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SYMMETRIC MEASURE-PRESERVING SYSTEMS

Abstract

A symmetric measure-preserving system is one where the measure Pr is preserved by two maps T and R where R is self-inverse and $T \circ R = T$. We discuss the existence of such systems and some consequences, including when unimodal maps are conjugate to the symmetric tent map.

1 Introduction

A continuous map $T : [0, 1] \to [0, 1]$ is called *unimodal with turning point* m if $m \in (0, 1)$ and T is continuous, strictly increasing on $[0, m]$ and strictly decreasing on $[m, 1]$. For the moment, let us call a unimodal map *two-to-one* if $T(0) = T(1) = 0$ and $T(m) = 1$. To each two-to-one map we can associate a unique continuous map $R : [0, 1] \to [0, 1]$ such that R is not the identity and $T \circ R = T$. The most well-known such pair of maps is $\tau(x) = \min(2x, 2(1-x))$ and $\rho(x) = 1 - x$.

For each probability measure Pr on the Borel subsets of $[0, 1]$ we may define the function $F : [0, 1] \to [0, 1]$ defined by $F(t) = \Pr([0, t])$. We will call F the *distribution function associated with* Pr.

Given a two-to-one map T one problem of interest is to characterize the probability measures Pr which are preserved by T. It is well-known that such measures exist. In the case of τ we know that Lebesgue measure on $[0, 1]$ is one such probability measure. We also note that this measure is preserved by ρ.

Suppose for a moment that given a two-to-one map T with turning point m and its associated map R that we can find a probability measure Pr which

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is preserved by both T and R. Let F be the distribution function associated with Pr. Since R preserves Pr we have

$$F(R(t)) = 1 - F(t) = \rho(F(t)),$$

and since T preserves Pr, for $x \in [0, m]$ we have

$$F(T(x)) = F(x) + 1 - F(R(x)) = 2F(x) = \tau(F(x)),$$

while for $x \in [m, 1]$ we have

$$F(T(x)) = F(T(R(x))) = \tau(F(R(x)) = \tau(\rho(F(x))) = \tau(F(x)).$$

Thus we have

$$\begin{aligned} F \circ R &= \rho \circ F \\ F \circ T &= \tau \circ F \end{aligned}$$

Note that to this point we only use symmetry. Suppose in addition we know that F is strictly increasing and continuous. Such would be the case if the probability measure was non-atomic and assigned positive probability to all sub-intervals of $[0, 1]$. In this case we have

$$\begin{aligned} R &= F^{-1} \circ \rho \circ F \\ T &= F^{-1} \circ \tau \circ F, \end{aligned}$$

and we would have proven that T and τ (and R and ρ) are topologically conjugate.

Conversely, suppose that T is two-to-one with turning point m, R is the associated map with $T \circ R = T$, and for some homeomorphism $F : [0, 1] \to [0, 1]$ we have $F \circ T = \tau \circ F$ and $F \circ R = \rho \circ F$. Then F is the distribution function of a probability measure preserved by both T and R. To see why, let Pr be the probability measure on the Borel subsets of $[0, 1]$ whose distribution function is F. Such Pr exists by the Carathéodory extension theorem. It is sufficient to check that the measures of intervals of the form $[0, y]$ are preserved. Since T is two-to-one there is a unique $x \in [0, m]$ with $T(x) = y$, and we have $T^{-1}([0, y]) = [0, x] \cup [R(x), 1]$, and $R^{-1}([0, y]) = [R(y), 1]$. Note that $R(m) = m$ so that $F(m) = F(R(m)) = \rho(F(m)) = 1 - F(m)$ so $F(m) = 1/2$. Hence

$$\begin{aligned} \Pr([0, x] \cup [R(x), 1]) &= F(x) + 1 - F(R(x)) = F(x) + \rho(F(R(x)) \\ &= 2F(x) = \tau(F(x)) = F(T(x)) = \Pr([0, y]), \end{aligned}$$

and

$$\Pr([R(y), 1]) = 1 - F(R(y)) = \rho(F(R(y)) = F(y) = \Pr([0, y]).$$

In this paper we

1. Generalize the idea of two-to-one maps to abstract measure spaces.
2. In the case where the measure space is a compact metric space, show that there are non-atomic probability measures preserved by two-to-one maps (suitably defined) which are also preserved by the reflection map R.
3. In the case where the compact metric space is $[0, 1]$, give conditions on two-to-one maps which ensure that this measure will give positive probability to any subinterval of $[0, 1]$.
4. In the case of $[0, 1]$, look at what happens if we have non-atomic probability measures which give probability 0 to some subintervals of $[0, 1]$.

2 Some Additional Definitions and Examples

We will call the quintuple $(\Omega, \mathcal{F}, \Pr, T, R)$ a *symmetric measure-preserving system* if

P0: $(\Omega, \mathcal{F}, \Pr)$ is a probability space;

P1: $(\Omega, \mathcal{F}, \Pr, T)$ is a measure-preserving system;

P2: $(\Omega, \mathcal{F}, \Pr, R)$ is a measure-preserving system;

P3: $\{\omega \in \Omega : R(\omega) \neq \omega\} \in \mathcal{F}$ and $\Pr(\{\omega \in \Omega : R(\omega) \neq \omega\}) > 0$;

P4: $R(R(\omega)) = \omega$ for all $\omega \in \Omega$;

P5: $T \circ R = T$.

We shall call a measurable map R of $(\Omega, \mathcal{F}, \Pr)$ a **reflection** of $(\Omega, \mathcal{F}, \Pr)$ if it satisfies (P3) and (P4). If we have no measure in mind, we shall call a measurable map R of $(\Omega, \mathcal{F})$ a **reflection** of $(\Omega, \mathcal{F})$ if R is not the identity map and $R \circ R$ is the identity map.

Two examples of symmetric measure-preserving systems are

- $\Omega = [0, 1]$;
- $\mathcal{F} =$ the Borel subsets of $[0, 1]$;

- $\Pr(E)$ = the ordinary Lebesgue measure of E;
- $T(x) = \min(2x, 2(1-x))$;
- $R(x) = 1 - x$;

and

- $\Omega = [0, 1]$;
- $\mathcal{F}$ = the Borel subsets of $[0, 1]$;
- $\Pr(E) = \int_E \frac{1}{\pi\sqrt{x - x^2}}\, dx$';
- $T(x) = 4x(1-x)$;
- $R(x) = 1 - x$.

Note that the probability measure in the second example is non-atomic and gives positive probability to all subintervals of $[0, 1]$. This provides one example of the situation discussed in the previous section.

We now proceed to generalize our earlier idea of two-to-one. Note that we drop the requirement that the map be onto.

Suppose that $\mathcal{F}$ is a σ-algebra on the set Ω and that T is a measurable map from Ω to Ω. We shall say that T is **two-to-one** if there are measurable sets Ω_l and Ω_r and a reflection R of $(\Omega, \mathcal{F})$ with the properties that

- $\Omega = \Omega_l \cup \Omega_r$;
- $\Omega_l \cap \Omega_r$ is the set of fixed points of R;
- $T \circ R = T$;
- The restriction of T to each of Ω_l and Ω_r is one-to-one;
- If $F \in \mathcal{F}$ then $T(F \cap \Omega_l) \in \mathcal{F}$ and $T(F \cap \Omega_r) \in \mathcal{F}$.

Since we can show that there is exactly one such R for any two-to-one map T, we will refer to R as **the reflection associated with** T. Also note that the sets Ω_l and Ω_r cannot be empty and that R maps each of these sets onto the other. Two-to-one maps are a natural generalization of unimodal maps.

We have seen examples of two-to-one maps on $[0, 1]$. Here are some examples on the closed unit disk and on the unit circle in the complex plane.

Suppose that a and b are complex numbers with $|a|^2 = |b|^2 + 1$. The fractional linear transformation $f(z) = (az + b)/(\bar{b}z + \bar{a})$ maps the unit disk onto itself and maps the unit circle onto itself. The map $T(z) = (f(z))^2$ maps the unit circle onto itself and maps the unit disk onto itself. In each case T is two-to-one. To see why, take $R(z) = f^{-1}(-f(z))$. R is a fractional linear transformation which maps the unit circle to the unit circle and the unit disk to the unit disk. As a map of the unit disk to itself, R has exactly one fixed point at $z = -b/a$, and this fixed point does not lie on the unit circle. What is interesting about this example is that as a map of the unit circle to itself, R has no fixed points, in contrast with the examples on $[0, 1]$.

3 Constructing Symmetric Measures

In this section we assume that T is two-to-one and that R is the reflection associated with T. As we shall not consider more than one two-to-one map at a time, this should cause no confusion. We will give conditions on T which assure the existence of a probability measure Pr such that the system $(\Omega, \mathcal{F}, \Pr, T, R)$ is a symmetric measure-preserving system.

Let $\mathcal{I}_T$ denote the invariant σ-algebra of T and let $\mathcal{I}'_T = \{G \in \mathcal{I}_T : T(G) = G\}$. In some cases, Theorem 3 below can be used to show that $\mathcal{I}'_T$ only contains the empty set, as we shall see in the next section.

Lemma 1. *Suppose that $G \in \mathcal{I}'_T$ and $G \neq \emptyset$. Let μ be a probability measure on $(\Omega, \mathcal{F})$ and suppose that $\mu(G) = 1$. Then the set function ν defined on $\mathcal{F}$ by*

$$\nu(E) = \frac{1}{2}\mu(T(E \cap G \cap \Omega_l)) + \frac{1}{2}\mu(T(E \cap G \cap \Omega_r))$$

is a probability measure on $(\Omega, \mathcal{F})$ with $\nu(G) = 1$, $\nu \circ T^{-1} = \mu$ and $\nu \circ R^{-1} = \nu$.

Proof. It is clear that ν is well-defined and non-negative, since T carries elements of $\mathcal{F}$ to elements of $\mathcal{F}$. Next note that $T(G \cap \Omega_l) = T(G \cap \Omega_r) = G$, so $\nu(G) = 1$, and that since the restriction of T to each of Ω_l and Ω_r is one-to-one, ν is countably additive. Hence ν is a probability measure on $\mathcal{F}$.

Note that $R(G) = R^{-1}(G) = R^{-1}(T^{-1}(G)) = (T \circ R)^{-1}(G) = T^{-1}(G) = G$, and $R(\Omega_l) = \Omega_r$, so $\nu \circ R^{-1} = \nu$.

Finally we show that $\nu \circ T^{-1} = \mu$. First observe that for any set $E \in \mathcal{F}$ we have

$$T(T^{-1}(G \cap E) \cap \Omega_l) = G \cap E = T(T^{-1}(G \cap E) \cap \Omega_r).$$

To see why, recall that T maps G onto G. Therefore

$$T(T^{-1}(G \cap E)) = G \cap E.$$

Therefore, $g \in G \cap E$ if and only if there is some $g' \in T^{-1}(G \cap E)$ such that $T(g') = g$. Now, $g' \in T^{-1}(G \cap E)$ if and only if $R(g') \in T^{-1}(G \cap E)$. Since either $g' \in \Omega_l$ and $R(g') \in \Omega_r$ or vice versa, T maps both $T^{-1}(G \cap E) \cap \Omega_l$ and $T^{-1}(G \cap E) \cap \Omega_r$ onto $G \cap E$, as claimed.

Therefore, for any $E \in \mathcal{F}$,

$$\begin{aligned}
2\nu(T^{-1}(E)) &= 2\nu(T^{-1}(E) \cap G) \\
&= 2\nu(T^{-1}(E \cap G)) \\
&= \mu(T(T^{-1}(G \cap E) \cap \Omega_l)) + \mu(T(T^{-1}(G \cap E) \cap \Omega_r)) \\
&= 2\mu(G \cap E) \\
&= 2\mu(E),
\end{aligned}$$

which finishes the proof of the lemma. □

Lemma 2. *Suppose that $G \in \mathcal{I}'_T$ and $G \neq \emptyset$. Let μ be a probability measure on $(\Omega, \mathcal{F})$ and suppose that $\mu(G) = 1$. There is a sequence μ_n of R-invariant probability measures on $(\Omega, \mathcal{F})$ such that $\mu_n(G) = 1$ and $\mu_n \circ T^{-1} = \mu_{n-1}$ for $n = 1, 2, \ldots$.*

PROOF. We give a recursive construction.

Put $\mu_0 = (\mu + \mu \circ R^{-1})/2$. Since $R \circ R$ is the identity map on Ω, μ_0 is R-invariant. Since $R^{-1}(G) = G$ we have $\mu_0(G) = 1$.

Suppose now that n is a positive integer and $\mu_0, \ldots, \mu_{n-1}$ have been constructed to satisfy Lemma 2. Define μ_n by

$$\mu_n(E) = \frac{1}{2}\mu_{n-1}(T(E \cap G \cap \Omega_l)) + \frac{1}{2}\mu_{n-1}(T(E \cap G \cap \Omega_r)).$$

Then Lemma 1 shows that μ_n satisfies the conditions of Lemma 2 as well. □

Theorem 3. *Suppose that Ω is a compact metric space, that $\mathcal{F}$ is the Borel sigma algebra and that T and R are continuous. Suppose that $G \in \mathcal{I}'_T$ and $G \neq \emptyset$. Then there is a probability measure* Pr *on $(\Omega, \mathcal{F})$ having* $\Pr(G) = 1$ *which is invariant under both T and R. Furthermore, if R has at most one fixed point and T and R have no fixed points in common, then* Pr *is non-atomic.*

PROOF. Let μ_n be the sequence of measures constructed in Lemma 2. Put $\sigma_n = n^{-1}(\mu_0 + \cdots + \mu_{n-1})$ for $n = 1, 2, \ldots$. Each σ_n is invariant under R and R is continuous, so any limit point of the sequence σ_n will also be invariant under R. Since

$$\sigma_n = n^{-1}(\mu_n \circ T^{-n} + \mu_n \circ T^{-n+1} + \cdots + \mu_n \circ T^{-1})$$

it is easy to show that any limit point of the sequence σ_n will also be T invariant. (See Theorem 6.9 of Walters [1982] for the case of Borel measures on $[0,1]$.)

Now suppose that R has at most one fixed point and R and T have no fixed points in common. We first show that no periodic point of T may be an atom of Pr. Suppose that ω is a periodic point of T with period n. Let $p = \Pr(\{\omega\}) > 0$. Note that the inverse image of an atom under T is never empty, and therefore, contains either 1 or 2 points. Observe that

1. $\omega \in T^{-n}(\omega)$;

2. $T^{-n}(\omega)$ contains ω and at least one other point, and has probability p.

3. Each element of $T^{-n}(\omega)$ is an atom, and these atoms each have a probability which is less than p.

Therefore $p > 0$ is not possible, meaning there are no periodic atoms.

Now we show that no non-periodic point may be an atom either. Begin with the purported atom ω. For each positive integer n the elements of $T^{-n}(\omega)$ are atoms, and since no atom is a periodic point, the sets $T^{-n}(\omega)$, $n = 1, 2, \ldots$ are disjoint. Since these sets all have the same probability, they must have probability 0 which contradicts our assumption that ω is an atom. □

Corollary 4. *Suppose that $T : [0,1] \to [0,1]$ is a continuous, onto, unimodal map with $T(0) = 0 = T(1)$. Then there is a non-atomic probability measure on the Borel sets of $[0,1]$ and a continuous reflection R of $[0,1]$ so that $([0,1], \mathcal{B}, \Pr, T, R)$ is a symmetric measure-preserving system.*

4 Applications

Next we will show how Theorem 3 can be used to analyze the behavior of some symmetric unimodal maps of $[0,1]$ to itself.

Lemma 5. *Let $\mathcal{I}$ be a closed bounded interval, let a be the left endpoint of $\mathcal{I}$ and let b be in the interior of $\mathcal{I}$. Suppose $f : \mathcal{I} \to \mathcal{I}$*

1. *is continuous;*

2. *satisfies $f(x) > x$ on $(a, b]$;*

3. *satisfies $f(f(b)) > a$.*

Then for each $y \in (a, b]$ there is some integer $k \geq 2$ for which $f^{(k)}(b) > y$.

PROOF. Suppose not. Then for each positive integer k we have $y \geq f^{(k+1)}(b) = f(f^{(k)}(b)) > f^{(k)}(b)$ so $p \equiv \lim_{k\to\infty} f^{(k)}(b) \in (f^{(2)}(b), y] \subset (-a, b]$ is a fixed point of f. This contradicts our assumption that f has no fixed points in $(a, b]$. □

Theorem 6. *Suppose that* $([0,1], \mathcal{B}, \Pr, T, R)$ *is a symmetric measure–preserving system and that*

1. $\Pr$ *has no atoms;*
2. T *is unimodal with turning point* m*;*
3. $T(x) > x$ *on* $(0, m]$*;*
4. $T(0) = T(1) = 0$.

Then for any $a \in [0,1]$*, if* $T(a) < 1$ *then* $\Pr([T(a), 1]) > 0$.

PROOF. Suppose not. Then $\Pr([0, T(a)]) = 1$. We will use Lemma 5 to derive a contradiction. It is sufficient to examine the case $a \in (0, m]$.

Note that since $\Pr$ and T are both invariant under R and R is self-inverse, $\Pr(A) = 0$ implies $\Pr(T(A)) = 0$. Since T is continuous and maps both 0 and 1 to 0, and $\Pr([T(a), 1]) = 0$, for every $k \geq 1$ we have $\Pr([0, T^{(k+1)}(a)]) = 0$. From Lemma 5 for some such k we have $T^{(k+1)}(a) > a$ so $[0, a] \subset [0, T^{(k+1)}(a)]$. This implies $\Pr([0, a]) = 0$, which in turn implies $\Pr([0, T(a)]) = 0$, which is our contradiction. □

Corollary 7. *Suppose that* $m \in (0, 1)$ *and*

1. $T : [0,1] \to [0,1]$ *is unimodal with turning point* m*;*
2. $T(x) > x$ *on* $(0, m]$*;*
3. $T(0) = T(1) = 0$*;*
4. $T(m) < 1$, $T(T(m)) > 0$.

Then $\mathcal{I}'_T$ *contains only the empty set.*

PROOF. Suppose not. We will now apply Theorem 3. Let $\Pr$ be the probability measure on the Borel subsets of $[0,1]$ which is preserved by both R and T. Note that $\Pr$ is not atomic as m is the only fixed point of R and m is not a fixed point of T. Hence $\Pr([T(m), 1]) = \Pr(T^{-1}([T(m), 1])) = \Pr(\{m\}) = 0$. This contradicts Theorem 6. □

Next we consider the question of when symmetric measure-preserving systems are isomorphic. Following Walters [1982] we say that two symmetric measure-preserving systems $(\Omega_i, \mathcal{F}_i, \Pr_i, T_i, R_i)$, $i = 1, 2$ are **isomorphic** if there exist $M_i \in \mathcal{F}_i$ with $\Pr_i(M_i) = 1$ for $i = 1, 2$ such that

(a) $T_i(M_i) \subset M_i$ for $i = 1, 2$;

(b) There is an invertible measure-preserving transformation $\Phi : M_1 \to M_2$ with

$$\Phi(T_1(\omega)) = T_2(\Phi(\omega))$$
$$\Phi(R_1(\omega)) = R_2(\Phi(\omega))$$

for all $\omega \in M_1$.

Recall the symmetric tent map system, $([0,1], \mathcal{B}([0,1]), \lambda, \tau, \rho)$, defined in the introduction. Here is a formalization of the situation described in the introduction.

Theorem 8. *Suppose that $T : [0,1] \to [0,1]$ is a continuous unimodal map with turning point m, $T(0) = T(1) = 0$, $T(m) = 1$, and reflection R. Suppose that $([0,1], \mathcal{B}, \Pr, T, R)$ is a symmetric measure-preserving system and the distribution function of* Pr *is a homeomorphism of $[0,1]$ onto $[0,1]$. Then $([0,1], \mathcal{B}, \Pr, T, R)$ is isomorphic to $([0,1], \mathcal{B}, \lambda, \tau, \rho)$.*

Since the key in this theorem is having the distribution function of Pr be an increasing function, the following corollary to Theorem 6 is of interest.

Corollary 9. *Suppose that $([0,1], \mathcal{B}, \Pr, T, R)$ is a symmetric measure–preserving system and*

1. Pr *has no atoms;*
2. *T is unimodal with turning point m;*
3. *$T(m) = 1$ and $T(0) = T(1) = 0$;*
4. *For every interval $I \subset [0,1]$ there is some positive integer k so that $m \in T^{(k)}(I)$.*

Then the distribution function of Pr *is a homeomorphism of $[0,1]$ onto $[0,1]$. In particular, $([0,1], \mathcal{B}, \Pr, T, R)$ is isomorphic to $([0,1], \mathcal{B}, \lambda, \tau, \rho)$.*

Proof. Suppose not. Note that we must have $T(x) > x$ on $(0, m]$. Let F denote the distribution function of Pr. Then for some $0 \leq a < b \leq 1$ we have $F(a) = F(b)$, so $\Pr([a,b]) = 0$. Let I denote $[a,b]$, and choose k so that $m \in T^{(k)}(I) \equiv I_k$. Note that since T is continuous I_k is a closed interval, and I_k has probability 0. It is also clear that I_k has non-empty interior. Let $J_k = I_k \cup R(I_k)$. J_k is a closed interval with probability 0 which contains m

in its interior. Hence $T(J_k)$ is an interval of probability 0 with right endpoint 1 and non-empty interior. This contradicts Theorem 6. □

We are, however, in a position to assert the existence of symmetric measure-preserving systems. Using Corollary 4 and the idea of the proof of Corollary 9, it is easy to see

Theorem 10. *Suppose*

1. $T : [0,1] \to [0,1]$ *is continuous;*
2. T *is unimodal with turning point* m*;*
3. $T(m) = 1$, $T(0) = T(1) = 0$*;*
4. *For every interval* $I \subset [0,1]$ *there is some positive integer* k *so that* $m \in T^{(k)}(I)$.

Then there is a continuous reflection of $[0,1]$*, denote it by* R*, and non-atomic probability measure* Pr *on* $\mathcal{B}$ *which assigns positive probability to all intervals, such that* $([0,1], \mathcal{B}, \Pr, T, R)$ *is a symmetric measure-preserving system which is isomorphic to* $([0,1], \mathcal{B}, \lambda, \tau, \rho)$.

Condition 4 in the theorem is satisfied in many cases. See the discussion of homtervals and stable periodic orbits in Collet and Eckmann [1980].

5 Symmetry in $([0,1], \mathcal{B}, \Pr)$

Suppose that we are given a probability measure Pr on the Borel subsets, $\mathcal{B}$, of $[0,1]$. We would like to construct transformations T and R so that $(\Omega, \mathcal{B}, \Pr, T, R)$ is symmetric. We have seen that this is easily done if the distribution function of Pr is continuous and strictly increasing. Suppose then we only require that it be continuous.

Theorem 11. *If* Pr *is a non-atomic probability measure on the Borel sets of* $[0,1]$ *then there is a symmetric measure-preserving system* $([0,1], \mathcal{B}, \Pr, T, R)$.

The proof is presented as a series of lemmas. As before, put $F(t) = \Pr([0,t])$. Put $F^{-1}(y) = \sup\{x : F(x) \leq y\}$ and $R(t) = F^{-1}(1 - F(t))$ for all $t \in [0,1]$. Then we have:

Lemma 12. *There exists* $\Omega_0 \subset \mathcal{B}$ *with* $\Pr(\Omega_0) = 1$ *such that* $R(R(\omega)) = \omega$ *for all* $\omega \in \Omega_0$.

Proof. We shall take Ω_0 to be the complement of the union of all intervals where F is constant. Precisely, we define

$$\mathcal{J} = \{[a,b] \subset [0,1] : a < b, F(a) = F(b),\\ x < a < b < y \text{ implies } F(x) < F(a) < F(y)\}$$

Since the elements of $\mathcal{J}$ are disjoint closed subintervals of $[0,1]$ of positive length, $\mathcal{J}$ is countable, and the union of its elements is not $[0,1]$ since each element of $\mathcal{J}$ has probability 0. Let Ω_0 be the complement of the union of the elements of $\mathcal{J}$. It is clear that $\Pr(\Omega_0) = 1$ and that F is strictly increasing on Ω_0.

It is easy to see that for all $x \in [0,1]$ we have $F(F^{-1}(x)) = x$. What we need to know is that if $x \in \Omega_0$ then $F^{-1}(F(x)) = x$. To see this, observe that for all x we have $x \leq F^{-1}(F(x))$, so we suppose that $x \in \Omega_0$ and $x < F^{-1}(F(x))$. However, since $F(x) = F(F^{-1}(F(x)))$, this would imply that both x and $F^{-1}(F(x))$ were in Ω_0^c, a contradiction.

Now it is a simple matter to check that if $x \in \Omega_0$ then $R(R(x)) = x$. □

Lemma 13. *Suppose that $g : [a,b] \to [0,1]$ is monotone and continuous. Let $h = F^{-1} \circ g$. For $z \in (a,b]$ put $c_z = h(z^-)$ and put $c_a = a$. For $z \in [a,b)$ put $d_z = h(z^+)$ and put $d_b = b$. Then for any $z \in [a,b]$, we have $F(c_z) = F(d_z)$.*

Proof. Simply observe that since F and g are continuous, $F(c_z) = g(z) = F(d_z)$. □

First we apply Lemma 13 to prove:

Lemma 14. *R preserves* Pr.

Proof. It is sufficient to prove that for any $b \in [0,1]$, $\Pr([b,1]) = \Pr(R^{-1}([b,1]))$.

First notice that F^{-1} is strictly increasing and continuous from the right. Since F itself is non-decreasing and continuous we conclude that R is non-increasing and continuous from the left. Let $b \in [0,1]$ be given and put $t_b = \sup(\{x : R(x) \geq b\})$. It is straightforward to check that $R(t_b) \geq b$ and that $R^{-1}([b,1]) = [0, t_b]$.

Since $\Pr([b,1]) = 1 - F(b)$ and $\Pr(R^{-1}([b,1])) = \Pr([0,t_b]) = F(t_b)$, it is sufficient to show that $1 - F(b) = F(t_b)$. This is easily done by applying Lemma 13 with $g(x) = 1 - F(x)$ and $z = t_b$, and observing that $R(t_b^+) \leq b \leq R(t_b) = R(t_b^-)$. □

We now focus our attention on constructing T which preserves Pr and which satisfies $T = T \circ R$. We omit the straightforward proof of the following:

Lemma 15. *$m \equiv F^{-1}(1/2)$ is the unique fixed point of R.*

Define the function T as follows:

$$T(x) = \begin{cases} F^{-1}(2F(x)) & \text{if } x \in [0,m] \\ F^{-1}(2(1-F(x))) & \text{if } x \in [m,1] \end{cases}$$

Lemma 16. $T = T \circ R$

PROOF. It is easy to check that for any $x \in [0,1]$ that $F(x) = 1 - F(R(x))$.

Suppose that $x \in [0,m]$. Then $R(x) \geq R(m) = m$ so $R(x) \in [m,1]$. So, $T(x) = F^{-1}(2F(x)) = F^{-1}(2(1 - F(R(x)))) = T(R(x))$. Similarly, if $x \in [m,1]$ then $R(x) \leq R(m) = m$ so $R(x) \in [0,m]$ and $T(x) = F^{-1}(2(1 - F(x))) = F^{-1}(2F(R(x))) = T(R(x))$. □

Lemma 17. T *preserves* Pr.

PROOF. It will be sufficient to prove that for any $b \in [0,1]$ that $\Pr([b,1]) = \Pr(T^{-1}([b,1]))$.

Fix such a b and put $a_b = \inf(\{x : T(x) \geq b\})$ and $c_b = \sup(\{x : T(x) \geq b\})$. Observe that T is right continuous on $[0,m]$, left continuous on $[m,1]$, and $T(m) = 1$. Therefore $a_b \leq m \leq c_b$ and $T^{-1}([b,1]) = [a_b, c_b]$. Once we show that $F(b) = 2F(a_b)$ and $F(b) = 2(1 - F(c_b))$ we will be done, since averaging these equations gives $F(b) = F(a_b) + 1 - F(c_b)$, which in turn shows

$$\begin{aligned} \Pr([b,1]) &= 1 - F(b) = 1 - [F(a_b) + 1 - F(c_b)] \\ &= F(c_b) - F(a_b) = \Pr([a_b, c_b]). \end{aligned}$$

(Note the use of our assumption that Pr is non-atomic.)

To see that $F(b) = 2(F(a_b))$ apply Lemma 13 with $g(x) = 2F(x)$ on $[0,m]$ and $z = a_b$, and to see that $F(b) = 2(1 - F(c_b))$ apply Lemma 13 with $g(x) = 2(1 - F(x))$ on $[m,1]$ with $z = c_b$. □

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