) - w\| < \varepsilon,$$

for every finite (Σ_{Mc}, Δ)-partition $\mathcal{P}$ of Ω.

Now we compare the $\mathcal{B}_{Mc}$-integral with the Fremlin-integral. We need the following Lemma that may be proved in a standard way (for the case $\Omega = [0, 1]$ and $X = \mathbb{R}$ see [7], p. 323).

Lemma 4. *Let $f: \Omega \to X$ be a function and let $N \subset \Omega$. If $\mu(N) = 0$, then for each $\varepsilon > 0$ there exists a gauge Δ in N such that $\sigma(\|f\|, \mathcal{P}) < \varepsilon$, for each $(\mathcal{B}_{Mc}, \Delta)$-partition $\mathcal{P}$ tagged in N.*

Theorem 2. *A function $f: \Omega \to X$ is $\mathcal{B}_{Mc}$-integrable on Ω if and only if it is Fremlin-integrable on Ω.*

PROOF. Let f be Fremlin-integrable on Ω. Since for any gauge Δ, each $(\mathcal{B}_{Mc}, \Delta)$-partition is also a (Σ_{Mc}, Δ)-partition, therefore f is $\mathcal{B}_{Mc}$-integrable on Ω.

For the converse, let $\varepsilon > 0$ be fixed and let Δ be a gauge such that

$$\left\| \sigma(f, \mathcal{P}) - (\mathcal{B}_{Mc}) \int_{\Omega} f \right\| < \frac{\varepsilon}{4} \,, \tag{4}$$

for each $(\mathcal{B}_{Mc}, \Delta)$- partition $\mathcal{P}$ of Ω.
Now let $\mathcal{Q} = \{(E_i, \omega_i) : \; i = 1, ..., n\}$ be a (Σ_{Mc}, Δ)-partition of Ω. Put $m = \max_{i=1,\ldots,n} \|f(\omega_i)\|$ and take $0 < \eta < (4nm)^{-1}\varepsilon$.
The proof will be inductive.

Assume that for some $1 \leq q < n$ we have already sets $A_1, \ldots, A_q \in \mathcal{B}^U$, and open sets $U_1, \ldots, U_q$ satisfying for each $j \leq q$ the following properties:

$$E_j \cup \{w_j\} \subseteq U_j \subseteq \Delta(\omega_j), \qquad \omega_j \in \overline{A_j}\,, \qquad A_j \subseteq U_j \setminus \overline{\bigcup_{k<j} A_k}\,, \tag{5}$$

$$\mu(U_j \setminus E_j) < \frac{\eta}{n+2}\,, \qquad \mu\left(\left[U_j \setminus \overline{\bigcup_{k<j} A_k}\right] \setminus A_j\right) < \frac{\eta}{n+2} \tag{6}$$

and

$$\mu(E_j \triangle A_j) < \eta\,. \tag{7}$$

Having these sets, we take an open set U_{q+1} such that

$$E_{q+1} \cup \{w_{q+1}\} \subseteq U_{q+1} \subseteq \Delta(\omega_{q+1})$$

and

$$\mu(U_{q+1} \setminus E_{q+1}) < \frac{\eta}{n+2}\,.$$

Moreover by Proposition 5, there exists $A_{q+1} \in \mathcal{B}^U$, such that

$$\omega_{q+1} \in \overline{A_{q+1}}\,, \qquad A_{q+1} \subseteq U_{q+1} \setminus \overline{\bigcup_{k \leq q} A_k}$$

and

$$\mu\left(\left[U_{q+1} \setminus \overline{\bigcup_{k \leq q} A_k}\right] \setminus A_{q+1}\right) < \frac{\eta}{n+2}\,.$$

Since the sets E_j are disjoint we have

$$\begin{aligned}
&\mu(E_{q+1}\triangle A_{q+1})\\
&\leq \mu\left(\left[E_{q+1}\bigcap\left(U_{q+1}\setminus\overline{\textstyle\bigcup_{k\leq q}A_k}\right)\right]\setminus A_{q+1}\right)\\
&+\mu\left(\left[E_{q+1}\bigcap\left(U_{q+1}\bigcap\overline{\textstyle\bigcup_{k\leq q}A_k}\right)\right]\setminus A_{q+1}\right)+\mu(U_{q+1}\setminus E_{q+1})\\
&< \tfrac{2\eta}{n+2}+\mu\left(E_{q+1}\bigcap\overline{\textstyle\bigcup_{k\leq q}A_k}\right)\\
&\leq \tfrac{2\eta}{n+2}+\mu(E_1\triangle A_1)+...+\mu(E_q\triangle A_q)\\
&<\eta\,.
\end{aligned}$$

As the first step is similar to the inductive one (we set $A_0=\emptyset$), the construction is over. Since $A_i\in\mathcal{B}^U$, $i=1,...,n$, we have $A_i=\bigcup_{j=1}^{p_i}C_j^{(i)}$ where $C_j^{(i)}$ are non overlapping $\mathcal{B}$ sets. Moreover, by property (5), $\{(C_j^{(i)},\omega_i): i=1,...,n,\ j=1,...,p_i\}$ is a $(\mathcal{B}_{Mc},\Delta)$-partition.

If $\mu(\Omega\setminus\bigcup_{i=1}^n A_i)=0$, we are done. Otherwise by (5), we have that

$$\Omega\setminus\bigcup_{i=1}^n A_i\subset\bigcup_{i=1}^n\Delta(\omega_i)\bigcup N,$$

where $\mu(N)=0$. For each $w\in\Omega\setminus\bigcup_{i=1}^n A_i$, $w\notin N$, we choose an index $i(w)\in\{1,...,n\}$ such that $w\in\Delta(\omega_{i(w)})$. Moreover we apply Lemma 4 to the set N and we find a gauge $\Delta^{(0)}$ in N such that $\Delta^{(0)}(w)\subset\Delta(w)$ for each $w\in N$ and $\sigma(\|f\|,\mathcal{S})<\varepsilon/4$, for each $(\mathcal{B}_{Mc},\Delta)$-partition $\mathcal{S}$ tagged in N.

Now we define a gauge $\Delta^{(1)}$ in Ω in the following way: $\Delta^{(1)}(w)=\Delta(w)$ if $w\in\bigcup_{i=1}^n A_i$, $\Delta^{(1)}(w)=\Delta(w)\bigcap\Delta(\omega_{i(w)})$ if $w\notin(\bigcup_{i=1}^n A_i\bigcup N)$, and $\Delta^{(1)}(w)=\Delta^{(0)}(w)$ if $w\in N\bigcap(\Omega\setminus\bigcup_{i=1}^n A_i)$. Then we apply Proposition 3 with $\Delta=\Delta^{(1)}$, and we extend $\mathcal{R}=\{(C_j^{(i)},\omega_i): i=1,...,n,\ j=1,...,p_i\}$ to a $(\mathcal{B}_{Mc},\Delta^{(1)})$-partition of Ω by adding the couples $\{(D_j,w_j):\ j=1,...,r\}$. We may always assume that for $1\leq j\leq s$, $w_j\notin N$, while for $s\leq j\leq r$, $w_j\in N$. Moreover we set $\mathcal{S}=\{(D_j,w_j):\ s\leq j\leq r\}$. By construction $\mathcal{R}\bigcup\{(D_j,\omega_{i(w_j)}):\ j=1,...,s\}\bigcup\mathcal{S}$ is a $(\mathcal{B}_{Mc},\Delta)$-partition of Ω.

Then by (4), (5) and (6) we have

$$
\begin{aligned}
&\left\| \sum_{i=1}^{n} \mu(E_i) f(\omega_i) - (\mathcal{B}_{Mc}) \int_\Omega f \, d\mu \right\| \\
&\quad \le \left\| \sum_{i=1}^{n} (\mu(E_i) - \mu(A_i)) f(\omega_i) \right\| + \left\| \sum_{i=1}^{n} \sum_{j=1}^{p_i} f(\omega_i) \mu(C_j^{(i)}) + \right. \\
&\qquad \left. + \sum_{j=1}^{s} \mu(D_j) f(\omega_{i(w_j)}) + \sum_{j=s}^{r} \mu(D_j) f(\omega_i) - (\mathcal{B}_{Mc}) \int_\Omega f \, d\mu \right\| \\
&\qquad + \sigma(\|f\|, \mathcal{S}) + \left\| \sum_{j=1}^{s} \mu(D_j) f(\omega_{i(w_j)}) \right\| \\
&\quad < m \cdot \sum_{i=1}^{n} \mu(E_i \triangle A_i) + m \cdot \mu(\Omega \setminus \bigcup_{i=1}^{n} A_i) + \frac{\varepsilon}{2} \\
&\quad < \frac{3\varepsilon}{4} + m \cdot \mu \left((\bigcup_{i=1}^{n} E_i) \setminus (\bigcup_{i=1}^{n} A_i) \right) \\
&\quad < \frac{3\varepsilon}{4} + m \cdot \sum_{i=1}^{n} \mu(E_i \setminus A_i) < \varepsilon \, .
\end{aligned}
$$

Then the function f is Fremlin integrable. □

Remark 3. We observe that in the previous proof we use the fact that the family $\mathcal{B}^U$ separates points off closed sets. In [3] it is proved that if Ω is compact it is possible to use in the construction of Riemann sums only suitable "intervals", instead of measurable sets. Indeed let $\mathcal{A}$ be an algebra of measurable sets such that whenever $F \subseteq G$, F is closed and G is open there is an $A \in \mathcal{A}$ such that $F \subseteq A \subseteq G$; the 'intervals' considered by Fremlin are the elements of a family $\mathcal{C} \subseteq \mathcal{A}$ such that every member of $\mathcal{A}$ is a finite disjoint union of members of $\mathcal{C}$. We note that in general a family of sets that separates points off closed sets not necessarily satisfies the above condition of Fremlin (see [1]).

References

[1] B. Bongiorno, L. Di Piazza and K. Musial, *An alternate approach to the McShane integral*, Real Anal. Ex., **25(1)** (1999/2000), 829–848.

[2] D. H. Fremlin, *Integration of Vector-Valued Functions*, Atti Semin. Mat. Fis. Univ. Modena **42** (1994), 205–211.

[3] D. H. Fremlin, *The generalized McShane integral*, Illinois J. Math. **39** (1995), 39–67.

[4] D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Illinois J. Math. **38** (1994), 471–479.

[5] D. H. Fremlin, *Topological Riesz space and measure theory*, Cambridge Univ. Press (1974).

[6] D. H. Fremlin, J. Mendoza, *On the integration of vector-valued functions*, Illinois J. Math. **38** (1994), 127–147.

[7] R. Gordon, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Graduate Studies in Math. Vol. 4 Am. Math. Soc. (1991).

[8] R. Henstock, *Theory of integration*, Butterworths, London (1963).

[9] J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J. **7** (1957), 418–446.

[10] H. Lebesgue, *Integral, longueur, maire*, Ann. Mat. Pura Appl. **7** (1902), 231–359.

[11] N. W. Leng, L. P. Yee, *Nonabsolute integral on measure spaces*, Bull. London Math. Soc. **32** (2000), 34–38.

[12] E. J. McShane, *Unified integration*, Academic Press, New York (1983).

[13] K. M. Ostaszewski, *Henstock integration in the plane*, Mem. Amer. Math. Soc., 353 (1986).

[14] S. Schwabik, Y. Guoju, *Topics in Banach space integration*, Series in Real Analysis, **10**, Hackensack, NJ (2005).

[15] V. A. Skvortsov, *Variations and variational measures in integration theory and some applications*, J. Math. Sci., **9(5)** (1998), 3293–3322.