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Gogi Pantsulaia, Department of Mathematics, Georgian Technical University, Tbilisi, 0175, Georgia
I.Vekua Institute of Applied Mathematics, Tbilisi State University, Tbilisi 0143, Georgia. email: gogipantsulaia@yahoo.com

ON UNIFORMLY DISTRIBUTED SEQUENCES OF AN INCREASING FAMILY OF FINITE SETS IN INFINITE-DIMENSIONAL RECTANGLES

Abstract

The concepts of uniformly distributed sequences of an increasing family of finite sets and Riemann integrability are considered in terms of the "Lebesgue measure" on infinite-dimensional rectangles in R^{∞} and infinite-dimensional versions of famous results of Lebesgue and Weyl are proved.

1 Introduction

Following [5], a sequence $s_1, s_2, s_3, \cdots$ of real numbers from an interval $[a, b]$ is said to be equidistributed or uniformly distributed on that interval if the proportion of terms contained in a subinterval $[c, d]$ is proportional to the length of that subinterval. Such sequences are studied in Diophantine approximation theory and have applications to Monte Carlo integration (see, for example, [5], [6], [12]).

Let $\mathcal{R}$ be the class of all infinite dimensional rectangles R of the form

$$R = \prod_{i=1}^{\infty}[a_i, b_i], \ -\infty < a_i < b_i < +\infty$$

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with $0 < \prod_{i=1}^{\infty}(b_i - a_i) < +\infty$, where

$$\prod_{i=1}^{\infty}(b_i - a_i) := \lim_{n\to\infty}\prod_{i=1}^{n}(b_i - a_i).$$

In [1], a translation invariant Borel measure λ was constructed on $\mathbb{R}^{\infty}$ such that

$$\lambda(R) = \prod_{i=1}^{\infty}(b_i - a_i)$$

for $R \in \mathcal{R}$.

The purpose of the present paper is to consider the concept of a uniform distribution in infinite-dimensional rectangles which can be used to calculate Riemann integrals over such rectangles. Similar topics are discussed in [10].

The paper is organized as follows.

In Section 2, some auxiliary notions and facts due to Weyl [13] are considered. In Section 3, the main results of the paper are proved. In particular, the infinite-dimensional versions of the famous results due to Lebesgue [9] and Weyl [13] are proved.

2 Auxiliary notions and propositions

Definition 2.1. A bounded sequence $s_1, s_2, s_3, \cdots$ of real numbers is said to be equidistributed or uniformly distributed on an interval $[a, b]$ if for any subinterval $[c, d]$ of $[a, b]$ we have

$$\lim_{n\to\infty}\frac{\#(\{s_1, s_2, s_3, \cdots, s_n\} \cap [c, d])}{n} = \frac{d-c}{b-a},$$

where # denotes a counting measure.

Remark 2.1. For $a \le c < d \le b$, let $][c, d][$ denote a subinterval of $[a, b]$ that has one of the following forms : $[c, d], [c, d[,]c, d[$ or $]c, d]$. Then it is obvious that a bounded sequence $s_1, s_2, s_3, \cdots$ of real numbers is equidistributed or uniformly distributed on an interval $[a, b]$ iff, for any subinterval $][c, d][$ of $[a, b]$, we have

$$\lim_{n\to\infty}\frac{\#(\{s_1, s_2, s_3, \cdots, s_n\}\cap][c, d][)}{n} = \frac{d-c}{b-a}.$$

Definition 2.2 (Weyl [13])**.** A sequence $s_1, s_2, s_3, \cdots$ is said to be equidistributed modulo 1 or uniformly distributed modulo 1 if the sequence $(s_n - [s_n])_{n\in\mathbb{N}}$ of the fractional parts of $(s_n)_{n\in N}$'s is equidistributed (equivalently, uniformly distributed) on the interval $[0, 1]$.

Example 2.1 ([5], Exercise 1.12, p. 16)**.** The sequence of all multiples of an irrational α

$$0, \alpha, 2\alpha, 3\alpha \cdots$$

is uniformly distributed modulo 1.

Example 2.2 ([5], Exercise 1.13, p. 16)**.** The sequence

$$\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \cdots, \frac{0}{k}, \cdots, \frac{k-1}{k}, \cdots$$

is uniformly distributed modulo 1.

Example 2.3. The sequence of all multiples of an irrational α by successive prime numbers

$$2\alpha, 3\alpha, 5\alpha, 7\alpha, 11\alpha, \cdots$$

is equidistributed modulo 1. This is the famous theorem of analytic number theory proved by I. M. Vinogradov in 1935 (see [16]).

Notation In the sequel, and as distinct from N. Bourbaki's well known notion, by $\mathbb{N}$ we understand the set $\{1, 2, \cdots\}$.

Remark 2.2. If $(s_k)_{k\in\mathbb{N}}$ is uniformly distributed modulo 1, then

$$((s_k - [s_k])(b-a) + a)_{k\in N}$$

is uniformly distributed on an interval $[a, b)$.

The following assertion contains an interesting application of uniformly distributed sequences for the calculation of Riemann integrals.

Lemma 2.1 (Weyl [13])**.** *The following two conditions are equivalent:*

(i) $(a_n)_{n\in\mathbb{N}}$ *is equidistributed modulo* 1*;*

(ii) For every Riemann integrable function f *on* $[0, 1]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(a_j) = \int_{[0,1]} f(x)dx.$$

3 On uniformly distributed sequences of an increasing family of finite sets in infinite-dimensional rectangles

Let $s_1, s_2, s_3, \cdots$ be uniformly distributed on an interval $[a,b]$. Setting $Y_n = \{s_1, s_2, s_3, \cdots, s_n\}$ for $n \in \mathbb{N}$, $(Y_n)_{n\in\mathbb{N}}$ will be an increasing sequence of finite subsets of the $[a,b]$ that, for any subinterval $[c,d]$ of the $[a,b]$, the following equality

$$\lim_{n\to\infty} \frac{\#(Y_n \cap [c,d])}{\#(Y_n)} = \frac{d-c}{b-a}$$

will be valid. This remark raises the following:

Definition 3.1. An increasing sequence $(Y_n)_{n\in\mathbb{N}}$ of finite subsets of $[a,b]$ is said to be equidistributed or uniformly distributed on an interval $[a,b]$ if, for any subinterval $[c,d]$ of $[a,b]$, we have

$$\lim_{n\to\infty} \frac{\#(Y_n \cap [c,d])}{\#(Y_n)} = \frac{d-c}{b-a}.$$

Definition 3.2. Let $\prod_{k\in\mathbb{N}}[a_k,b_k] \in \mathcal{R}$. A set U is called an elementary rectangle in $\prod_{k\in\mathbb{N}}[a_k,b_k]$ if it admits the following representation

$$U = \prod_{k=1}^{m}][c_k,d_k][\times \prod_{k\in\mathbb{N}\setminus\{1,\cdots,m\}} [a_k,b_k],$$

where $a_k \le c_k < d_k \le b_k$ for $1 \le k \le m$.

It is obvious that

$$\lambda(U) = \prod_{k=1}^{m}(d_k - c_k) \times \prod_{k=m+1}^{\infty}(b_k - a_k),$$

for the elementary rectangle U.

Definition 3.3. An increasing sequence $(Y_n)_{n\in\mathbb{N}}$ of finite subsets of an infinite-dimensional rectangle $\prod_{k\in\mathbb{N}}[a_k,b_k] \in \mathcal{R}$ is said to be uniformly distributed on $\prod_{k\in N}[a_k,b_k]$ if for every elementary rectangle U in $\prod_{k\in\mathbb{N}}[a_k,b_k[$ we have

$$\lim_{n\to\infty} \frac{\#(Y_n \cap U)}{\#(Y_n)} = \frac{\lambda(U)}{\lambda(\prod_{k\in\mathbb{N}}[a_k,b_k[)}.$$

Theorem 3.1. *Let $\prod_{k\in\mathbb{N}}[a_k,b_k]\in\mathcal{R}$. Let $(x_n^{(k)})_{n\in\mathbb{N}}$ be uniformly distributed on the interval $[a_k,b_k]$ for $k\in\mathbb{N}$. We set*

$$Y_n=\prod_{k=1}^{n}(\cup_{j=1}^{n}x_j^{(k)})\times\prod_{k\in\mathbb{N}\setminus\{1,\cdots,n\}}\{a_k\}.$$

Then $(Y_n)_{n\in\mathbb{N}}$ is uniformly distributed in the rectangle $\prod_{k\in\mathbb{N}}[a_k,b_k]$.

Proof. Let U be an elementary rectangle in $\prod_{k\in\mathbb{N}}[a_k,b_k]$.

Since $(x_n^{(k)})_{n\in\mathbb{N}}$ is uniformly distributed on the interval $[a_k,b_k]$ for $k\in\mathbb{N}$, by Remark 2.1 we have

$$\lim_{n\to\infty}\frac{\#(\{x_1^{(k)},x_2^{(k)},\cdots,x_n^{(k)}\}\cap][c_k,d_k][)}{n}=\frac{d_k-c_k}{b_k-a_k}.$$

Hence

$$\lim_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)}=\lim_{n\to\infty}\prod_{k=1}^{m}\frac{\#(\{x_1^{(k)},x_2^{(k)},\cdots,x_n^{(k)}\}\cap][c_k,d_k][)}{n}=$$
$$=\prod_{k=1}^{m}\lim_{n\to\infty}\frac{\#(\{x_1^{(k)},x_2^{(k)},\cdots,x_n^{(k)}\}\cap][c_k,d_k][)}{n}=$$
$$=\prod_{k=1}^{m}\frac{d_k-c_k}{b_k-a_k}=\frac{\lambda(U)}{\lambda(\prod_{k\in N}[a_k,b_k])}.$$

□

Remark 3.1. In the context of Theorem 3.1, it is natural to ask *whether there exists an increasing sequence of finite subsets $(Y_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)}=\frac{\lambda(U)}{\lambda(\prod_{k\in\mathbb{N}}[a_k,b_k])}$$

for every infinite-dimensional rectangle $U=\prod_{k\in\mathbb{N}}X_k\subset\prod_{k\in\mathbb{N}}[a_k,b_k]$, where, for each $k\in\mathbb{N}$, X_k is a finite sum of pairwise disjoint subintervals of $[a_k,b_k]$?

Let us show that the answer to this question is negative.

Indeed, assume the contrary and let $(Y_n)_{n\in\mathbb{N}}$ be such an increasing sequence of finite subsets in $\prod_{k\in N}[a_k,b_k]$. Then we have

$$\cup_{n\in\mathbb{N}}Y_n=\{(x_i^{(k)})_{i\in\mathbb{N}}:k\in\mathbb{N}\}.$$

For $k \in \mathbb{N}$, we set $X_k = [a_k, b_k] \setminus x_k^{(k)}$. Then it is clear that

$$\lambda(\prod_{k \in \mathbb{N}} X_k) = \lambda(\prod_{k \in \mathbb{N}} [a_k, b_k])$$

and

$$\frac{\#(Y_n \cap \prod_{k \in \mathbb{N}} X_k)}{\#(Y_n)} = 0$$

for $k \in \mathbb{N}$, which implies

$$\lim_{n \to \infty} \frac{\#(Y_n \cap \prod_{k \in N} X_k)}{\#(Y_n)} = 0 < 1 = \frac{\lambda(\prod_{k \in N} X_k)}{\lambda(\prod_{k \in \mathbb{N}} [a_k, b_k])}.$$

Definition 3.4. Let $\prod_{k \in \mathbb{N}} [a_k, b_k] \in \mathcal{R}$. A family of pairwise disjoint elementary rectangles $\tau = (U_k)_{1 \le k \le n}$ of $\prod_{k \in \mathbb{N}} [a_k, b_k]$ is called the Riemann partition of $\prod_{k \in \mathbb{N}} [a_k, b_k]$ if $\cup_{1 \le k \le n} U_k = \prod_{k \in \mathbb{N}} [a_k, b_k]$.

Definition 3.5. Let $\tau = (U_k)_{1 \le k \le n}$ be the Riemann partition of $\prod_{k \in N} [a_k, b_k]$. Let $\ell(Pr_i(U_k))$ be the length of the i-th projection $Pr_i(U_k)$ of U_k for $i \in \mathbb{N}$. We set

$$d(U_k) = \sum_{i \in N} \frac{\ell(Pr_i(U_k))}{2^i (1 + \ell(Pr_i(U_k)))}.$$

It is obvious that $d(U_k)$ is the diameter of the elementary rectangle U_k for $k \in \mathbb{N}$ with respect to the Tikhonov metric ρ defined as follows

$$\rho((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}) = \sum_{k \in \mathbb{N}} \frac{|x_k - y_k|}{2^k (1 + |x_k - y_k|)}$$

for $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty$.

A number $d(\tau)$ defined by

$$d(\tau) = \max\{d(U_k) : 1 \le k \le n\}$$

is called the mesh or the norm of the Riemann partition τ.

Definition 3.6. Let $\tau_1 = (U_i^{(1)})_{1 \le i \le n}$ and $\tau_2 = (U_j^{(2)})_{1 \le j \le m}$ be the Riemann partitions of $\prod_{k \in N} [a_k, b_k]$. We say that $\tau_2 \le \tau_1$ iff

$$(\forall j)((1 \le j \le m) \to (\exists i_0)(1 \le i_0 \le n \ \& \ U_j^{(2)} \subseteq U_{i_0}^{(1)})).$$

Definition 3.7. Let f be a real-valued bounded function defined on $\prod_{i \in \mathbb{N}} [a_i, b_i]$. Let $\tau = (U_k)_{1 \le k \le n}$ be the Riemann partition of $\prod_{k \in \mathbb{N}} [a_k, b_k]$ and $(t_k)_{1 \le k \le n}$ be a sample such that, for each k, $t_k \in U_k$. Then:

(i) a sum $\sum_{k=1}^{n} f(t_k)\lambda(U_k)$ is called a Riemann sum of f with respect to the Riemann partition $\tau = (U_k)_{1\le k\le n}$ together with the sample $(t_k)_{1\le k\le n}$;

(ii) a sum $S_\tau = \sum_{k=1}^{n} M_k\lambda(U_k)$ is called an upper Darboux sum with respect to the Riemann partition τ where $M_k = \sup_{x\in U_k} f(x)(1 \le k \le n)$;

(iii) a sum $s_\tau = \sum_{k=1}^{n} m_k\lambda(U_k)$ is called a lower Darboux sum with respect to the Riemann partition τ where $m_k = \inf_{x\in U_k} f(x)(1 \le k \le n)$.

Definition 3.8. Let f be a real-valued bounded function defined on $\prod_{i\in N}[a_i, b_i[$. We say that f is Riemann-integrable on $\prod_{i\in \mathbb{N}}[a_i, b_i]$ if there exists a real number s such that for every positive real number ϵ there exists a real number $\delta > 0$ such that, for every Riemann partition $(U_k)_{1\le k\le n}$ of $\prod_{k\in\mathbb{N}}[a_k, b_k]$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1\le k\le n}$, we have

$$\Big|\sum_{k=1}^{n} f(t_k)\lambda(U_k) - s\Big| < \epsilon.$$

The number s is called a Riemann integral and denoted by

$$(R)\int_{\prod_{k\in\mathbb{N}}[a_k,b_k]} f(x)d\lambda(x).$$

Definition 3.9. A function f is called a step function on $\prod_{k\in\mathbb{N}}[a_k, b_k]$ if it can be written as

$$f(x) = \sum_{k=1}^{n} c_k \mathcal{X}_{U_k}(x),$$

where $\tau = (U_k)_{1\le k\le n}$ is any Riemann partition of $\prod_{k\in\mathbb{N}}[a_k, b_k]$, $c_k \in R$ for $1 \le k \le n$ and $\mathcal{X}_A$ is the indicator function of A.

Theorem 3.2. *Let f be a continuous function on $\prod_{k\in\mathbb{N}}[a_k, b_k]$ with respect to the Tikhonov metric ρ. Then f is Riemann-integrable on $\prod_{k\in\mathbb{N}}[a_k, b_k]$.*

PROOF. It is obvious that, for every Riemann partition $\tau = (U_k)_{1\le k\le n}$ of $\prod_{k\in\mathbb{N}}[a_k, b_k]$ and for every sample $(t_k)_{1\le k\le n}$ with $t_k \in U_k(1 \le k \le n)$, we have

$$s_\tau \le \sum_{k=1}^{n} f(t_k)\lambda(U_k) \le S_\tau.$$

Note that if τ_1 and τ_2 are two Riemann partitions of $\prod_{k\in\mathbb{N}}[a_k, b_k]$ such that $\tau_2 \le \tau_1$, then

$$s_{\tau_1} \le s_{\tau_2} \le \sum_{k=1}^{n} f(t_k)\lambda(U_k) \le S_{\tau_2} \le S_{\tau_1}.$$

Let us show the validity of the condition

$$(\forall\epsilon)(\epsilon > 0 \to (\exists r)(\forall\tau)(d(\tau) < r \to S_\tau - s_\tau < \epsilon)),$$

which yields $\inf_\tau S_\tau = \sup_\tau s_\tau$.

Following the Tikhonov theorem, $\prod_{k\in\mathbb{N}}[a_k, b_k]$ is a compact set in the Polish group $\mathbb{R}^\infty$ equipped with the Tikhonov metric ρ.

Following Cantor's well known result, the function f is uniformly continuous on $\prod_{k\in N}[a_k, b_k]$. Hence, for $\epsilon > 0$, there exists $r > 0$ such that

$$(\forall x, y)(x, y \in \prod_{k\in\mathbb{N}}[a_k, b_k] \& \rho(x, y) < r \to |f(x) - f(y)| \le \frac{\epsilon}{\lambda(\prod_{k\in N}[a_k, b_k])}).$$

Thus, for every Riemann partition $\tau = (U_k)_{1\le k\le n}$ with $d(\tau) < r$, we get

$$S_\tau - s_\tau \le \frac{\epsilon}{\lambda(\prod_{k\in\mathbb{N}}[a_k, b_k[)} \times \sum_{1\le k\le n} \lambda(U_k) = \epsilon.$$

Thus $\inf_\tau S_\tau = \sup_\tau s_\tau$.

Finally, setting $\delta = r$ and $s = \inf_\tau S_\tau$, we deduce that for every Riemann partition $(U_k)_{1\le k\le n}$ of the $\prod_{k\in\mathbb{N}}[a_k, b_k]$ with $d(\tau) < \delta$ and for every sample $(t_k)_{1\le k\le n}$ with $t_k \in U_k (1 \le k \le n)$, we have

$$\Big|\sum_{k=1}^{n} f(t_k)\lambda(U_k) - s\Big| \le S_\tau - s_\tau \le \epsilon.$$

This ends the proof of Theorem 3.2. □

We have the following infinite-dimensional version of the Lebesgue theorem (see [8], Lebesgue Theorem, p. 359).

Theorem 3.3. *Let f be a bounded real-valued function on $\prod_{k\in\mathbb{N}}[a_k, b_k] \in \mathcal{R}$. Then f is Riemann integrable on $\prod_{k\in N}[a_k, b_k]$ if and only if f is λ-almost continuous on $\prod_{k\in\mathbb{N}}[a_k, b_k]$.*

PROOF. We first prove the necessity. Let f be a Riemann integrable function on $\prod_{k\in\mathbb{N}}[a_k, b_k] \in \mathcal{R}$.

Then, for every $\epsilon > 0$ and $\mu > 0$, there exists a Riemann partition $\tau = (U_k)_{1\le k\le n}$ such that

$$\epsilon \times \mu \ge S_\tau - s_\tau \ge \sum_{1\le k\le n}(M_k - m_k)\lambda(U_k) \ge$$

$$\sum_{k\in I_1}(M_k - m_k)\lambda(U_k) \ge \mu \sum_{k\in I_1}\lambda(U_k), \tag{3.1}$$

where $I_1 = \{k : 1 \le k \le n \ \&\ U_k$ contains at least one inner point p belonging to the set $E_\mu\}$, where

$$E_\mu = \{x : x \in \prod_{k\in\mathbb{N}}[a_k, b_k] \ \&\ \omega(f, x) \ge \mu\}$$

and

$$\omega(f, x) = \lim_{\delta\to 0} \sup_{x', x" \in V(x,\delta)\cap\prod_{k\in\mathbb{N}}[a_k,b_k]} \left|f(x^{'}) - f(x^{"})\right|.$$

Here, for $x \in \mathbb{R}^\infty$ and $\delta > 0$, $V(x, \delta)$ is defined by

$$V(x, \delta) = \{y : y \in \prod_{k\in\mathbb{N}}[a_k, b_k] \ \&\rho(x, y) \le \delta\}.$$

Since, for $k \in I_1$, p is an inner point of U_k, there exists $V(p, \delta(k, p))$ such that $V(p, \delta(k, p)) \subseteq U_k$. Note that

Since $\omega(f, p) \ge \mu$, we have

$$M_k - m_k \ge M_p - m_p \ge \omega(f, p) \ge \mu,$$

where

$$M_p = \sup_{x\in V(p,\delta(k,p))} f(x),\ \ m_\delta = \inf_{x\in V(p,\delta(k,p))} f(x).$$

From (3.1) we get

$$\epsilon \ge \sum_{k\in I_1}\lambda(U_k).$$

Other points of E_μ, which are not inner points of the elements of the partition τ, can be placed on the boundary of the elements of τ, whose λ-measure is zero.

Thus, for $\mu > 0$, we have

$$\lambda(E_\mu) \le \sum_{k\in I_1}\lambda(U_k) + \lambda(\cup_{1\le k\le n}\partial(U_k)) \le \frac{\epsilon}{\mu},$$

which yields $\lambda(E_\mu) = 0$. Since the set E of all points of discontinuity of f admits the representation $E = \cup_{k=1}^{\infty} E_{\frac{1}{k}}$, we deduce that $\lambda(E) = 0$.

This ends the proof of necessity and we continue with the proof of the sufficiency.

For $K \in \mathbb{R}^+$, suppose we have $|f(x)| \leq K$ whenever $x \in \prod_{k\in\mathbb{N}}[a_k, b_k]$. Suppose that f is λ-almost continuous on $\prod_{k\in N}[a_k, b_k]$. For $\epsilon > 0$, let μ be a positive number such that

$$4\mu\lambda(\prod_{k\in\mathbb{N}}[a_k, b_k]) < \epsilon.$$

Since, for a set E of all points of discontinuity of f on $\prod_{k\in\mathbb{N}}[a_k, b_k]$ we have $\lambda(E) = 0$, we easily claim that $\lambda(E_\mu) = 0$. Since E_μ is closed in $\prod_{k\in\mathbb{N}}[a_k, b_k]$, we claim that E_μ is compact. Hence, for $\epsilon > 0$, there exists a finite family of open elementary rectangles in $\prod_{k\in\mathbb{N}}[a_k, b_k]$ whose union covers E_μ such that

$$\lambda(\cup_{1\leq k\leq n} U_k) < \frac{\epsilon}{4K}.$$

Finally, we have

$$\prod_{k\in\mathbb{N}}[a_k, b_k] = \cup_{1\leq k\leq n} U_k \cup F,$$

where F is a compact subset in $\prod_{k\in N}[a_k, b_k]$.

It is obvious that, for every point $x \in F$, we have $\omega(f, x) < \mu$. Since F is compact, we can choose $\delta > 0$ such that for every $x, x^{'} \in F$ the condition $\rho(x, x^{'}) < \delta$ yields $\left|f(x) - f(x^{'})\right| < 2\lambda$.

Since F is a finite union of elementary rectangles in $\prod_{k\in\mathbb{N}}[a_k, b_k]$ (this follows from the fact that the class of all elementary rectangles in $\prod_{k\in\mathbb{N}}[a_k, b_k]$ is a ring), there exists a partition $\tau_1 = (F_i)_{2\leq i\leq m}$ of F such that, for i with $2 \leq i \leq m$, F_i is an elementary rectangle in $\prod_{k\in\mathbb{N}}[a_k, b_k]$ with $d(F_i) < \delta$. Then $\tau = \{\cup_{1\leq k\leq n} U_k, F_2, \cdots, F_m\}$ will be Riemann partition of $\prod_{k\in\mathbb{N}}[a_k, b_k]$ such that

$$S_\tau - s_\tau = (M_1 - m_1)\lambda(\cup_{1\leq k\leq n} U_k) + \sum_{1\leq i\leq m}(M_i - m_i)\lambda(F_k) \leq$$

$$\frac{\epsilon}{2} + 2\mu\lambda(\prod_{k\in\mathbb{N}}[a_k, b_k])) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Remark 3.2. Theorem 3.2 is a simple consequence of Theorem 3.3. Therefore, using Theorem 3.3, one can extend the concept of Riemann integrability theory to the case of functions defined in the topological vector space $\mathbb{R}^\infty$ of all real-valued sequences equipped with Tikhonov topology.

In the sequel, we need some important notions and well-known results from general topology and measure theory.

Definition 3.10. A topological Hausdorff space X is called normal if given any disjoint closed sets E and F, there are neighborhoods U of E and V of F that are also disjoint.

Lemma 3.1 (Urysohn [15]). *A topological space X is normal if and only if any two disjoint closed sets can be separated by a function. That is, given disjoint closed sets E and F, there is a continuous function f from X to $[0,1]$ such that the preimages of 0 and 1 under f are E and F, respectively.*

Remark 3.3. Since all compact Hausdorff spaces are normal, we deduce that $\prod_{k \in N}[a_k, b_k]$ equipped with Tikhonov topology is normal. By Urysohn's lemma we deduce that any two disjoint closed sets in $\prod_{k \in \mathbb{N}}[a_k, b_k]$ can be separated by a function.

Definition 3.11. A Borel measure μ, defined on a Hausdorff topological space X is called Radon if

$$(\forall Y)(Y \in \mathcal{B}(X) \ \& \ 0 \leq \mu(Y) < +\infty \to \mu(Y) = \sup_{\substack{K \subseteq Y \\ \text{K is compact in X}}} \mu(K)).$$

Lemma 3.2 (Ulam [14]). *Every probability Borel measure defined on a Polish metric space is Radon.*

In the sequel, we denote by $\mathcal{C}(\prod_{k \in \mathbb{N}}[a_k, b_k])$ a class of all continuous (with respect to the Tikhonov topology) real-valued functions on $\prod_{k \in N}[a_k, b_k]$.

Theorem 3.4. *For $\prod_{i \in \mathbb{N}}[a_i, b_i] \in \mathcal{R}$, let $(Y_n)_{n \in \mathbb{N}}$ be an increasing family of its finite subsets. Then $(Y_n)_{n \in \mathbb{N}}$ is uniformly distributed in $\prod_{k \in \mathbb{N}}[a_k, b_k]$ if and only if the equality*

$$\lim_{n \to \infty} \frac{\sum_{y \in Y_n} f(y)}{\#(Y_n)} = \frac{(R) \int_{\prod_{k \in N}[a_k, b_k]} f(x) d\lambda(x)}{\lambda(\prod_{i \in N}[a_i, b_i])}$$

holds for every $f \in \mathcal{C}(\prod_{k \in \mathbb{N}}[a_k, b_k])$.

PROOF. We begin by proving the necessity. Let $(Y_n)_{n\in\mathbb{N}}$ be uniformly distributed on $\prod_{k\in\mathbb{N}}[a_k,b_k]$ and let $f(x)=\sum_{k=1}^m c_k\mathcal{X}_{U_k}(x)$ be a step function. Then we have

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}=\lim_{n\to\infty}\frac{\sum_{y\in Y_n}\sum_{k=1}^m c_k\mathcal{X}_{U_k}(y)}{\#(Y_n)}=$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^m c_k\#(U_k\cap Y_n)}{\#(Y_n)}=\sum_{k=1}^m c_k\lim_{n\to\infty}\frac{\#(U_k\cap Y_n)}{\#(Y_n)}=$$

$$\sum_{k=1}^m c_k\frac{\lambda(U_k)}{\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i])}=\frac{(R)\int_{\prod_{k\in N}[a_k,b_k]}f(x)d\lambda(x)}{\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i])}.$$

Now, let $f\in\mathcal{C}(\prod_{k\in\mathbb{N}}[a_k,b_k])$. By Theorem 3.2 we deduce that f is Riemann-integrable. From the definition of a Riemann integral we deduce that, for every positive ϵ, there exist two step functions f_1 and f_2 on $\prod_{i\in\mathbb{N}}[a_i,b_i]$ such that

$$f_1(x)\le f(x)\le f_2(x)$$

and

$$(R)\int_{\prod_{i\in N}[a_i,b_i]}(f_1(x)-f_2(x))d\lambda(x)<\epsilon.$$

Then we have

$$(R)\int_{\prod_{i\in\mathbb{N}}[a_i,b_i]}f(x)d\lambda(x)-\epsilon\le(R)\int_{\prod_{i\in\mathbb{N}}[a_i,b_i]}f_1(x)d\lambda(x)=$$

$$\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i])\times\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f_1(y)}{\#(Y_n)}\le\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i])\times\underline{\lim}_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}\le$$

$$\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i])\times\overline{\lim}_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}\le\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f_2(y)}{\#(Y_n)}\le$$

$$\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i])\times(R)\int_{\prod_{i\in\mathbb{N}}[a_i,b_i]}f_2(x)d\lambda(x)\le(R)\int_{\prod_{i\in\mathbb{N}}[a_i,b_i]}f(x)d\lambda(x)+\epsilon.$$

The latter relation yields the existence of a limit $\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}$ such that

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n}f(y)}{\#(Y_n)}=\frac{(R)\int_{\prod_{k\in N}[a_k,b_k]}f(x)d\lambda(x)}{\lambda(\prod_{i\in N}[a_i,b_i])}.$$

This ends the proof of the necessity.

To begin the proof of the sufficiency, assume that $(Y_n)_{n\in\mathbb{N}}$ is an increasing sequence of subsets of $\prod_{k\in N}[a_k, b_k]$ such that the equality

$$\lim_{n\to\infty} \frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)} = \frac{(R)\int_{\prod_{k\in N}[a_k,b_k[} f(x)d\lambda(x)}{\lambda(\prod_{i\in N}[a_i, b_i])}$$

holds for every $f \in \mathcal{C}(\prod_{k\in\mathbb{N}}[a_k, b_k])$.

Let U be any elementary rectangle in $\prod_{i\in\mathbb{N}}[a_i, b_i]$.

For $\epsilon > 0$, by Ulam's lemma we can choose a compact set

$$F \subset \prod_{k\in\mathbb{N}}[a_k, b_k] \setminus [U]_T,$$

such that $\lambda((\prod_{k\in\mathbb{N}}[a_k, b_k] \setminus [U]_T) \setminus F) < \frac{\epsilon}{2}$, where $[U]_T$ denotes the completion of the set U by the Tikhonov topology in $\prod_{k\in\mathbb{N}}[a_k, b_k]$. Then, by Urysohn's lemma there is a continuous function g_2 from $\prod_{k\in\mathbb{N}}[a_k, b_k]$ to $[0, 1]$ such the preimages of 0 and 1 under g_2 are F and $[U]_T$, respectively. Then, for $x \in \prod_{k\in N}[a_k, b_k]$, we have

$$\mathcal{X}_U(x) \leq g_2(x)$$

and

$$(R)\int_{\prod_{k\in N}[a_k,b_k]} (g_2(x) - \mathcal{X}_U(x))d\lambda(x) \leq \frac{\epsilon}{2},$$

where $\mathcal{X}_U$ is an indicator of U defined on $\prod_{k\in\mathbb{N}}[a_k, b_k]$.

Now let us consider the set $[\prod_{k\in\mathbb{N}}[a_k, b_k] \setminus U]_T$. Using Ulam's lemma, we can choose a compact set

$$F_1 \subset \prod_{k\in\mathbb{N}}[a_k, b_k] \setminus [\prod_{k\in\mathbb{N}}[a_k, b_k] \setminus U]_T$$

such that

$$\lambda((\prod_{k\in\mathbb{N}}[a_k, b_k] \setminus [\prod_{k\in N}[a_k, b_k] \setminus U]_T) \setminus F_1) < \frac{\epsilon}{2}.$$

Then, by Urysohn's lemma there is a continuous function g_1 from $\prod_{k\in\mathbb{N}}[a_k, b_k]$ to $[0, 1]$ such that the preimages of 0 and 1 under g_1 are $[\prod_{k\in\mathbb{N}}[a_k, b_k] \setminus U]_T$ and F_1, respectively. Then, for $x \in \prod_{k\in\mathbb{N}}[a_k, b_k]$, we have

$$g_1(x) \leq \mathcal{X}_U(x)$$

and

$$(R)\int_{\prod_{k\in N}[a_k,b_k]} (\mathcal{X}_U(x) - g_1(x))d\lambda(x) \leq \frac{\epsilon}{2}.$$

Now, we deduce that for every elementary rectangle U in $\prod_{i\in\mathbb{N}}[a_i,b_i]$ there exists two continuous functions g_1 and g_2 on $\prod_{i\in\mathbb{N}}[a_i,b_i]$ such that

$$g_1(x) \le \mathcal{X}_U(x) \le g_2(x)$$

and

$$(R)\int_{\prod_{i\in\mathbb{N}}[a_i,b_i]}(g_2(x)-g_1(x))d\lambda(x) \le \epsilon.$$

Then we have

$$\lambda(U)-\epsilon \le (R)\int_{\prod_{i\in N}[a_i,b_i]} g_2(x)d\lambda(x)-\epsilon \le (R)\int_{\prod_{i\in N}[a_i,b_i]} g_1(x)d\lambda(x) =$$

$$\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i]) \times \lim_{n\to\infty}\frac{\sum_{y\in Y_n} g_1(y)}{\#(Y_n)} \le \lambda(\prod_{i\in\mathbb{N}}[a_i,b_i]) \times \underline{\lim}_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)} \le$$

$$\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i]) \times \overline{\lim}_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)} \le \lambda(\prod_{i\in\mathbb{N}}[a_i,b_i]) \times \lim_{n\to\infty}\frac{\sum_{y\in Y_n} g_2(y)}{\#(Y_n)} =$$

$$(R)\int_{\prod_{i\in N}[a_i,b_i]} g_2(x)d\lambda(x) \le (R)\int_{\prod_{i\in N}[a_i,b_i]} g_1(x)d\lambda(x)+\epsilon \le \lambda(U)+\epsilon.$$

Since ϵ was taken arbitrary, we deduce that

$$\lambda(\prod_{i\in\mathbb{N}}[a_i,b_i]) \times \lim_{n\to\infty}\frac{\#(Y_n\cap U)}{\#(Y_n)} = \lambda(U).$$

This ends the proof of Theorem 3.4. □

Now by the scheme used in the proof of Theorem 3.4, one can get the validity of an infinite-dimensional analog of Lemma 3.1. In particular, the following assertion is valid.

Theorem 3.5. *For $\prod_{i\in\mathbb{N}}[a_i,b_i] \in \mathcal{R}$, let $(Y_n)_{n\in\mathbb{N}}$ be an increasing family its finite subsets. Then $(Y_n)_{n\in\mathbb{N}}$ is uniformly distributed in the $\prod_{k\in\mathbb{N}}[a_k,b_k]$ if and only if the equality*

$$\lim_{n\to\infty}\frac{\sum_{y\in Y_n} f(y)}{\#(Y_n)} = \frac{(R)\int_{\prod_{k\in N}[a_k,b_k]} f(x)d\lambda(x)}{\lambda(\prod_{i\in N}[a_i,b_i])}$$

holds for every Riemann integrable function f on $\prod_{k\in\mathbb{N}}[a_k,b_k]$).

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