RESEARCH

Real Analysis Exchange
Vol. 36(2), 2010/2011, pp. 373–404

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TRIANGLE INTEGRAL - A NONABSOLUTE INTEGRATION PROCESS SUITABLE FOR PIECEWISE LINEAR SURFACES

Abstract

We present a two-dimensional nonabsolute gauge integral which satisfies several convergence theorems and a general divergence theorem, and at the same time admits a change of variables formula valid up to affine transformations - thus applicable to piecewise linear surfaces. Our approach is based on a modification of the M_1-integral presented in [6], using triangle-based partitions.

1 Introduction

The problem of defining an integration process for which

$$\int_a^b F' = F(b) - F(a) \tag{1.1}$$

is satisfied for each differentiable real valued function F defined in a compact interval $[a, b]$ was first solved, independently, by Denjoy and Perron in the

Mathematical Reviews subject classification: Primary: 26A39
Key words: nonabsolute integration, Henstock-Kurzweil integration
Received by the editors September 27, 2010
Communicated by: Luisa Di Piazza
*The research for this paper was supported by CAPES.

beginning of the twentieth century; it was known to Lebesgue, when he introduced his integral in 1904, that it would fail to satisfy this property, since for some differentiable functions the derivative is not even integrable. Take for example the function F defined in $[0,1]$ by

$$F(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & \text{if } x > 0; \\ 0, & \text{if } x = 0. \end{cases}$$

In the fifties, Henstock and Kurzweil developed another integration process which was shown to be equivalent to Denjoy's and Perron's, but with a much simpler definition based on gauges and Riemann sums. The integrals defined by Denjoy, Perron and Henstock-Kurzweil are often referred to as nonabsolute integrals, since they lack the strong property of the Lebesgue integral which says that if f is integrable, then $|f|$ is also integrable. We refer to [2] for the definitions of the mentioned integrals.

The problem of generalizing the Lebesgue integral in order to obtain a large class of functions which satisfy (1.1) is naturally extended to the case where the domain has more dimensions. It is desirable to define an integration process in some domain $U \subset \mathbb{R}^n$ which satisfies the following version of the divergence theorem:

$$\int_U divF = \int_{\partial U} F \cdot N, \text{for each differentiable } F : \mathbb{R}^n \to \mathbb{R}^n. \tag{1.2}$$

To clarify the notation, $divF$ denotes the *divergence of* F, defined by $divF \doteq \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$, and $F \cdot N$ is the scalar product of F with the outward-pointing normal vector N, which is defined over the boundary ∂U of U. If (1.2) holds, it is implied that $divF$ must be integrable for each differentiable function F. If the domain U is relatively simple - take, for example, a simplicial complex - the integral to the right is expected to converge for each differentiable F, since $F \cdot N$ is continuous in each face of U. The natural extension of the definition of integral given by Henstock and Kurzweil for the multidimensional case does not satisfy (1.2) even when U is an interval (that is, the cartesian product of compact intervals in the real line).[1]

Several attempts have been made to modify the definition of Henstock and Kurzweil for more dimensions in order to obtain (1.2), maintaining the gauge approach. One line of development was to impose on the admissible interval-based partitions of the domain some regularity conditions (see for example [6] and [13]); this way, the simplicity of working with intervals is kept and it is

[1] The advantage of this natural extension is that a Fubini type theorem is satisfied (see [14]). In fact, there is a degree of incompatibility between the divergence theorem and Fubini's Theorem; we refer to [9], [15] and the upcoming [8] for a study on this subject.

possible to obtain satisfactory convergence theorems for the integrals defined in the referred articles. On the other hand, the M_1-integral defined in [6], as well as the Henstock-Kurzweil multidimensional integral, have the common flaw of being sensitive to rotations (see [9] and [10]); thus these integrals are not suitable for applications to manifolds, since we do not have a satisfactory change of variables formula. The other line of development is based on taking partitions of the domain into sets which are more complicated than intervals (see, for example, [4] and [5]). This way a change of variables formula is obtained - up to C^1-transformations for the integral defined by Kurzweil in [4]. However, the price for admitting complicated partitions is that it becomes hard to prove even simple properties for the integrals defined in the referred articles. In this context Pfeffer obtained good results in [16]; in this paper Pfeffer defines an integral using partitions of the domain into BV sets, also with some regularity control. We should mention that there is a number of interesting and more recent results for nonabsolute multidimensional integrals which are defined leaving aside the simple partition-controlled-by-gauge approach; the interested reader should check, for example, the PU-integral from [3], which is based on partitions of the unity, and the distributional integral from [19].

In the present paper, still maintaining the gauge approach, we introduce an integral which is a modification of the M_1-integral - we call it Δ-integral. Our admissible partitions will be into triangles - which are, in a way, simpler[2] than intervals. This way, all the basic properties of the M_1-integral remain true for the Δ-integral, and we can easily obtain a change of variables formula valid up to affine transformations - which makes this integral suitable for application to piecewise linear surfaces ([17]). We recall that any smooth manifold admits a (unique) piecewise linear structure (cf. [20]). We will show in Section 4 that the Δ-integral is in fact strictly less general than the M_1-integral. In Sections 5, 6, 7 and 8, more advanced properties are proved for the Δ-integral, including a general divergence theorem and several convergence theorems.

For further investigation on the divergence theorem, the authors recommend De Pauw's survey [1]; not only is the meaning of the integral symbol in (1.2) considered, but also the class of functions F and the domains U. In the present paper we will restrict our study to domains that are finite unions of triangles. See the comments on further steps in the last section.

[2]The idea of trying triangle-based partitions for nonabsolute integration in order to try to avoid rotation problems arose from a discussion on the subject between Prof. Pavel Krejčí from the Institute of Mathematics of the Academy of Sciences of the Czech Republic and one of the authors; Prof. Krejčí's argument that triangles might help, since they are more simple than intervals and are also independent from coordinate systems, inspired the present work.

2 The Δ-integral

Let us establish some notation. When we speak of a partition $K_1, \dots, K_n$ of a nonempty closed subset K of $\mathbb{R}^2$, we mean that each K_j is closed and has positive Lebesgue measure, $K_1 \cup \dots \cup K_n = K$, and the sets $K_1, \dots, K_n$ do not overlap (which means that the Lebesgue measure of $K_i \cap K_j$, that we shall denote by $|K_i \cap K_j|$, equals zero whenever $i \neq j$). For partitions we use also the notation $\{K_j\}_{j \in \Gamma}$, where Γ is simply a finite and nonempty index set. When we speak of a triangle in $\mathbb{R}^2$, we are always referring to the *closed* set.

Let I be a triangle in $\mathbb{R}^2$ and δ be a gauge in I. We say that a set of the form $\mathcal{P} = \{(I^j, t^j)\}_{j \in \Gamma}$ is a Δ-*partition* of I if $\{I^j\}_{j \in \Gamma}$ is a partition of I into triangles with $t^j \in I^j$ for each $j \in \Gamma$, and we say that it is δ-*fine* when in addition we have for each $j \in \Gamma$ that $I^j \subset B_{\delta(t^j)}(t^j)$ (the open ball centered in t^j with radius $\delta(t^j)$).

The *irregularity* of $\mathcal{P}$ is defined by

$$irr(\mathcal{P}) \doteq \sum_{j \in \Gamma} q(I^j), \tag{2.1}$$

where $q(I^j) = per(I^j)\, diam(I^j)$ (the perimeter times the diameter of the triangle I^j), and for each given $C > 0$, we say that $\mathcal{P}$ is C-*regular* when $irr(\mathcal{P}) \leq C$.

To obtain a satisfactory definition for the integral, we must verify Cousin's Lemma for this kind of partition.

Lemma 2.1 (Cousin)**.** *Let I be a triangle, δ a gauge in I and $C \geq q(I)$. Then there exists a δ-fine, C-regular Δ-partition of I.*

Proof. Let us first note the following for an arbitrary triangle J. By linking the midpoints of each side of J by three line segments, we induce a partition of J into four triangles $J_1, \dots, J_4$, which are congruent to each other and similar to J. By doing so we have

$$q(J) = \sum_{j=1}^{4} q(J_j). \tag{2.2}$$

Let us prove Cousin's Lemma for $C = q(I)$; the general case follows trivially. Suppose that I does *not* admit a δ-fine, C-regular Δ-partition. Then if we take a partition of I into four congruent triangles $I_1^1, \dots, I_4^1$, as it was described above, we have by (2.2) that at least one of these triangles, say, $I_{n_1}^1$, does not admit a δ-fine, $\frac{C}{4}$-regular Δ-partition.

Taking likewise a partition of $I_{n_1}^1$ into four triangles $I_1^2, \dots, I_4^2$, at least one of them, $I_{n_2}^2$, does not admit a δ-fine, $\frac{C}{4^2}$-regular Δ-partition.

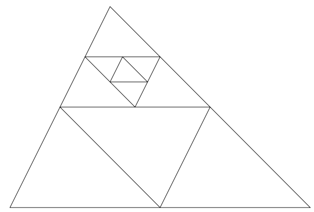

Figure 1: Proof of Cousin's Lemma.

Proceeding this way, as shown in Figure 1, we obtain a sequence of triangles $(I_{n_j}^j)_j$ such that, for each j, $I_{n_j}^j$ does not admit a δ-fine, $\frac{C}{4^j}$-regular Δ-partition, and $I \supset I_{n_1}^1 \supset I_{n_2}^2 \supset \dots$. Since $diam(I_{n_j}^j) \xrightarrow{j} 0$, there exists $t \in I$ such that $\cap_{j\in\mathbb{N}} I_{n_j}^j = \{t\}$. There is k satisfying $I_{n_k}^k \subset B_{\delta(t)}(t)$, which leads us into contradiction, since $\{(I_{n_k}^k, t)\}$ is a δ-fine, $\frac{C}{4^k}$-regular Δ-partition of $I_{n_k}^k$. $\square$

Definition 2.2. *Let I be a triangle. We say that $f : I \to \mathbb{R}$ is* Δ-integrable *if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$ and each $C \geq q(I)$, there is a gauge δ in I which satisfies, for each Δ-partition of I, the following condition:*

$$\text{if } \mathcal{P} \text{ is } \delta\text{-fine and } C\text{-regular, then } |S(f,\mathcal{P}) - A| < \epsilon.$$

In that case, we write

$$(\Delta)\int f \doteq (\Delta)\int_I f \doteq A.$$

We point out that the difference between the definition of the Δ-integral and the definition of the M_1-integral from [6] is restricted to the fact that, for the M_1-integral, the considered domain I is an interval and the considered partitions are into intervals instead of triangles; the regularity condition is the same, for the M_1-integral the diameter and the perimeter of each *interval* of the partition $\mathcal{P}$ is considered to calculate $irr(\mathcal{P})$.

The regularity condition imposed to the Δ-partitions is necessary to obtain the divergence Theorem for the Δ-integral in a most general setting, as we shall see (Theorem 8.1). The next result gives us a change of variables formula for the Δ-integral, which makes it suitable for piecewise linear surfaces:

Proposition 2.3. *Let I and J be triangles, T an affine operator in $\mathbb{R}^2$ which satisfies $T[I] = J$, and $f : J \to \mathbb{R}$ a Δ-integrable function. Then $f \circ T$ is Δ-integrable in I and*

$$(\Delta)\int_I f \circ T = |\det T|^{-1} (\Delta)\int_J f.$$

Proof. Let $\epsilon > 0$ and $C \geq q(I)/||T||$. then there exists a gauge η in J such that for each η-fine, $(||T||C)$-regular Δ-partition $\mathcal{Q}$ of J we have

$$\left| S(f, \mathcal{Q}) - (\Delta)\int_J f \right| < |\det T|\epsilon. \tag{2.3}$$

Consider in I the gauge $\delta \doteq \eta \circ T$, and suppose that $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ is a δ-fine, C-regular Δ-partition of I. Then if we denote $T < \mathcal{P} > \doteq \{(T[I^j], T(t^j)) : j \in \Gamma\}$ we have that $|T[I^j]| = |\det T|\,|I^j|$ implies

$$S(f, T < \mathcal{P} >) = |\det T| S(f \circ T, \mathcal{P}).$$

Note that for an arbitrary triangle K inequality $q(T[K]) \leq ||T||q(K)$ holds, thus $irr(T < \mathcal{P} >) \leq ||T||irr(\mathcal{P})$. Then $T < \mathcal{P} >$ is a δ-fine, $(||T||C)$-regular Δ-partition of J. Then by inequality (2.3) we have that

$$\left| S(f \circ T, \mathcal{P}) - |\det T|^{-1}(\Delta)\int_I f \right| =$$

$$= \left| |\det T|^{-1} S(f, T < \mathcal{P} >) - |\det T|^{-1}(\Delta)\int_I f \right| < \epsilon,$$

which concludes our proof. □

It is unknown to the authors whether it is possible to generalize the above proposition for more general transformations; see the commentaries in Section 9.

3 Basic properties

Throughout this section, I always denotes a triangle, except when indicated.

Proposition 3.1 (Linearity)**.** *Let f and g be Δ-integrable functions in I and $\lambda \in \mathbb{R}$. Then $f + g$ and λf are Δ-integrable, and*

$$(\Delta)\int (f + g) = (\Delta)\int f + (\Delta)\int g, \text{ and}$$

$$(\Delta)\int (\lambda f) = \lambda(\Delta)\int f.$$

In particular, the zero function is Δ-integrable and $(\Delta)\int 0 = 0$.

Proof. Let $\epsilon > 0$ and $C \geq q(I)$. There exist gauges δ_1 and δ_2 in I such that, for each δ_1-fine, C-regular Δ-partition $\mathcal{P}$ we have

$$\left| \sum_{(J,t)\in\mathcal{P}} f(t)|J| - (\Delta)\int f \right| < \frac{\epsilon}{2}$$

and for each δ_2-fine, C-regular Δ-partition $\mathcal{Q}$ we have

$$\left| \sum_{(J,t)\in\mathcal{Q}} g(t)|J| - (\Delta)\int g \right| < \frac{\epsilon}{2}.$$

Thus if $\mathcal{P}$ is a $min\{\delta_1, \delta_2\}$-fine, C-regular Δ-partition of I, then it satisfies

$$\begin{aligned} \left| \sum_{(J,t)\in\mathcal{P}} (f(t)+g(t))|J| - \left((\Delta)\int f + (\Delta)\int g \right) \right| &\leq \left| \sum_{(J,t)\in\mathcal{P}} f(t)|J| - (\Delta)\int f \right| \\ &+ \left| \sum_{(J,t)\in\mathcal{P}} g(t)|J| - (\Delta)\int g \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and the first part of the proof is concluded. The second part derives trivially form the identity $S(\lambda f, \mathcal{P}) = \lambda S(f, \mathcal{P})$. □

Like the Lebesgue integral, the Δ-integral satisfies the following:

Proposition 3.2. *If $f : I \to \mathbb{R}$ equals zero almost everywhere, then f is Δ-integrable and $(\Delta)\int f = 0$.*

Proof. Let $\epsilon > 0$ and $C \geq q(I)$, and consider the set $X \doteq \{x \in I : f(x) \neq 0\}$. X can be written as a disjoint union of the form $\cup_n X_n$, where

$$X_n \doteq \{x \in I : n-1 < f(x) \leq n\},\ n \in \mathbb{N}.$$

Note that $|X_n| = 0$ for each natural number n, since $|X| = 0$. Then there exists an open subset U_n of I satisfying

$$X_n \subset U_n,\ |U_n| < \frac{1}{2^n n}\epsilon.$$

For each $x \in X_n$, there exists $\delta(x) > 0$ such that $B_{\delta(x)}(x) \subset U_n$. this defines a gauge in X. Extending δ arbitrarily to all I we have, for each δ-fine, C-regular Δ-partition $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ of I, that

$$\begin{aligned}
|S(f,\mathcal{P}) - 0| &= \left|\sum_{j\in\Gamma} f(t^j)|I^j|\right| \\
&= \left|\sum_{j\in\Gamma_0} f(t^j)|I^j| + \sum_{n\in\mathbb{N}}\sum_{j\in\Gamma_n} f(t^j)|I^j|\right| \\
&\leq \sum_{n\in\mathbb{N}}\sum_{j\in\Gamma_n} |f(t^j)|\,|I^j| \;\leq\; \sum_{n\in\mathbb{N}} n \sum_{j\in\Gamma_n} |I^j| \\
&\leq \sum_{n\in\mathbb{N}} n|U_n| \;\leq\; \epsilon,
\end{aligned}$$

where $\Gamma_0 \doteq \{j \in \Gamma : t^j \in I \setminus X\}$ e $\Gamma_n \doteq \{j \in \Gamma : t^j \in X_n\}$. Thus f is Δ-integrable and $(\Delta)\!\int f = 0$. $\square$

As a consequence, by linearity of the Δ-integral and using the same argument of Proposition 3.2, we obtain the following corollary:

Corollary 3.3. *If f is Δ-integrable in I and $g : I \to \mathbb{R}$ satisfies $g(x) = f(x)$ for almost each $x \in I$, then g is Δ-integrable in I and*

$$(\Delta)\!\int g = (\Delta)\!\int f.$$

The Δ-integral also preserves order, in the following sense:

Proposition 3.4. *If f and g are Δ-integrable in I and $f(x) \leq g(x)$ for almost all $x \in I$, then*

$$(\Delta)\!\int f \leq (\Delta)\!\int g.$$

Proof. By the linearity, it suffices to show that $(\Delta)\!\int h \geq 0$, where $h = g - f$. By Corollary 3.3, we can assume that $h(x) \geq 0$ for *all* $x \in I$. But in that case, for each Δ-partition $\mathcal{P}$ of I, the Riemann sum $S(h,\mathcal{P})$ is non-negative, and the result follows from the fact that $(\Delta)\!\int h$ can be approximated by Riemann sums of that kind. $\square$

Proposition 3.5 (Cauchy Criterion)**.** *A function f defined in I is Δ-integrable if and only if for each $\epsilon > 0$ and each sufficiently large C there exists a gauge δ in I such that, for each two δ-fine, C-regular partitions $\mathcal{P}$ and $\mathcal{P}'$ of I, we have*

$$|S(f,\mathcal{P}) - S(f,\mathcal{P}')| < \epsilon.$$

Proof. ($\Leftarrow$) Let us fix a sufficiently large C and consider a sequence $(\delta_n)_n$ of gauges in I satisfying:

1. for each $x \in I$ we have that $\delta_1(x) \geq \delta_2(x) \geq \dots$;
2. for each two δ_n-fine, C-regular partitions $\mathcal{P}$ and $\mathcal{P}'$ of I, we have

$$|S(f,\mathcal{P}) - S(f,\mathcal{P}')| < \frac{1}{n}.$$

Fix, for each n, a δ_n-fine, C-regular Δ-partition $\mathcal{P}_n$ of I. Note that for each n, k we have

$$|S(f,\mathcal{P}_n) - S(f,\mathcal{P}_{n+k})| < \frac{1}{n}, \tag{3.1}$$

and it follows that $(S(f,\mathcal{P}_n))_n$ is a Cauchy sequence. Writing $\lim_n S(f,\mathcal{P}_n) \doteq A$ and taking the limit in k in (3.1) we obtain

$$|S(f,\mathcal{P}_n) - A| \leq \frac{1}{n}.$$

For each $\epsilon > 0$, take a natural number n_0 which satisfies $\frac{2}{n_0} < \epsilon$. Considering $\delta \doteq \delta_{n_0}$, we have for each δ-fine, C-regular Δ-partition $\mathcal{P}$ of I that

$$|S(f,\mathcal{P}) - A| \leq |S(f,\mathcal{P}) - S(f,\mathcal{P}_{n_0})| + |S(f,\mathcal{P}_{n_0}) - A| \leq \frac{1}{n_0} + \frac{1}{n_0} < \epsilon,$$

from which follows that f is Δ-integrable and $(\Delta)\int f = A$.

The other implication is straightforward. □

Proposition 3.6 (Additivity)**.** *Let I be a triangle, $I = K \cup L$, where K and L are non-overlapping triangles. Then f is Δ-integrable in I if and only if f is Δ-integrable in K and L, and in that case*

$$(\Delta)\int_K f + (\Delta)\int_L f = (\Delta)\int_I f. \tag{3.2}$$

Proof. Suppose that f is Δ-integrable in I. Let $\epsilon > 0$ and $C \geq q(I)$, and let us consider a gauge δ in I which satisfies

$$\left| S(f, \mathcal{P}) - (\Delta)\int_I f \right| < \epsilon, \tag{3.3}$$

for each δ-fine, $2C$-regular Δ-partition $\mathcal{P}$ of I. Let $\mathcal{P}_K^1$ and $\mathcal{P}_K^2$ be δ-fine, C-regular Δ-partitions of K, and let $\mathcal{P}_L$ be a δ-fine, C-regular Δ-partition of L. Then $\mathcal{P}_I^1 \doteq \mathcal{P}_K^1 \cup \mathcal{P}_L$ and $\mathcal{P}_I^2 \doteq \mathcal{P}_K^2 \cup \mathcal{P}_L$ are δ-fine, $2C$-regular Δ-partitions of I satisfying

$$S(f, \mathcal{P}_K^1) - S(f, \mathcal{P}_K^2) = S(f, \mathcal{P}_I^1) - S(f, \mathcal{P}_I^2).$$

By (3.3) we have that $|S(f, \mathcal{P}_I^1) - S(f, \mathcal{P}_I^2)| < 2\epsilon$, thus Cauchy criterion 3.5 guarantees the Δ-integrability of f in K. Analogously we prove that f is Δ-integrable in L.

Suppose now that f is Δ-integrable in K and L. Let $\epsilon > 0$ and $C \geq q(I)$, and consider gauges δ_K in K and δ_L in L which satisfy

$$\left| S(f, \mathcal{P}) - (\Delta)\int_K f \right| < \frac{\epsilon}{2},$$

for each δ_K-fine, $3C$-regular Δ-partition $\mathcal{P}$ of K, and

$$\left| S(f, \mathcal{Q}) - (\Delta)\int_L f \right| < \frac{\epsilon}{2},$$

for each δ_L-fine, $3C$-regular Δ-partition $\mathcal{Q}$ of L. Define a gauge δ in I by

$$\delta(x) \doteq \begin{cases} \min\{\delta_K(x), d(x, L)\}, & \text{if } x \in K \setminus L; \\ \min\{\delta_L(x), d(x, K)\}, & \text{if } x \in L \setminus K; \\ \min\{\delta_K(x), \delta_L(x)\}, & \text{if } x \in K \cap L. \end{cases} \tag{3.4}$$

Let $\mathcal{P}$ be a δ-fine, C-regular Δ-partition of I. Then for each $(J, t) \in \mathcal{P}$ such that J intercepts $K \cap L$ we have that $t \in K \cap L$; if we have moreover that $J \cap K$ is *not* a triangle, then $J \cap K$ can be partitioned into up to three triangles which intercept in t. This can be done by linking t to the vertices of J which are in K, as is shown in Figure 2.

The same holds evidently when $J \cap L$ is not a triangle. By these observations, by the definition of δ and taking in account that $q(M) \leq q(N)$ when M and N are triangles with $M \subset N$, we modify partition $\mathcal{P}$ in order o obtain a Δ-partition $\mathcal{Q} = \{(I^j, t^j) : j \in \Gamma\}$ of I which satisfies

1. $S(f, \mathcal{P}) = S(f, \mathcal{Q})$;

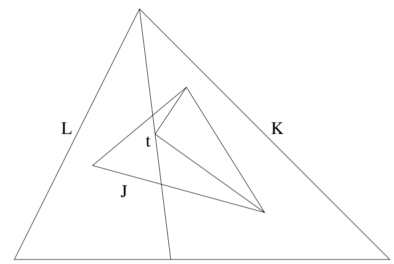

Figure 2: Additivity.

2. for each $j \in \Gamma$, $I^j \subset K$ or $I^j \subset L$;

3. denoting $\Gamma_K \doteq \{j \in \Gamma : I^j \subset K\}$, we have that $\mathcal{Q}_K \doteq \{(I^j, t^j) : j \in \Gamma_K\}$ is a δ_K-fine, $3C$-regular Δ-partition of K;

4. denoting $\Gamma_L \doteq \{j \in \Gamma : I^j \subset L\}$, we have that $\mathcal{Q}_L \doteq \{(I^j, t^j) : j \in \Gamma_L\}$ is a δ_L-fine, $3C$-regular Δ-partition of L.

Then we have that

$$\begin{aligned} \left|S(f,\mathcal{P}) - \left((\Delta)\int_K f + (\Delta)\int_L f\right)\right| &= \left|S(f,\mathcal{Q}) - \left((\Delta)\int_K f + (\Delta)\int_L f\right)\right| \\ &\le \left|S(f,\mathcal{Q}_K) - (\Delta)\int_K f\right| + \\ &\quad + \left|S(f,\mathcal{Q}_L) - (\Delta)\int_L f\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

thus f is Δ-integrable in I and (3.2) holds. □

In comparison to the M_1-integral from [6], proving additivity properties for the Δ-integral is a little bit more tricky, since we do not have (as we do for intervals) that the intersection of two overlapping triangles is always a triangle. It is a straightforward technical exercise to extend the proof above to the case where I is partitioned into a finite quantity of triangles $I_1, \ldots, I_n$. Some difficulty arises when we define δ in (3.4), but it is solved when we take

$$\delta(x) \doteq \begin{cases} \min\{\delta_j(x), d(x, \cup_{k\neq j} I_k)\}, & \text{if } x \in I_j \setminus (\cup_{k\neq j} I_k); \\ \min\{\delta_j(x) : x \in I_j\}, & \text{if } x \text{ is in two or more } I_j. \end{cases} \tag{3.5}$$

We will say that a subset of $\mathbb{R}^2$ is a Δ-*elementary set* when it is a (nonempty) finite union of triangles; it is straightforward that Δ-elementary sets can always be written as a finite union of *non-overlapping* triangles. For instance, two-dimensional simplicial complexes are Δ-elementary sets.

Taking into account that if I and J are triangles with $J \subset I$, then $I \setminus J$ is a Δ-elementary set, we have the following Corollary of Proposition 3.6:

Corollary 3.7. *If f is Δ-integrable in I (or $\mathbb{R}^2$) and J is a triangle contained in I (or $\mathbb{R}^2$), then f is Δ-integrable in J.*

4 Δ-integration in Δ-elementary sets and comparison with the M_1-integral

The additivity properties of the Δ-integral allows us to naturally extend its definition to functions defined in Δ-elementary domains, as we shall see throughout this Section.

Definition 4.1. *Let K be a Δ-elementary set in R^2, and suppose that $\{I_1, \ldots, I_n\}$ is a partition of K into triangles. We will say that $f : K \to \mathbb{R}$ is* Δ-integrable (in K) *if for each $j = 1, \ldots, n$ the restriction of f to I_j is Δ-integrable in the sense of Definition 2.2. In that case, we define*

$$(\Delta)\int f \doteq (\Delta)\int_K f \doteq \sum_{i=1}^{n} (\Delta)\int_{I_i} f.$$

The additivity of the Δ-integral guarantees that the definition above is independent of the choice of $I_1, \ldots, I_n$. In effect, if $\{J_1, \ldots, J_m\}$ is another partition of K into triangles, then the set

$$(\cup_{i=1}^{n} \partial I_i) \cup (\cup_{j=1}^{m} \partial J_j)$$

of borderlines of the triangles I_j and J_k determine a partition of K into a finite quantity of Δ-elementary sets, and each one of these sets can be partitioned into finite triangles. The conclusion follows when we apply the additivity property for the triangle case (Proposition 3.6).

The linearity of the Δ-integral (Proposition 3.1), as well as the properties described in Lemma 3.2, Corollary 3.3 and Proposition 3.4 are clearly extended to functions defined in K.

Proposition 3.6 can be generalized to a Δ-elementary set in the context that follows:

Proposition 4.2. *Let K be a Δ-elementary set, $f : K \to \mathbb{R}$ and suppose that $K = M \cup N$, where M and N are non-overlapping Δ-elementary sets. Then f is Δ-integrable in K if and only if f is Δ-integrable in M and N, and in that case we have*

$$(\Delta)\int_K f + (\Delta)\int_M f = (\Delta)\int_N f.$$

Proof. It follows directly from the fact that, if $\{M_1, \dots, M_m\}$ is a partition of M into triangles and $\{N_1, \dots, N_n\}$ is a partition of N into triangles then $\{M_1, \dots, M_m, N_1, \dots, N_n\}$ is a partition of K into triangles. $\square$

Δ-integration in Δ-elementary sets could be defined exactly as in Definition 2.2, only replacing the triangular domain I by a Δ-elementary one. These two definitions are compatible according to the following characterization:

Proposition 4.3. *Let K be a Δ-elementary set, $f : K \to \mathbb{R}$ and $A \in \mathbb{R}$. Then the following assertions are equivalent:*

1. *f is Δ-integrable and $(\Delta)\int f = A$;*
2. *for each $\epsilon > 0$ and each sufficiently large $C > 0$, there exists a gauge δ in K which satisfies, for each Δ-partition $\mathcal{P}$ of K, the following condition:*

$$\text{if } \mathcal{P} \text{ is } \delta\text{-fine and } C\text{-regular, then } |S(f, \mathcal{P}) - A| < \epsilon. \tag{4.1}$$

Δ-partitions K, δ-fineness and C-regularity must be interpreted in the same way as for Δ-partitions of a triangle. Note that Cousin's Lemma holds also for the Δ-elementary set K, in the following sense: if δ is a gauge in K and K can be partitioned into triangles $I_1, \dots, I_n$, then for each $C \geq \sum_{i=1}^n q(I_i)$ there exists a δ-fine, C-regular Δ-partition K. To see this we just have to apply the triangle version of Cousin's Lemma (Lemma 2.1) to each I_i. The *sufficiently large* $C > 0$, as specified in item *2.* of the above Proposition, can be for example those $C > 0$ which are greater than $\sum_{i=1}^n q(I_i)$, so that Cousin's Lemma guarantees the existence of δ-fine, C-regular Δ-partitions of K.

Proof (of Proposition 4.3). Let us refer by now to the functions f which satisfy *2.* as Δ'*-integrable*, and for these functions let us define the Δ' *integral of* f by $(\Delta')\int f \doteq A$.

Suppose f is Δ-integrable and $(\Delta)\int f = A$. Then K can be partitioned into triangles $I_1, \dots, I_n$ such that f is Δ-integrable in each I_i, and denoting

$$(\Delta)\int_{I_1} f = A_1, \dots, (\Delta)\int_{I_n} f = A_n,$$

we have that $A = A_1 + \cdots + A_n$. Let $\epsilon > 0$ be given, and let $C > 0$ be sufficiently large. For each i there is a gauge δ_i in I_i satisfying, for each δ_i-fine, $3C$-regular Δ-partition $\mathcal{P}$ of I_i,

$$|S_{I_i}(f, \mathcal{P}) - A_i| < \frac{\epsilon}{n}. \tag{4.2}$$

We can assume that δ_i satisfies the following additional condition: for each $x \in \partial I_i$, $B_{\delta(x)}(x)$ does not intersect the sides of I_i that do not contain x.

Define a gauge δ in K by

$$\delta(x) \doteq \begin{cases} \min\{\delta_i(x), d(x, \partial I_i)\}, & \text{if } x \in int(I_i) \text{ for some } i; \\ \min\{\delta_i(x) : i = 1, \ldots, n,\ x \in I_i\}, & \text{otherwise.} \end{cases}$$

If we fix a δ-fine, C-regular Δ-partition $\mathcal{P} = \{(J^j, t^j) : j \in \Gamma\}$ of K, then, for each i, j satisfying $|J^j \cap I_i| > 0$, we have that

$$J^j \cap I_i \neq \emptyset \Rightarrow t^j \in I_i.$$

Then for such i, j, $J^j \cap I_i$ can be partitioned into n_{ij} triangles $J_1^{ij}, \ldots, J_{n_{ij}}^{ij}$ by the line segments which link t^j to the vertices of J^j which are in I_i (n_{ij} is a natural number from one to three, see the proof of Proposition 3.6). Thus,

$$\mathcal{Q} \doteq \{(J_l^{ij}, t^j) : i = 1, \ldots, n,\ j \in \Gamma,\ |J^j \cap I_i| > 0,\ l = 1, \ldots n_{ij}\}$$

is a δ-fine Δ-partition of K. Moreover, for each $i = 1, \ldots, n$,

$$\mathcal{Q}_i \doteq \{(J_l^{ij}, t^j) \in \mathcal{Q} : J_l^{ij} \subset I_i\}$$

is a δ-fine, $3C$-regular Δ-partition of I_i, therefore satisfying (4.2). Since $\mathcal{Q} = \cup_{i=1}^n \mathcal{Q}_i$ and $S(f, \mathcal{P}) = S(f, \mathcal{Q})$, it follows that

$$|S(f, \mathcal{P}) - A| \leq |S(f, \mathcal{Q}_1) - A_1| + \cdots + |S(f, \mathcal{Q}_n) - A_n| < \epsilon,$$

so f is Δ'-integrable and $(\Delta')\!\int f \doteq A$.

Suppose now that f is Δ'-integrable and $(\Delta')\!\int f = A$ and fix a triangle $I \subset K$. Consider a sufficiently large $C > 0$,[3] let $\epsilon > 0$, and let δ be a gauge in K such that, for each δ-fine, $2C$-regular Δ-partition $\mathcal{P}$ of K, we have that

$$|S(f, \mathcal{P}) - A| < \frac{\epsilon}{2}. \tag{4.3}$$

[3] If K can be partitioned into triangles $I_1, \ldots, I_n$, it suffices to take $C \geq \max\left\{q(I), \sum_{i=1}^n q(I_i)\right\}$.

Suppose that $\mathcal{Q}$ and $\mathcal{R}$ are δ-fine, C-regular Δ-partitions of I. $K \setminus I$ admits a δ-fine, C-regular Δ-partition $\mathcal{S}$; then the Δ-partitions of K defined by

$$\mathcal{Q}' \doteq \mathcal{Q} \cup \mathcal{S} \text{ and } \mathcal{R}' \doteq \mathcal{R} \cup \mathcal{S}$$

are δ-fine and $2C$-regular, thus satisfy inequality (4.3) and it follows that

$$|S(f, \mathcal{Q}') - S(f, \mathcal{R}')| < \epsilon.$$

Since $|S(f, \mathcal{Q}') - S(f, \mathcal{R}')| = |S(f, \mathcal{Q}) - S(f, \mathcal{R})|$, by Cauchy criterion (Proposition 3.5) we have that f is Δ-integrable in I. Therefore f is Δ-integrable in K, and Proposition 4.2 guarantees that $(\Delta)\int f = A$. □

Using the characterization above, it is straightforward to prove that the Δ-integral also satisfies the Cauchy criterion for functions defined in Δ-elementary sets; it is just an adaptation of the proof of proposition 3.5.

It is possible now to compare the Δ-integral with the M_1-integral for interval domains:

Proposition 4.4. *Let $K \subset \mathbb{R}^2$ be an interval. If $f : K \to \mathbb{R}$ is Δ-integrable, then f is also M_1-integrable and*

$$(M_1)\int f = (\Delta)\int f.$$

PROOF. Let $\epsilon > 0$ be given, let $C > 0$ be sufficiently large, and suppose that δ is a gauge in K such that, for each δ-fine, $4C$-regular Δ-partition $\mathcal{P}$ of K, we have that

$$\left| S(f, \mathcal{P}) - (\Delta)\int_K f \right| < \epsilon. \tag{4.4}$$

Let $\mathcal{Q} = \{(K^j, t^j) : j \in \Gamma\}$ be a δ-fine, C-regular M_1-partition of K. For each j, the line segments which link t^j to the vertices of K^j induce a partition K^j into triangles $K_1^j, \cdots, K_{n_j}^j$, where n_j is an integer from two to four, as seen in Figure 3.

Then the Δ-partition $\mathcal{P}_0$ defined by

$$\mathcal{P}_0 \doteq \cup_{j \in \Gamma} \cup_{i=1}^{n_j} \{(K_i^j, t^j)\}$$

is $4C$-regular, since for each j, i we have $q(K_i^j) \le |K^j| diam(K^j)$. The conclusion follows from (4.4) and from the equality $S(f, \mathcal{P}_0) = S(f, \mathcal{Q})$. □

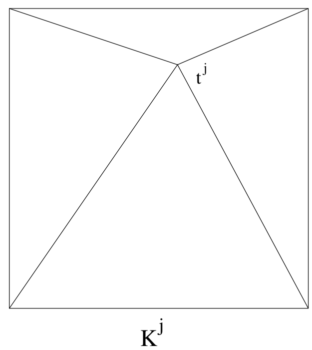

Figure 3: Comparison between the Δ-integral and the M_1-integral.

It was proven in [9] (see also the upcoming paper [10]) that the M_1-integral is sensitive to rotations, that is, there are some M_1-integrable functions which, if we compose with a certain rotation, cease to be M_1-integrable. Therefore, since the Δ-integral has a change of variables formula valid for linear transformations (Proposition 2.3), the converse to Proposition 4.4 does not hold in general.

5 Saks-Henstock Lemma, almost everywhere derivability of Δ-primitives and measurability of Δ-integrable functions

Lemma 5.1 (Saks-Henstock). *Let K be a Δ-elementary set and $f : K \to \mathbb{R}$ be a Δ-integrable function, and consider a gauge δ in K such that f's Δ-integrability condition (4.1) is satisfied for $\epsilon > 0$ and a sufficiently large $C > 0$. Suppose that $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ is a δ-fine, C-regular Δ-partition of K.*

Then for each nonempty $\Gamma' \subset \Gamma$ we have that

$$\left| \sum_{j\in\Gamma'} \left[f(t^j)|I^j| - (\Delta) \int_{I^j} f \right] \right| \leq \epsilon.$$

Proof. Let $r > 0$. Fix for each $j \in \Gamma \setminus \Gamma'$ a gauge δ_j in I^j, smaller than δ in I^j, and a δ_j-fine, $q(I^j)$-regular Δ-partition $\mathcal{P}_j$ of I^j which satisfy

$$\left| S_{I^j}(f, \mathcal{P}_j) - (\Delta)\int_{I^j} f \right| < \frac{r}{\#(\Gamma \setminus \Gamma')}, \tag{5.1}$$

where $\#(\Gamma \setminus \Gamma')$ denotes the number of elements of $\Gamma \setminus \Gamma'$. Consider the following partition of I:

$$\tilde{\mathcal{P}} \doteq \{(I^j, t^j) : j \in \Gamma'\} \cup \left(\cup_{j\in\Gamma\setminus\Gamma'}\mathcal{P}_j\right).$$

It is easily seen that $\tilde{\mathcal{P}}$ is δ-fine, and that

$$irr(\tilde{\mathcal{P}}) = \sum_{j\in\Gamma'} q(I^j) + \sum_{j\in\Gamma\setminus\Gamma'} irr(\tilde{\mathcal{P}})$$

$$\leq \sum_{j\in\Gamma'} q(I^j) + \sum_{j\in\Gamma\setminus\Gamma'} q(I^j) \leq C.$$

By the Δ-integrability of f we have that

$$\left|S_I(f, \tilde{\mathcal{P}}) - (\Delta)\int_K f\right|$$

$$= \left|\sum_{j\in\Gamma'} f(t^j)|I^j| + \sum_{j\in\Gamma\setminus\Gamma'} S_{I^j}(f, \mathcal{P}_j) - \sum_{j\in\Gamma'}(\Delta)\int_{I_j} f - \sum_{j\in\Gamma\setminus\Gamma'}(\Delta)\int_{I_j} f\right| \leq \epsilon.$$

Then from inequality (5.1) it follows that

$$\left|\sum_{j\in\Gamma'} f(t^j)|I^j| - \sum_{j\in\Gamma'}(\Delta)\int_{I_j} f\right| \leq \epsilon + r.$$

Taking the limit when r goes to zero, we conclude our proof. □

Saks-Henstock Lemma can also be presented as follows:

Corollary 5.2 (Saks-Henstock)**.** *Let K be a Δ-elementary set and $f : K \to \mathbb{R}$ be a Δ-integrable function, and consider a gauge δ in K such that f's Δ-integrability condition (4.1) is satisfied for $\epsilon > 0$ and a sufficiently large $C > 0$. Suppose that $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ is a δ-fine, C-regular Δ-partition of K. Then*

$$\sum_{j\in\Gamma}\left|f(t^j)|I^j| - (\Delta)\int_{I^j} f\right| \leq 2\epsilon.$$

Proof. Just observe that

$$\sum_{j\in\Gamma}\left|f(t^j)|I^j| - (\Delta)\int_{I^j} f\right| =$$

$$= \sum_{j\in\Gamma_1}\left(f(t^j)|I^j| - (\Delta)\int_{I^j} f\right) - \sum_{j\in\Gamma_2}\left(f(t^j)|I^j| - (\Delta)\int_{I^j} f\right),$$

where $\Gamma_1 \doteq \left\{j\in\Gamma : f(t^j)|I^j| - (\Delta)\int_{I^j} f \geq 0\right\}$ and $\Gamma_2 \doteq \Gamma\setminus\Gamma_1$, and apply twice Saks-Henstock Lemma 5.1. □

The concept of derivative of a triangle function is an useful tool for obtaining good convergence results for the Δ-integral. In abstract integration theory, usually the concept of derivative is applicable to interval functions (see for example [18]). For our convenience, we adapt the concept to triangle functions. We denote by $Sub_\Delta(K)$ the set of all triangles which are contained in the Δ-elementary set K. A *triangle function in* K is a real-valued function defined on $Sub_\Delta(K)$. By the additivity property of the Δ-integral (Proposition 4.2), for each Δ-integrable function f in K it is possible to define a triangle function by

$$J\in Sub_\Delta(K)\mapsto (\Delta)\int_J f.$$

We will refer to this mapping as the Δ-*primitive* of f.

Given $\rho>0$, a nonempty closed set $B\subset\mathbb{R}^2$ is said to be ρ-*regular* if

$$\sup\{|B|/|J| : J \text{ is a square which contains } B\} > \rho.$$

In particular, when B is an interval, B is ρ-regular if and only if $l(B)/L(B) > \rho$, where $l(B)$ and $L(B)$ denote respectively the largest and the shortest sides of B.

Definition 5.3. *Let $K\subset\mathbb{R}^2$ be a Δ-elementary set and $F : Sub_\Delta(I)\to\mathbb{R}$ be a triangle function in K. We say that F is* derivable in $x\in K$ *if there exists $a\in\mathbb{R}$ such that for each $\rho>0$, we have that each decreasing sequence of ρ-regular triangles $I_j\subset Sub_\Delta(I)$ which converges to x satisfies*

$$\frac{F(I_j)}{|I_j|}\to a.$$

In that case, we write $F'(x)\doteq a$.

By *decreasing sequence of triangles $(I_j)_j$ converging to x*, we mean that $I_1\supset I_2\supset\dots$ and $\cap_j I_j = \{x\}$. It is usual to write

$$\lim_{J\to x}\frac{F(J)}{|J|} = a;$$

this notation means precisely that for each $\rho>0$ we have that each decreasing sequence of ρ-regular triangles I_j converging to x satisfies $\frac{F(I_j)}{|I_j|}\to a$. The next important result relates Δ-integration with derivation:

Proposition 5.4. *Let f be Δ-integrable in the Δ-elementary set K, denote by F the Δ-primitive of f. Then*

$$\lim_{J \to x} \frac{F(J)}{|J|} = f(x) \tag{5.2}$$

for almost all $x \in K$.

The proof involves Vitali's Covering theorem. We recall that, given a nonempty subset X of $\mathbb{R}^2$, we say that a family $\mathcal{C}$ of closed subsets of $\mathbb{R}^2$ *covers X in the sense of Vitali* if for each $x \in X$ there exists $\rho = \rho(x) > 0$ and a sequence $(B_n)_n$ of ρ-regular elements of $\mathcal{C}$ which satisfy $B_1 \supset B_2 \supset \ldots$ and $\cap_n B_n = \{x\}$. Vitali's Covering Theorem states the following:

Theorem 5.5 (Vitali's Covering Theorem)**.** *Let X be a nonempty subset of $\mathbb{R}^2$. If $\mathcal{C}$ covers X in the sense of Vitali, then there exists an at most countable subset $\{B_n : n \in \Lambda\}$ of $\mathcal{C}$ such that B_n are pairwise disjoint and $|X \setminus \cup_n B_n| = 0$.*

We will also need the following Lemma:

Lemma 5.6. *For each $r > 0$ there is an $A = A(r) > 0$ which satisfies $q(J) \leq A|J|$ for each r-regular triangle J.*

Proof (of Lemma 5.6). Let J be an r-regular triangle. Then there exists a square D containing J and satisfying $|J| \geq r|D|$. Then

$$q(J) = perim(J)diam(J) < 3\,diam(J)^2 \leq 3\,diag(D)^2 = 6|D| \leq \frac{6}{r}|J|,$$

thus $A \doteq \frac{6}{r}$ satisfies the desired condition. □

Proof (of Proposition 5.4). Let $X \doteq \{x \in K : \text{(5.2) is not satisfied}\}$. Then for each $x \in X$, we have that F is not derivable in x or F is derivable in x but $F'(x) \neq f(x)$; in either case, for each such x there are $\rho(x) > 0$ and $\eta(x) > 0$ such that for each neighbourhood V of x there exists a $\rho(x)$-regular triangle $J = J(x, V) \subset K$ satisfying $x \in J \subset V$ and

$$|F(J) - f(x)|J|| > \eta(x)|J|. \tag{5.3}$$

For each $m, n \in \mathbb{N}$ let $X_{mn} \doteq \{x \in X : \rho(x) > 1/m,\ \eta(x) > 1/n\}$. Note that $\cup_{m,n \in \mathbb{N}} X_{mn} = X$, so it suffices to show that for each given $m, n \in \mathbb{N}$ we have $|X_{mn}| = 0$. By Lemma 5.6 there exists $A > 0$ such that, for each $\frac{1}{m}$-regular triangle J, we have

$$q(J) \leq A|J|. \tag{5.4}$$

Fix $B > |K|$ and let $\epsilon > 0$. Since f is Δ-integrable in K, by the Corollary 5.2 of Saks-Henstock Lemma there exists a gauge δ in K such that for each δ-fine, AB-regular Δ-partition $\mathcal{P}$ *in* K we have

$$\sum_{(J,t)\in\mathcal{P}} |F(J) - f(t)|J|| < \epsilon. \tag{5.5}$$

We can assume that $\delta(x) \leq 1$ for each $x \in K$, and that B is large enough, so that there exist δ-fine AB-regular Δ-partitions of K.

The family

$$\mathcal{C} \doteq \{J(x,V) : x \in X_{mn},\ V \text{ is a neighbourhood of } x \text{ contained in } B_{\delta(x)}(x)\}$$

is a covering of X_{mn} in the sense of Vitali, then by Vitali's Covering Theorem 5.5 there are $J_1 = J_1(x_1, V_1), \ldots, J_k = J_k(x_k, V_k) \in \mathcal{C}$, pairwise disjoint, satisfying

$$m^*(X_{mn}) < \sum_{i=1}^{k} |J_i| + \epsilon, \tag{5.6}$$

where m^* denotes exterior measure. (5.4) guarantees that

$$\sum_{i=1}^{k} q(J_i) \leq A \sum_{i=1}^{k} |J_i| \leq AB, \tag{5.7}$$

then by (5.3), (5.5) and (5.6) we have that

$$\begin{aligned} m^*(X_{mn}) &< \sum_{i=1}^{k} |J_i| + \epsilon < \sum_{i=1}^{k} \frac{|F(J_i) - f(x_i)|J_i||}{\eta(x_i)} + \epsilon \\ &< n \sum_{i=1}^{k} |F(J_i) - f(x_i)|J_i|| + \epsilon < (n+1)\epsilon, \end{aligned}$$

and thus $|X_{mn}| = 0$. □

It is now possible to show that Δ-integrable functions are measurable. We will refer to functions defined in a Δ-elementary set K which can be written in the form

$$f(x) = \chi_{I^1}(x)a_1 + \cdots + \chi_{I^n}(x)a_n,$$

where $\{I^1, \ldots, I^n\}$ is a partition of K into triangles and $a_1, \ldots, a_n$ are constants, as *triangle-step functions*.

Proposition 5.7. *If f is Δ-integrable in a Δ-elementary set K, then f is almost everywhere the limit of a sequence of triangle-step functions. In particular, f is measurable.*

Proof. In the case that K is a triangle, for each $n \in \mathbb{N}$, we can take a partition of K into 4^n congruent triangles $I_n^1, \ldots, I_n^{4^n}$, as shown in Figure 4.

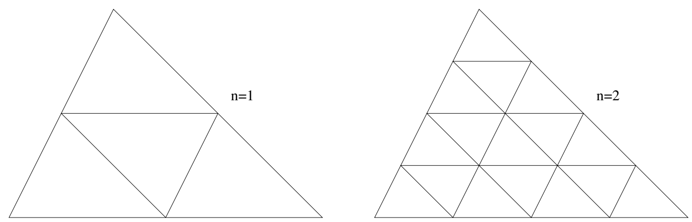

Figure 4: Partitions of K.

Define, for each $x \in I_n^j$,

$$f_n(x) \doteq \frac{F(I_n^j)}{|I_n^j|},$$

where F is the Δ-primitive of f. Proposition 5.4 guarantees that $f_n(x) \stackrel{n}{\to} f(x)$ for almost all $x \in K$. The result can be naturally extended to the Δ-elementary case, since Δ-elementary sets can be finitely partitioned into triangles. $\square$

6 Some convergence theorems

We present in this section versions of the monotone convergence theorem, dominated convergence theorem and Fatou's Lemma for the Δ-integral.

Theorem 6.1 (Monotone convergence)**.** *Let K be a Δ-elementary set, f a real valued function defined in K, $(f_n)_n$ a sequence of Δ-integrable functions in K and $A \in \mathbb{R}$, and suppose that the following conditions are satisfied:*

1. *$f_n(x) \stackrel{n}{\to} f(x)$, for almost every $x \in K$;*
2. *$f_1(x) \leq f_2(x) \leq \ldots$ for almost every $x \in K$;*
3. *$(\Delta)\int f_n \stackrel{n}{\to} A$.*

Then f is Δ-integrable and $(\Delta)\int f = A$.

Proof. By Corollary 3.3, it suffices to prove this when conditions *1.* and *2.* are satisfied for *every* $x \in K$. Let $\epsilon > 0$ be given, let $C > 0$ be sufficiently large, and consider $n_0 \in \mathbb{N}$ satisfying, for each $n \geq n_0$,

$$\left|(\Delta)\int f_n - A\right| < \epsilon. \tag{6.1}$$

For each $x \in K$, there is a positive integer $k(x) > n_0$ such that $|f_{k(x)}(x) - f(x)| < \epsilon$. By the Corollary 5.2 of Saks-Henstock Lemma, for each positive integer n there is a gauge δ_n in K which satisfies, for each δ_n-fine, C-regular Δ partition $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ of K,

$$\sum_{j\in\Gamma}\left|f_n(t^j)|I^j| - (\Delta)\int_{I^j} f_n\right| < \frac{\epsilon}{2^n}. \tag{6.2}$$

Let $\delta(x) \doteq \delta_{k(x)}(x)$ and suppose that $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ is a δ-fine, C-regular Δ-partition of K. Then

$$\begin{aligned}
|S(f,\mathcal{P}) - A| &\leq \sum_{j\in\Gamma}|f(t^j) - f_{k(t^j)}(t^j)|\,|I^j| \\
&\quad + \sum_{j\in\Gamma}\left|f_{k(t^j)}(t^j)|I^j| - (\Delta)\int_{I^j} f_{k(t^j)}\right| \\
&\quad + \left|\sum_{j\in\Gamma}(\Delta)\int_{I^j} f_{k(t^j)} - A\right| \\
&= \epsilon|I| + \epsilon + \left|\sum_{j\in\Gamma}(\Delta)\int_{I^j} f_{k(t^j)} - A\right|.
\end{aligned}$$

It suffices to show now that $\left|\sum_{j\in\Gamma}(\Delta)\int_{I^j} f_{k(t^j)} - A\right| \leq \epsilon$. Since each sequence $\left((\Delta)\int_{I^j} f_n\right)_n$ is nondecreasing and bounded from above by A, for each $j \in \Gamma$ there exists $A^j \in \mathbb{R}$ with $(\Delta)\int_{I^j} f_n \xrightarrow{n} A^j$. Then

$$(\Delta)\int f_n = \sum_{j\in\Gamma}(\Delta)\int_{I^j} f_n \xrightarrow{n} \sum_{j\in\Gamma} A^j,$$

which implies by hypothesis that $\sum_{j\in\Gamma} A^j = A$.

Denoting $l \doteq l(\mathcal{P}) \doteq min\{k(t^j);\, j \in \Gamma\}$, we have that

$$(\Delta)\int f_l = \sum_{j\in\Gamma}(\Delta)\int_{I^j} f_l \leq \sum_{j\in\Gamma}(\Delta)\int_{I^j} f_{k(t^j)} \leq \sum_{j\in\Gamma} A^j = A,$$

from which follows that

$$\left|\sum_{j\in\Gamma}(\Delta)\int_{I^j} f_{k(t^j)} - A\right| \le \left|(\Delta)\int f_l - A\right| \le \epsilon.$$

□

Lemma 6.2. *Let g, h, f_1 and f_2 be Δ-integrable in the Δ-elementary set K. If $g(x) \le f_i(x) \le h(x)$, for $i = 1,2$ for almost all $x \in K$, then the functions $min\{f_1, f_2\}$ and $max\{f_1, f_2\}$ are also Δ-integrable.*

PROOF. Again it suffices to prove for when the inequality holds for every $x \in K$. Suppose also that K is a triangle; it is straightforward to extend to the Δ-elementary case. Let us consider first the case where $g = 0$. For each triangle $J \subset K$, define $F^*(J) \doteq max\left\{(\Delta)\int_J f_1, (\Delta)\int_J f_2\right\}$. The function F^* satisfies, for each triangle $J \subset K$ and each partition $\mathcal{P}$ of J into triangles,

$$F^*(J) \le \sum_{L\in\mathcal{P}} F^*(L). \tag{6.3}$$

If $\mathcal{P}$ is a partition of K, we have moreover that

$$0 \le \sum_{L\in\mathcal{P}} F^*(L) \le (\Delta)\int h,$$

thus we can consider

$$0 \le A \doteq sup\left\{\sum_{L\in\mathcal{P}} F^*(L) : \mathcal{P} \text{ is a partition of } K\right\} \le (\Delta)\int h.$$

Let $f = max\{f_1, f_2\}$, and let us show that $(\Delta)\int f = A$. Let $\epsilon > 0$ be given, let $C > 0$ be sufficiently large, and fix a partition $\mathcal{P}_1$ of K which satisfies

$$\sum_{L\in\mathcal{P}_1} F^*(L) > A - \epsilon. \tag{6.4}$$

By the Corollary 5.2 of Saks-Henstock Lemma, there is a gauge δ in K such that for each δ-fine, C-regular Δ-partition $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ of K we have that

1. $\sum_{j\in\Gamma}\left|f(t^j)|I^j| - (\Delta)\int_{I^j} f\right| \le 2\epsilon$, for $i = 1, 2$;
2. $\mathcal{P}$ is *finer* than $\mathcal{P}_1$, that is, for each $j \in \Gamma$ there is $(L, t) \in \mathcal{P}_1$ with $I^j \subset L$.

Condition 2. guarantees, by (6.3), that

$$0 \le A - \sum_{j\in\Gamma} F^*(I^j) \le A - \sum_{L\in\mathcal{P}_1} F^*(L) < \epsilon. \tag{6.5}$$

For each $J \in Sub_\Delta(K)$ and each $i = 1, 2$, denote the set

$$sup\left\{\sum_{(K,t)\in\mathcal{Q}} \left|f_i(t)|K| - (\Delta)\int_K f_i\right| : \mathcal{Q} \text{ is a } \delta\text{-fine, } C\text{-regular } \Delta\text{-partition of } J\right\}$$

by $B_i(J)$. Note that, for $i = 1, 2$, $B_i(K) < \epsilon$ and

$$\sum_{L\in\mathcal{P}} B_i(L) \le B_i(J), \tag{6.6}$$

for each $J \in Sub_\Delta(K)$ and each partition $\mathcal{P}$ of J into triangles.

Let $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ be a δ-fine, C-regular Δ-partition of K. For each $j \in \Gamma$ and $i = 1, 2$ we have $f_i(t^j)|I^j| \le (\Delta)\int_{I^j} f_i + B_i(I^j) \le F^*(I^j) + B_1(I^j) + B_2(I^j)$, thus

$$f(t^j)|I^j| \le F^*(I^j) + B_1(I^j) + B_2(I^j). \tag{6.7}$$

A similar computation gives us

$$F^*(I^j) - B_1(I^j) - B_2(I^j) \le f(t^j)|I^j|. \tag{6.8}$$

Then by (6.7), (6.8) and (6.6), we have

$$\left|\sum_{j\in\Gamma}\left[f(t^j)|I^j| - F^*(I^j)\right]\right| \le \left|\sum_{j\in\Gamma}(B_1(I^j) + B_2(I^j))\right| \le B_1(I) + B_2(I) \le 2\epsilon.$$

By (6.5) we have

$$\left|\sum_{j\in\Gamma} f(t^j)|I^j| - A\right| \le \left|\sum_{j\in\Gamma}\left[f(t^j)|I^j| - F^*(I^j)\right]\right| + \left|\sum_{j\in\Gamma} F^*(I^j) - A\right| \le 3\epsilon,$$

as desired.

For the case where $g \ne 0$, since $0 \le f_i(x) - g(x) \le h(x) - g(x)$, $i = 1, 2$, it suffices to apply what we have already shown to conclude that $\max\{f_1, f_2\}$ is Δ-integrable. $\min\{f_1, f_2\}$ is also Δ-integrable, since

$$\min\{f_1, f_2\} = -\max\{-f_1, -f_2\}.$$

□

Theorem 6.3 (Dominated convergence)**.** *Let K be a Δ-elementary set, f a real valued function defined in K and $(f_n)_n$ a sequence of Δ-integrable functions in K, and suppose that the following conditions are satisfied:*

1. *$f_n(x) \overset{n}{\to} f(x)$, for almost every $x \in K$;*
2. *there exist Δ-integrable functions g, h in K such that, for each n, we have $g(x) \leq f_n(x) \leq h(x)$ for almost every $x \in K$.*

Then f is Δ-integrable and

$$(\Delta)\int f_n \to (\Delta)\int f.$$

Proof. We restrict ourselves again to the case where *1.* and *2.* are satisfied for every $x \in K$. By Lemma 6.2, for each n the function $\min\{f_1, \ldots, f_n\}$ is Δ-integrable, and by the monotone convergence Theorem 6.1 we have that for each positive integer k the function $\inf\{f_n : n \geq k\}$ is also Δ-integrable. Likewise we show that for each positive integer k the function $\sup\{f_n : n \geq k\}$ is Δ-integrable. We can write then

$$(\Delta)\int \inf_{n\geq k} f_n \leq \inf_{n\geq k}(\Delta)\int f_n \leq \sup_{n\geq k}(\Delta)\int f_n \leq (\Delta)\int \inf_{n\geq k} f_n. \tag{6.9}$$

We recall that, for each $x \in K$, $f_n(x) \overset{n}{\to} f(x)$ if and only if

$$\lim_{k\to\infty}\left(\inf_{n\geq k} f_n(x)\right) = f(x) = \lim_{k\to\infty}\left(\sup_{n\geq k} f_n(x)\right).$$

Applying the monotone convergence Theorem to the sequence $(\inf_{n\geq k} f_n)_{k\in\mathbb{N}}$, we obtain that f is Δ-integrable and $(\Delta)\int f = \lim_{k\to\infty}(\Delta)\int \inf_{n\geq k} f_n$. It is clear that also $(\Delta)\int f = \lim_{k\to\infty}(\Delta)\int \sup_{n\geq k} f_n$. Combining with (6.9), we have that $(\Delta)\int f_n \to (\Delta)\int f$. □

From the proof of the dominated convergence theorem we obtain the following:

Lemma 6.4 (Fatou's Lemma)**.** *Let K be a Δ-elementary set, f be a real valued function defined in K and $(f_n)_n$ be a sequence of Δ-integrable nonnegative functions in K which satisfy $f_n(x) \to f(x)$ for almost all $x \in K$. If the sequence $\big((\Delta)\int f_n\big)_n$ is bounded, then f is Δ-integrable and*

$$(\Delta)\int f \leq \liminf_{n\to\infty}(\Delta)\int f_n.$$

7 Relation with the Lebesgue integral

The first remark we should make in order to compare the Δ-integral with the Lebesgue integral in a Δ-elementary domain K is that the simple functions are Δ-integrable, and the value of the Δ-integral coincides with the value of the Lebesgue integral. Indeed, consider a nonempty measurable set $E \subset K$, and let us show that χ_E is Δ-integrable and $(\Delta)\int \chi_E = |E|$; the result can be extended to simple functions by the linearity of the Δ-integral. Let $\epsilon > 0$ be given, and let $C > 0$ be sufficiently large. Then there is an open subset U of K satisfying $U \supset E$ and $|U| < |E| + \epsilon$, and it is possible to define a gauge δ in K satisfying

1. $B_{\delta(x)}(x) \subset U$, for each $x \in E$;
2. $\delta(x) \leq d(x, E)$, for each $x \in K \setminus E$.

For each δ-fine, C-regular Δ-partition $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ of K, let us denote $\Gamma_0 \doteq \{j \in \Gamma : t^j \notin E\}$ and $\Gamma_1 \doteq \{j \in \Gamma : t^j \in E\}$. By the way δ was defined, we have

$$|E| \leq S(\chi_E, \mathcal{P}) = \sum_{j \in \Gamma_0} 0.|I^j| + \sum_{j \in \Gamma_1} 1.|I^j| = \left|\cup_{j \in \Gamma_1} I^j\right| \leq |U| < |E| + \epsilon,$$

from which follows the desired result.

Also, each nonnegative Lebesgue integrable function is Δ-integrable, and the values of the Δ-integral and the Lebesgue integral coincide. This comes from the fact that nonnegative Lebesgue integrable functions can be approximated from below by a monotone sequence of positive simple functions, then we can apply the Monotone Convergence Theorem for the Δ-integral (Theorem 6.1).

In general, we have the following characterization of Lebesgue integrability:

Proposition 7.1. *Let K be a Δ-elementary set and $f : K \to \mathbb{R}$. Then f is Lebesgue integrable if and only if f and $|f|$ are Δ-integrable, and in that case*

$$(L)\int f = (\Delta)\int f.$$

For the proof we will need the following additional result:

Proposition 7.2. *Let K be a Δ-elementary set. If $f : K \to \mathbb{R}$ is measurable and $g(x) \leq f(x) \leq h(x)$ for almost all $x \in K$, where g and h are Δ-integrable, then f is Δ-integrable and*

$$(\Delta)\int g \leq (\Delta)\int f \leq (\Delta)\int h. \tag{7.1}$$

Proof (of Proposition 7.2). Again we restrict ourselves to the case where the inequalities are satisfied for all $x \in K$ and K is a triangle. Let $(\phi_n)_n$ be a sequence of triangle-step functions converging almost everywhere to f. Then by Lemma 6.2 $f_n \doteq \max\{g, \min\{h, \phi_n\}\}$ is Δ-integrable, $(f_n)_n$ converges almost everywhere to f and $g(x) \leq f_n(x) \leq h(x)$, for each n and each $x \in K$. Then, by dominated convergence Theorem 6.3, f is Δ-integrable, and (7.1) follows directly from Proposition 3.4. □

Proof (of Proposition 7.1). Suppose that f and $|f|$ are Δ-integrable. For each measurable set $E \subset K$, $\chi_E f$ is a measurable function satisfying

$$-|f| \leq \chi_E f \leq |f|,$$

then it follows from Proposition 7.2 that $\chi_E f$ is Δ-integrable. Then

$$f^+ \doteq \max\{0, f\} \text{ and } f^- \doteq -\min\{0, f\}$$

are measurable, nonnegative Δ-integrable functions, with $f = f^+ - f^-$.

Let us consider the sequence $(f_n^+)_n$ of truncations of f^+ defined for each positive integer n and each $x \in K$ by

$$f_n^+(x) \doteq \min\{n, f^+(x)\}.$$

Each f_n^+ is Lebesgue integrable, since it is bounded and measurable, and is Δ-integrable by Lemma 6.2. For each $x \in K$ we have moreover that $0 \leq f_1^+(x) \leq f_2^+(x) \leq \ldots$ e $f_n^+(x) \overset{n}{\to} f^+(x)$. By the monotone convergence Theorem for Lebesgue integrable functions, each f^+ is Lebesgue integrable, and by the remarks made at the beginning of this Section we have in that case

$$(L)\int f^+ = (\Delta)\int f^+.$$

Analogously we show that f^- is Lebesgue integrable and $(L)\int f^- = (\Delta)\int f^-$. By the linearity of the Δ-integral and the Lebesgue integral it follows that f is Lebesgue integrable and

$$(L)\int f = (\Delta)\int f.$$

Conversely, if f is Lebesgue integrable, then f^+ and f^- are also Lebesgue integrable, and since they are both nonnegative it follows that they are both Δ-integrable, and the values of the Δ-integrals coincide with values of the Lebesgue integral. It follows that

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-$$

are Δ-integrable and $(\Delta)\int f = (L)\int f$, $(\Delta)\int |f| = (L)\int |f|$ again by linearity of the Δ-integral. □

8 Divergence theorem

It is worth noting that, up to this point, the C-regularity of the Δ-partitions has not been needed to prove any result. In this work it is used exclusively to prove Theorem 8.1.

Theorem 8.1 (Divergence)**.** *Let Ω be a nonempty open subset of $\mathbb{R}^2$, $K \subset \Omega$ be Δ-elementary, and $F : \Omega \to \mathbb{R}^2$ be a differentiable function. Then $divF$ is Δ-integrable in K and*

$$(\Delta)\int_K divF = (L)\int_{\partial K} F \cdot N. \tag{8.1}$$

Note that the integral to the right in (8.1) always exist at these conditions by the continuity of $F \cdot N$ in each line segment which is a side of ∂K.

Proof. Let $\epsilon > 0$ be given, and let $C > 0$ be sufficiently large. By the differentiability of F, for each $x \in K$ there is a $\delta(x) > 0$ such that

$$y \in B_{\delta(x)}(x) \Rightarrow ||F(y) - F(x) - dF_x(y - x)|| \le \frac{\epsilon}{C}||y - x||, \tag{8.2}$$

where dF_x is the differential of F at the point x. δ naturally defines a gauge in K. Let $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ be a δ-fine, C-regular Δ-partition of K and define, for each $j \in \Gamma$,

$$G^j(y) \doteq F(t^j) + dF_{t^j}(y - t^j) \text{ and}$$

$$H^j(y) \doteq F(y) - F(t^j) - dF_{t^j}(y - t^j).$$

Note that, for each j, $F = G^j + H^j$. Then

$$\left|S_K(divF, \mathcal{P}) - (L)\int_{\partial I} F \cdot N\right| = \left|\sum_{j\in\Gamma}\left[divF(t^j)|I^j| - (L)\int_{\partial I^j} F \cdot N\right]\right|$$

$$= \left|\sum_{j\in\Gamma}\left[divF(t^j)|I^j| - (L)\int_{\partial I^j} (G^j + H^j) \cdot N\right]\right|$$

$$= \left|\sum_{j\in\Gamma}\left[divF(t^j)|I^j| - (L)\int_{\partial I^j} G^j \cdot N - (L)\int_{\partial I^j} H^j \cdot N\right]\right|.$$

A classic version of the divergence Theorem applied to the affine functions G^j gives us
$$(L)\int_{\partial I^j} G^j \cdot N = (L)\iint_{I^j} divG^j = divF(t^j)|I^j|.$$
Replacing it in the inequality above we have
$$\left|S(divF,\mathcal{P}) - (L)\int_{\partial I} F\cdot N\right| = \left|\sum_{j\in\Gamma}(L)\int_{\partial I^j} H^j\cdot N\right|.$$

Because of the differentiability condition expressed by inequality (8.2), for each j we have that
$$\left|(L)\int_{\partial I^j} H^j\cdot N\right| \le (L)\int_{\partial I^j} ||H^j(y)||dy \le \max_{y\in\partial I^j} ||H^j(y)||\,per(\partial I^j)$$
$$\le \max_{y\in\partial I^j}\frac{\epsilon}{C}||y-t^j||\,per(\partial I^j) \le \frac{\epsilon}{C}q(I^j). \tag{8.3}$$

Combining the inequalities we have
$$\left|S(divF,\mathcal{P}) - (L)\int_{\partial I} F\cdot N\right| = \left|\sum_{j\in\Gamma}(L)\int_{\partial I^j} H^j\cdot N\right| \le \sum_{j\in\Gamma}\left|(L)\int_{\partial I^j} H^j\cdot N\right|$$
$$\le \frac{\epsilon}{C}\sum_{j\in\Gamma} q(I^j) \le \epsilon,$$
which concludes our proof. □

We proved in Section 7 that the Δ-integral in Δ-elementary sets is *more general* than the Lebesgue integral, in the sense that each Lebesgue integrable function is also Δ-integrable and the values of the integrals coincide. By the other hand, there are Δ-integrable functions which are not Lebesgue integrable; in [8], as was mentioned, there is a discussion on the level of incompatibility between the divergence Theorem and Fubini's Theorem when one wishes to define an integration process which generalizes the Lebesgue integral for real valued functions defined on an interval of $\mathbb{R}^2$. For example, if we have an "integral"[4] T which generalizes the Lebesgue integral and integrates the divergence of all differentiable functions (which is the case of the

[4]Roughly speaking, by *integral* we mean a mapping from a certain subset of the real-valued functions into real numbers, satisfying some linearity, additivity and continuity conditions - see [8] for the precise definition. The classical one-dimensional definition can be found in [18].

Δ-integral), in the cited article it is shown how to define a function which is "T-integrable", but which is not integrable coordinate-wise when we apply one-dimensional Lebesgue integration; then clearly that function cannot be Lebesgue integrable.

9 Credits and next steps

Some of the presented proofs can be found for the one-dimensional Henstock-Kurzweil integral in [12]; our line of arguments, especially through Sections 5 to 7, is similar, making the necessary adaptations to our context. The proof of the divergence Theorem 8.1 is analogous to the one done for an integral introduced by Prof. Jean Mawhin in [13].

We conclude by pointing out some natural questions which arise for the Δ-integral. The first of course is whether it is possible to extend the definition to more dimensions; in that case, we would have to work with simplicial complex-shaped domains in R^n, and simplex-based partitions. The irregularity of a partition $\mathcal{P} = \{(I^j, t^j) : j \in \Gamma\}$ can be defined similarly as

$$irr(\mathcal{P}) \doteq \sum_{j \in \Gamma} q(I^j),$$

where $q(I^j) = m_{n-1}(\partial I^j)\, diam(I^j)$.[5] The first difficulty is proving Cousin's Lemma 2.1 for $n \geq 3$; our proof was based on equality (2.2), which is obtained by partitioning the original triangular domain into four congruent triangles, all of them similar to the domain. In three dimensions it is already impossible to partition a tetrahedron-like domain into smaller tetrahedra which are similar to the original one. Therefore we would need to obtain a different proof for Cousin's Lemma for higher dimensions, or to change the definition of the integral. If Cousin's Lemma is proved using the above notion of irregularity, the rest of the results presented in this paper can be generalized for higher dimensions, with some technical adaptations on the proofs. But we need Cousin's Lemma to define the integral.

The other important question is whether or not it is possible to obtain a change of variables formula which works for differentiable transformations. This is an (apparently!) non-trivial problem, related to the problem of studying the additivity properties of the Δ-integral when the sets into which we wish to partition our domain have piecewise smooth boundaries - that is, sets that are images of polyhedra by C^1-transformations. An affirmative answer would provide us with a nice nonabsolute integral suitable for differentiable

[5] $m_{n-1}(\partial I^j)$ denotes the $(N-1)$-dimensional surface area of the simplex I^j.

manifolds. It would also be possible to study generalizations of the divergence Theorem when the domains are sets which are more complicated than Δ-elementary sets.

Finally, in connection to the question mentioned in the previous paragraph, it is also of interest to compare the Δ-integral with the integral presented by Pfeffer in [16]. If these integrals happen to be equivalent, for example, Proposition 2.3 could be generalized (it would be valid also when we take lipeomorphisms[6] instead of just affine transformations), and at the same time our paper would end up simplifying some proofs of [16].

Acknowledgment. The authors would like to thank the referees for their useful remarks.

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