RESEARCH

Real Analysis Exchange
Vol. 24(1), 1998/99, pp. 373–390

Donald W. Spear, 201 E. Mimosa Circle, San Marcos, TX 78666

SETS WITH DIFFERENT DIMENSIONS IN [0,1]

Abstract

Given $0 < s < u < v < 1$ and $s < t < v$, a Cantor set Y is constructed in $[0, 1]$ with Hausdorff dimension s, packing dimension t, lower box dimension u and upper box dimension v. In the sense that t and u are independent, so are the packing and lower box dimensions. Although $Y = \{0\} \cup \bigcup_{\ell=0}^{\infty} Y_\ell$, the lower and upper box dimensions of each Y_ℓ are respectively s and t.

1 Introduction

The lower and upper box dimensions of a set E are denoted by $\underline{\dim}_B E$ and $\overline{\dim}_B E$. They are not able to see the fine structure of a set in the sense that

$$\underline{\dim}_B \overline{E} = \underline{\dim}_B E \text{ and } \overline{\dim}_B \overline{E} = \overline{\dim}_B E$$

where $\overline{E}$ denotes the closure of E. As a result there is a set E which is the union of a countable number of sets $\{E_i;\ i \in \mathbb{N}\}$ and the strict inequality

$$\underline{\dim}_B \bigcup_{i\in\mathbb{N}} E_i > \sup_{i\in\mathbb{N}}\{\underline{\dim}_B E_i\} \tag{1}$$

is necessary. However for just two sets E and F,

$$\underline{\dim}_B E \cup F = \sup\{\underline{\dim}_B E, \underline{\dim}_B F\}.$$

Similar remarks can be made for the upper box dimension.

In Section 5, we note that for the Hausdorff ($\dim_{\mathcal{H}}$) and the packing ($\dim_{\mathcal{P}}$) dimensions there must be equality in (1). The dimensions always have the

Mathematical Reviews subject classification: Primary 28A75; Secondary 28A78
Received by the editors January 30, 1998

relationships $\dim_{\mathcal{H}} E \leq \underline{\dim}_B E \leq \overline{\dim}_B E$ and $\dim_{\mathcal{H}} E \leq \dim_{\mathcal{P}} E \leq \overline{\dim}_B E$, but the lower box dimension and the packing dimension are independent.

If dim stands for any of the four dimensions, dim is monotone. Y is a Cantor set constructed in Section 7. The point 0 is such that if $\varepsilon > 0$, $\underline{\dim}_B Y \cap [0, \varepsilon) = u = \underline{\dim}_B Y$. If $x \in Y$, $\varepsilon > 0$ and $(x - \varepsilon, x + \varepsilon)$ misses $(0, \varepsilon)$, then $\underline{\dim}_B Y \cap (x - \varepsilon, x + \varepsilon) = s < u$. In this sense the local lower box dimension only agrees with the global lower box dimension at one point. The same holds for the upper box dimension, but is never possible for either the Hausdorff or the packing dimension.

2 Nested Intervals

For each positive integer $\wp$, we select a set $\mathcal{I}_\wp$ of nonoverlapping closed intervals from $[0, 1]$. The cardinality of $\mathcal{I}_\wp$ is called $c(\wp)$. Each interval in $\mathcal{I}_{\wp+1}$ is also in one and only one element of $\mathcal{I}_\wp$. Y is defined to be $\bigcap_{\wp=1}^{\infty} \bigcup \mathcal{I}_\wp$. We define a sequence $\{\delta(\wp);\ \wp \in \mathbb{N}\}$ for which $\delta(\wp + 1)/\delta(\wp) \in \mathbb{N} \setminus \{1\}$. Each interval in $\mathcal{I}_\wp$ is of length $\delta(\wp)^{-1}$.

In this paragraph the sequence $\delta(\wp)$ is defined. Permanently fix $0 < s < u < v < 1$ and $s < t < v$. With $\beta = \max\{[v/(1-v)], [2u/(1-u)]\} + 2$, set, for $j \in \mathbb{N}$, $\tau_{2j-1} = 2^{[(2j-1)\beta/u]}$ and $\tau_{2j} = 2^{[2j\beta/v]}$. A sequence of positive integers $\{n_i;\ i \in \mathbb{N}\}$ is defined with $n_i \geq 2$ in Section 6. If $\sum_1^h n_i < j \leq \sum_1^{h+1} n_i$, let $\varphi(j) = h + 1$. In order to simplify notations, set $n_0 = 0$ and $\varrho_{\theta-1} = \sum_1^{\theta-1} n_i$ where $\theta \in \mathbb{N}$. We are using the notation $\sum_j^k n_i = 0$ and $\prod_j^k n_i = 1$ if $k < j$. Let $\delta(1) = \tau_1$ and $\delta(\wp + 1) = \prod_{i=1}^{\wp+1} \tau_{\varphi(i)} = \delta(\wp)\tau_{\varphi(\wp+1)}$. The sequences in $\prod_1^\wp \{0, 1, \ldots, \tau_\varphi - 1\}$ or $\prod_1^\infty \{0, 1, \ldots, \tau_\varphi - 1\}$ are the digits of a number in $[0, 1]$. Let $Q : \prod_1^\wp \{0, 1, \ldots, \tau_\varphi - 1\}$ be defined by $Q(x) = \sum_1^\wp x(i)\delta(i)^{-1}$. Extend the domain of Q to $\prod_1^\infty \{0, 1, \ldots, \tau_\varphi - 1\}$ by setting $Q(x) = \sum_1^\infty x(i)\delta(i)^{-1}$. For $x \in \prod_1^\wp \{0, 1, \ldots, \tau_{\varphi(i)} - 1\}$, set

$$\Lambda_x = \Big\{ w \in \prod_1^\infty \{0, 1, \ldots, \tau_{\varphi(i)} - 1\};\ x = (w(1), \ldots, w(\wp)) \Big\}.$$

The intervals $\mathcal{I}_\wp$ are selected from

$$K_x = Q(x) + \left[0, \delta(\wp)^{-1}\right] \text{ where } x \in \prod_1^\wp \{0, 1, \ldots, \tau_\varphi - 1\}. \tag{2}$$

An exact definition of $\mathcal{I}_\wp$ is stated in (30). The map Q takes the sequences in Λ_x into K_x. The symbol $*$ indicates that $x * y$ is an extension of the sequence x. If x is in $\prod_1^\wp \{0, 1, \ldots, \tau_\varphi - 1\}$ and i is in $\{0, 1, \ldots, \tau_{\varphi(\wp+1)} - 1\}$, then $K_{x*i} \subset K_x$.

3 The Cardinality of $\mathcal{I}_\wp$

We use the sequences $\xi_j = 2^{j\beta}$, $\zeta_{2j-1} = 2^{[(2j-1)\beta s/u]}$ and $\zeta_{2j} = 2^{[2jt\beta/v]}$. The intervals in $\mathcal{I}_\wp$ are selected so that their cardinality is

$$c(\wp) = \sum_{j=0}^{\wp} \prod_{i=1}^{j} \xi_\varphi \cdot \prod_{i=j+1}^{\wp} \zeta_\varphi. \tag{3}$$

An interval of length $\delta(\wp)^{-1}$ contains $\tau_{\varphi(\wp+1)}$ intervals of length $\delta(\wp+1)^{-1}$. Since $(2j-1)\beta + 1 < [(2j-1)\beta/u]$ and $2j\beta + 1 < [2j\beta/v]$, we have $2\xi_j < \tau_j$. We can select $2\xi_{\varphi(\wp+1)}$ of the $\tau_{\varphi(\wp+1)}$ intervals. If $m < n_1$, the cardinality of $\mathcal{I}_m$ is $\xi_1^m + \sum_{j=0}^{m-1} \xi_1^j \zeta_1^{m-j}$. We need the cardinality of $\mathcal{I}_{m+1}$ to be

$$c(m+1) = 2\xi_1 \cdot \xi_1^m/2 + 2\zeta_1 \cdot \xi_1^m/2 + \zeta_1 \cdot \sum_{j=0}^{m-1} \xi_1^j \zeta_1^{m-j}.$$

2 ξ_1 intervals are selected in each of the first $\xi_1^m/2$ intervals of $\mathcal{I}_m$ and 2 ζ_1 intervals are selected in each of the second $\xi_1^m/2$ intervals of $\mathcal{I}_m$. This still leaves the last $\sum_{j=0}^{m-1} \xi_1^j \zeta_1^{m-j}$ intervals in $\mathcal{I}_m$ and ζ_1 intervals are selected in each of them. Thus $c(m+1) = \sum_{j=0}^{m+1} \xi_1^j \zeta_1^{m+1-j}$. A complete description of the construction is in Section 7. The term $\prod_1^\wp \xi_\varphi$ in the sum $c(\wp)$ dominates $c(\wp)$ in the sense that the ratio of $c(\wp)$ to $\prod_1^\wp \xi_\varphi$ is bounded.

Lemma 3.1. *For $k \in \{0\} \cup \mathbb{N}$ and $m \in \{0, 1, \ldots, n_{k+1}\}$,*

$$1 \le c(m+\varrho_k) \Big/ \prod_1^{m+\varrho_k} \xi_\varphi = \sum_{j=0}^{m+\varrho_k} \prod_{j+1}^{m+\varrho_k} (\zeta_\varphi/\xi_\varphi) < \max\{\lambda_j\}, \tag{4}$$

where

$$\lambda_j = \sum_0^\infty (\zeta_j/\xi_j)^i = \xi_j/(\xi_j - \zeta_j). \tag{5}$$

In order to see this note that $\zeta_j/\xi_j \in \{2^{[j\beta s/u]-j\beta}, 2^{[j\beta t/v]-j\beta}\}$. The supremum of the fractions $\{\zeta_j/\xi_j;\ j \in \mathbb{N}\}$ is realized for some j and the maximum of the λ_j's depends on the fractions ζ_j/ξ_j. The construction is designed so that the lower box dimension of Y is

$$\liminf_{j\to\infty} (\log \xi_j)/\log \tau_j = u$$

and the upper box dimension of Y is $\limsup_{j\to\infty}(\log \xi_j)/\log \tau_j = v$. The construction is designed so that Y is the union of $\{0\}$ and the sets $\{Y_\ell;\ \ell \in$

$\{0\} \cup \mathbb{N}\}$. The Hausdorff dimension of each Y_ℓ is $\liminf_{j\to\infty} \frac{\log \zeta_j}{\log \tau_j} = s$ and the packing dimension of each Y_ℓ is $\limsup_{j\to\infty} (\log \zeta_j)/\log \tau_j = t$.

What bounds are there on $(\log \zeta_{k+1})/\log \tau_{k+1}$ and $(\log \xi_{k+1})/\log \tau_{k+1}$? If $2|k$,

$$\frac{\log \zeta_{k+1}}{\log \tau_{k+1}} < \frac{(k+1)\beta s/u}{(k+1)\beta/u - 1} = s + \frac{s}{(k+1)\beta/u - 1} < s + \frac{s}{(k+1)\beta}$$
$$s - \frac{u}{(k+1)\beta} = \frac{(k+1)\beta s/u - 1}{(k+1)\beta/u} < \frac{\log \zeta_{k+1}}{\log \tau_{k+1}}$$
$$s - 1/(k+1) < (\log \zeta_{k+1})/\log \tau_{k+1} < s + 1/(k+1). \quad (6)$$

If $2|(k+1)$, then

$$t - 1/(k+1) < (\log \zeta_{k+1})/\log \tau_{k+1} < t + 1/(k+1). \quad (7)$$

If $2|k$,

$$u \le (\log \xi_{k+1})/\log \tau_{k+1} < u + 1/(k+1). \quad (8)$$

If $2|k+1$,

$$v \le (\log \xi_{k+1})/\log \tau_{k+1} < v + 1/(k+1). \quad (9)$$

4 Box Counting

For $r > 0$, set $N_r(F) = \#\{a \in \mathbb{Z};\ [ar, (a+1)r] \cap F \ne \emptyset\}$. The lower and upper box dimensions of F are defined to be

$$\underline{\dim}_B F = -\liminf_{r\downarrow 0} \frac{\log N_r(F)}{\log r} \text{ and } \overline{\dim}_B F = -\limsup_{r\downarrow 0} \frac{\log N_r(F)}{\log r}. \quad (10)$$

Let $B_r(x)$ be a closed ball and r be its radius. A set, $\mathcal{F} = \{B_{r_i}(x_i);\ i \in \mathcal{M}\}$, is said to h-pack J if its elements are pairwise disjoint, $x_i \in J$ and $2r_i \le h$. Let

$$P_h^\alpha J = \sup\left\{ \sum_{B_\varepsilon(x)\in\mathcal{R}} (2\varepsilon)^\alpha;\ \mathcal{R}\ h\text{-packs } J \right\} \text{ and } P_0^\alpha J = \inf_{h>0}\{P_h^\alpha J\}. \quad (11)$$

P_0^α is called the α packing premeasure of J. It is monotone $P_0^\alpha E \subseteq P_0^\alpha E \cup F$ and it is finitely subadditive $P_0^\alpha E \cup F \le P_0^\alpha E + P_0^\alpha F$, but it is not countably subadditive. Taylor and Tricot [4, p 683] point out that if E is the set of rational numbers in $[0,1]$, then $P_0^{1/2} E = \infty$, but each point has zero premeasure. At the end of this paper we will be able to conclude that for any ℓ, $P_0^{(t+v)/2} Y_\ell = 0$ but $P_0^{(t+v)/2} \bigcup_{\ell\in\mathbb{N}} Y_\ell = \infty$. In general P_0^α can not detect fine structure in the following sense.

Proposition 4.1. *If E is a subset of $\mathbb{R}$, then $P_0^\alpha E = P_0^\alpha \overline{E}$.*

Pick $1 > \varepsilon > 0$. For any $x \in \overline{E}$ and $r > 0$, there is $x' \in E$ at a distance less than $r\varepsilon$ from x and $B_{(1-\varepsilon)r}(x') \subseteq B_r(x)$. Let $h \downarrow 0$ and then $\varepsilon \downarrow 0$ in

$$(1-\varepsilon)^\alpha P_h^\alpha \overline{E} = \sup\left\{ \sum_{B_r(x)\in\mathcal{R}} (1-\varepsilon)^\alpha (2r)^\alpha;\ \mathcal{R}\ h\text{-packs}\ \overline{E} \right\} \le P_{(1-\varepsilon)h}^\alpha E.$$

Equality holds because $E \subseteq \overline{E}$. Given a set E with $0 < \operatorname{diam} E < 1$, $(\operatorname{diam} E)^\alpha$ as a function of α is decreasing with increasing α. If $\beta > \alpha$ and $0 < \varepsilon < 1$, then $P_\varepsilon^\beta E \le \varepsilon^{\beta-\alpha} P_\varepsilon^\alpha E$. When $P_0^\alpha E < \infty$, $P_0^\beta E$ is forced to be 0. When $P_0^\beta E > 0$, $P_0^\alpha E = \infty$.

Lemma 4.1. $\inf\{\alpha > 0;\ P_0^\alpha E = 0\} = \sup\{\alpha > 0;\ P_0^\alpha E > 0\}$.

The next proposition is a slight change from one of Tricot [5, p 59]. It shows that the upper box dimension can be defined by using the packing premeasures. Let $M_r(E)$ be the greatest number of nonoverlapping closed balls with radius r and center in E. The set function M_r can replace N_r in the definitions of the box dimensions.

Proposition 4.2. $\overline{\dim}_B E = \inf\{\alpha > 0;\ P_0^\alpha E = 0\} = \sup\{\alpha > 0;\ P_0^\alpha E > 0\}$.

If $\limsup_{r\downarrow 0} -\log M_r(E)/\log r > \gamma$, then $P_0^\gamma E \ge 1$ and

$$\sup\{\alpha > 0;\ P_0^\alpha E > 0\} \ge \limsup_{r\downarrow 0} -\log M_r(E)/\log r.$$

Now suppose $\alpha > \beta > \limsup_{r\downarrow 0} -\log M_r(E)/\log r$. There is $m \in \mathbb{N}$ such that if $0 < r \le 2^{-m}$, $r^{-\beta} > M_r(E)$. Choose $\mathcal{R}$ that 2^{-m}-packs E. For $\ell \ge m$, $M_{2^{-\ell-1}}(E) \ge \#\{B_r(x) \in \mathcal{R};\ 2^{-\ell-1} < r \le 2^{-\ell}\}$ and

$$\sum_{j=m}^{\infty} 2^{-j(\alpha-\beta)} 2^{\alpha+\beta} > \sum_{j=m}^{\infty} M_{2^{-j-1}}(E)(2^{-j+1})^\alpha \ge \sum_{B_r(x)\in\mathcal{R}} (2r)^\alpha.$$

So, $P_0^\alpha E = 0$ and $\inf\{\alpha > 0;\ P_0^\alpha E = 0\} \le \limsup_{r\downarrow 0} -\log M_r(E)/\log r$.

5 The Hausdorff and Packing Dimensions

The α spherical packing measure is

$$\mathcal{P}^\alpha E = \inf\left\{ \sum_{i=1}^{\infty} P_0^\alpha J_i;\ E = \bigcup_{i=1}^{\infty} J_i \right\}.$$

Suppose $\beta > \alpha$. When $\mathcal{P}^\alpha E < \infty$, $\mathcal{P}^\beta E$ is forced to be 0. When $\mathcal{P}^\beta E > 0$, $\mathcal{P}^\alpha E = \infty$. The packing dimension of a set E is

$$\dim_{\mathcal{P}} E = \inf\{\alpha \geq 0;\ \mathcal{P}^\alpha E < \infty\} = \sup\{\alpha \geq 0;\ \mathcal{P}^\alpha E > 0\}.$$

According to Falconer [2, p 48], $\dim_{\mathcal{P}} E \leq \overline{\dim}_B E$. This follows from Proposition 4.2. For $\alpha > 0$ and $\varepsilon > 0$, let

$$\mathcal{H}^\alpha_\varepsilon E = \inf\left\{\sum_{J\in\mathcal{R}} (\text{diam } J)^\alpha;\ E \subseteq \cup\mathcal{R},\ \text{diam } J \leq \varepsilon\right\}.$$

We could assume each J is an interval. The α Hausdorff measure is

$$\mathcal{H}^\alpha E = \sup_{\varepsilon>0} \mathcal{H}^\alpha_\varepsilon E = \lim_{\varepsilon\downarrow 0} \mathcal{H}^\alpha_\varepsilon E.$$

If $\beta > \alpha$ and $1 > \varepsilon > 0$, then $\mathcal{H}^\beta_\varepsilon E \leq \varepsilon^{\beta-\alpha}\mathcal{H}^\alpha_\varepsilon E$. When $\mathcal{H}^\alpha E < \infty$, $\mathcal{H}^\beta E = 0$. When $\mathcal{H}^\beta E > 0$, $\mathcal{H}^\alpha E = \infty$. The Hausdorff dimension of E is defined to be

$$\dim_{\mathcal{H}} E = \inf\{\alpha > 0;\ \mathcal{H}^\alpha E < \infty\} = \sup\{\alpha > 0;\ \mathcal{H}^\alpha E > 0\}.$$

Falconer [2, p 43] shows $\dim_{\mathcal{H}} E \leq \underline{\dim}_B E$. Let $\tilde{N}_r(E)$ be the smallest number of sets of diameter at most r which can cover E. If $\tilde{N}_r$ replaces N_r in (10), the same dimensions result [2, p 38]. For $0 < \varepsilon < 1$, $\mathcal{H}^\alpha_\varepsilon E \leq \tilde{N}_\varepsilon(E)\varepsilon^\alpha$ and

$$-\log \mathcal{H}^\alpha_\varepsilon E/\log\varepsilon \leq -\alpha - \log \tilde{N}_\varepsilon(E)/\log\varepsilon.$$

If $\mathcal{H}^\alpha E < \infty$, $\alpha \leq \underline{\dim}_B E$.

Both $\mathcal{H}^\alpha$ and $\mathcal{P}^\alpha$ are measures. Specifically, if $E = \bigcup_{\ell=0}^\infty E_\ell$, then

$$\mathcal{H}^\alpha E \leq \sum_{\ell=0}^\infty \mathcal{H}^\alpha E_\ell \text{ and } \mathcal{P}^\alpha E \leq \sum_{\ell=0}^\infty \mathcal{P}^\alpha E_\ell.$$

If $\mathcal{H}^\alpha E > 0$, there is ℓ so $\mathcal{H}^\alpha E_\ell > 0$. If $\alpha < \dim_{\mathcal{H}} E$, there is ℓ so $\alpha \leq \dim_{\mathcal{H}} E_\ell$. This forces $\dim_{\mathcal{H}} E \leq \sup\{\dim_{\mathcal{H}} E_\ell;\ \ell \in \{0, 1, \ldots\}\}$ and similarly for the packing dimension. Since each $E_\ell \subseteq E$,

$$\dim_{\mathcal{H}} E = \sup_{\ell\in\{0\}\cup\mathbb{N}} \dim_{\mathcal{H}} E_\ell \text{ and } \dim_{\mathcal{P}} E = \sup_{\ell\in\{0\}\cup\mathbb{N}} \dim_{\mathcal{P}} E_\ell.$$

Saint Raymond and Tricot [3, p 136] show that $\mathcal{H}^\alpha E \leq \mathcal{P}^\alpha E$. So $\dim_{\mathcal{H}} E \leq \dim_{\mathcal{P}} E$.

6 The Ratio $\log c(\wp)/\log \delta(\wp)$

According to Lemma 9.1 $\frac{\log c(\wp)}{\log \delta(\wp)}$ can be used instead of $\dfrac{\log N_r(Y)}{\log r}$ in calculating the box dimensions of Y. Make $n_1 = 2 + [2(\log \max\{\lambda_j\})/\log \tau_1]$ and for $k \geq 1$,

$$n_{k+1} = ([v/u]+1)n_1 \prod_2^k (j[v/u]+j+1) = ([v/u]+1)\sum_1^k in_i. \tag{12}$$

The λ's are defined in (5). Assume $n_1 \prod_2^k (j[v/u]+j+1) = \sum_1^k in_i$. Since

$$n_1 \prod_2^k (j[v/u]+j+1)((k+1)([v/u]+1)+1) = \sum_1^k in_i + (k+1)n_{k+1},$$

the second equality in (12) holds by induction.

The sequence δ does not grow too fast. In fact $\delta(\wp+1)/\delta(\wp) \overset{\wp\to\infty}{\longrightarrow} 1$.

Proposition 6.1. *For $k \in \mathbb{N}$ and any $\wp \in \{\varrho_k, \varrho_k+1, \ldots, \varrho_{k+1}-1\}$,*

$$(\log \tau_{\varphi(\wp+1)})/\log \delta(\wp) < 2v/uk. \tag{13}$$

We have $\dfrac{\log \tau_{\varphi(\wp+1)}}{\log \delta(\wp)} \leq \dfrac{(k+1)\beta/u}{\sum_1^k i\beta/v}$, because $n_i[i\beta/v] > i\beta/v$ and

$$[i\beta/v] \leq \log_2 \tau_i \leq [i\beta/u]. \tag{14}$$

Proposition 6.2. *The sequence δ grows fast enough that*

$$k+1 < (\log \delta(\varrho_{k+1}))/\log \delta(\varrho_k). \tag{15}$$

By the definition of n_{k+1}, $k\beta/u < ([v/u]+1)[(k+1)\beta/v]$ if and only if

$$k\sum_1^k i\beta n_i/u < n_{k+1}[(k+1)\beta/v]. \tag{16}$$

Inequality (15) is equivalent to $k+1 < 1+(n_{k+1}\log \tau_{k+1})/\log \delta(\varrho_k)$. Use (14).

Theorem 6.1. *For $k \in \{0\} \cup \mathbb{N}$ and $m \in \{0, 1, \ldots, n_{k+2}-1\}$,*

$$0 \leq \frac{\log c(m+\varrho_{k+1})}{\log \delta(m+\varrho_{k+1})} - \frac{\log \prod_{1+\varrho_k}^{m+\varrho_{k+1}} \xi_\varphi}{\log \delta(m+\varrho_{k+1})} < \frac{1}{k+1}. \tag{17}$$

For $m \in \{0, 1, \ldots, n_2 - 1\}$, $c(n_1 + m) = \sum_{j=0}^{n_1} \xi_1^j \zeta_1^{n_1-j} \zeta_2^m + \sum_{j=1}^{m} \xi_1^{n_1} \xi_2^j \zeta_2^{m-j}$ and

$$\frac{\log\left(c(m+n_1)/\xi_1^{n_1}\xi_2^m\right)}{\log \delta(m+n_1)} \leq \frac{\log \max\{\lambda_j\}}{m \log \tau_2 + n_1 \log \tau_1} < \frac{1}{2}$$

because

$$[\beta/u] n_1 > 2 \log_2 \max\{\lambda_j\}. \tag{18}$$

Now suppose $k \geq 1$. With regards to the denominator in (17),

$$\begin{aligned} \log_2 \delta(m + \varrho_{k+1}) &\geq n_{k+1}[(k+1)\beta/v] + \sum_{2}^{k} n_i [i\beta/v] + n_1[\beta/u] \\ &\geq n_{k+1}[(k+1)\beta/v] + k n_1 [\beta/u]. \end{aligned} \tag{19}$$

For $i \geq 2$, $n_1/u < n_i/v \Rightarrow n_1[\beta/u] \leq n_i[i\beta/v]$. Since $([v/u]+1)\sum_1^k i\beta n_i = \beta n_{k+1}$,

$$\log_2 \left(\frac{c(m + \varrho_{k+1})}{\prod_1^{m+\varrho_{k+1}} \xi_\varphi} \right) + \beta n_{k+1}/([v/u]+1) = \log_2 \left(\frac{c(m+\varrho_{k+1})}{\prod_{1+\varrho_k}^{m+\varrho_{k+1}} \xi_\varphi} \right).$$

Using (4), (18) and (19),

$$\frac{\log \left(c(m+\varrho_{k+1}) / \prod_{1+\varrho_k}^{m+\varrho_{k+1}} \xi_\varphi \right)}{\log \delta(m+\varrho_{k+1})} \leq \frac{\beta n_{k+1}/([v/u]+1) + n_1[\beta/u]/2}{n_{k+1}[(k+1)\beta/v] + k n_1 [\beta/u]} < \frac{1}{k+1},$$

because $(k+1)\beta u < ((k+1)\beta/v - 1)v$ implies $(k+1)\beta < ([v/u]+1)[(k+1)\beta/v]$.

Theorem 6.2. *The interval* $[u, v]$ *is the set of cluster points of*

$$\{(\log c(m+\varrho_k))/\log \delta(m+\varrho_k);\ k \in \mathbb{N},\ m \in \{0, 1, \ldots, n_{k+1}\}\}.$$

We show for $m \in \{0, 1, \ldots, n_{k+1}\}$,

$$3/k + v > 2/k + \max_{j \in \{k, k+1\}} \{(\log \xi_j)/\log \tau_j\} \geq (\log c(m+\varrho_k))/\log \delta(m+\varrho_k)$$

and

$$(\log c(m+\varrho_k))/\log \delta(m+\varrho_k) \geq \min_{j \in \{k,k+1\}} \{(\log \xi_j)/\log \tau_j\} - 1/k > u - 2/k.$$

This forces all cluster points to be in $[u, v]$. As a result of (24) and (25), both u and v must be cluster points.

Consider the second fraction in (17). Either

$$\left\{\frac{\log \xi_k^{n_k} + m\log \xi_{k+1}}{\log \delta(\varrho_k) + m\log \tau_{k+1}};\ m \in \{0,1,\dots,n_{k+1}\}\right\} \tag{20}$$

is increasing or it is decreasing. The change in the terms of this sequence can be written in abbreviated form as

$$\frac{w+(m+1)x}{z+(m+1)y} - \frac{w+mx}{z+my} = \frac{xz-wy}{(z+(m+1)y)(z+my)}. \tag{21}$$

The sign of (21) is independent of m. The two extremes in (20) are $\frac{\log \xi_k^{n_k}}{\log \delta(\varrho_k)}$ and $\dfrac{\log \xi_k^{n_k} + \log \xi_{k+1}^{n_{k+1}}}{\log \delta(\varrho_{k+1})}$. Use Proposition 6.2. If $k > 0$, then $(\log \xi_k^{n_k})/\log \delta(\varrho_{k+1}) <$ $1/(k+1)$ and

$$0 < \frac{\log \xi_{k+1}}{\log \tau_{k+1}} - \frac{n_{k+1}\log \xi_{k+1}}{\log \delta(\varrho_{k+1})} = \frac{\log \xi_{k+1}}{\log \tau_{k+1}}\frac{\log \delta(\varrho_k)}{\log \delta(\varrho_{k+1})} < \frac{1}{k+1}. \tag{22}$$

If $k=0$, $\dfrac{\log \xi_{k+1}}{\log \tau_{k+1}} - \dfrac{n_{k+1}\log \xi_{k+1}}{\log \delta(\varrho_{k+1})} = 0$. Starting with (17), we have

$$\begin{aligned}\min\left\{\frac{\log \xi_k}{\log \tau_k}, \frac{\log \xi_{k+1}}{\log \tau_{k+1}}\right\} - \frac{1}{k} &< \frac{\log c(m+\varrho_k)}{\log \delta(m+\varrho_k)} \\ &< \frac{2}{k} + \max\left\{\frac{\log \xi_k}{\log \tau_k}, \frac{\log \xi_{k+1}}{\log \tau_{k+1}}\right\}.\end{aligned} \tag{23}$$

By combining (17) and (22), bounds for the special case $m = n_{k+1}$ are seen in

$$\frac{\log \xi_{k+1}}{\log \tau_{k+1}} - \frac{1}{k+1} < \frac{\log c(\varrho_{k+1})}{\log \delta(\varrho_{k+1})} < \frac{\log \xi_{k+1}}{\log \tau_{k+1}} + \frac{1}{k+1}.$$

By (8), if $2|k$,

$$u - 1/(k+1) < (\log c(\varrho_{k+1}))/\log \delta(\varrho_{k+1}) < u + 2/(k+1). \tag{24}$$

By (9), if $2|k+1$,

$$v - 1/(k+1) < (\log c(\varrho_{k+1}))/\log \delta(\varrho_{k+1}) < v + 2/(k+1). \tag{25}$$

The greatest amount of change in (21) is

$$\frac{(\log \xi_{k+1})\log \delta(\varrho_k) - (n_k \log \xi_k)\log \tau_{k+1}}{(\log \delta(\varrho_{k+1}) + \log(\tau_{k+1}))\log \delta(\varrho_k)} \overset{k\to\infty}{\longrightarrow} 0. \tag{26}$$

The elements of $\{(\log c(\wp))/\log \delta(\wp);\ \wp \in \mathbb{N}\}$ which are in $[u,v]$ form a dense subset of $[u,v]$.

7 The Construction

In this section, we choose $\Omega_{\ell,\wp} \subseteq \prod_1^{\wp}\{0,1,\ldots,\tau_\varphi - 1\}$ and set

$$Y = \{0\} \cup \bigcup_{\ell=0}^{\infty} \bigcap_{\wp=\ell+1}^{\infty} \bigcup_{x\in\Omega_{\ell,\wp}} K_x.$$

K_x is defined in (2). At $\ell = 0$, set $\Omega_0 = \Omega_{0,1} * \prod_2^{\infty}\{0,1,\ldots,\zeta_\varphi - 1\}$ where

$$\Omega_{0,\wp} = \{\xi_1, \xi_1+1,\ldots,\xi_1+\zeta_1-1\} * \prod_{i=2}^{\wp}\{0,1,\ldots,\zeta_{\varphi(i)}-1\}.$$

If $j > k$, we want $\prod_{i=j}^{k} E_i$ to be the empty set $\emptyset$, i. e., $\emptyset * A = A = A * \emptyset$. We define Y_0 to be all the points $Q(x) = \sum_{i=1}^{\infty} x(i)\delta(i)^{-1}$ where $x \in \Omega_0$. Thus,

$$Y_0 = \bigcap_{\wp=1}^{\infty} \bigcup_{x\in\Omega_{0,\wp}} K_x \subset \xi_1\tau_1^{-1} + [0, \zeta_1\tau_1^{-1}].$$

The intervals in $\mathcal{I}_\wp$ for a general $\wp$ are defined in (30), but the first selection is

$$\mathcal{I}_1 = \{w\tau_1^{-1} + [0,\tau_1^{-1}];\ w \in \{0,1,\ldots,\xi_1+\zeta_1-1\}\}.$$

Before the second selection is made, $\{0,1,\ldots,\xi_1+\zeta_1-1\}$ is split into two sets.

$$\Gamma_1 = \{0,1,\ldots,\xi_1-1\} \text{ and } \Omega_{0,1} = \{\xi_1,\xi_1+1,\ldots,\xi_1+\zeta_1-1\}.$$

Then Γ_1 is split into the two sets

$$F_1 = \{0,1,\ldots,\xi_1/2-1\} \text{ and } A_1 = \{\xi_1/2, \xi_1/2+1,\ldots,\xi_1-1\}.$$

Order the sequences in $\prod_1^{\wp}\{0,1,\ldots,\tau_\varphi-1\}$ lexicographically, i. e., $x < w$ means $\exists\ j = \min\{i \in \{1,\ldots,\wp\};\ x(i) \neq w(i)\}$ and $x(j) < w(j)$. For $\ell \in \{1,\ldots,\beta\}$, set $\Gamma_\ell = \{0,1,\ldots,\xi_1 2^{-\ell+1}-1\} * \prod_2^{\ell}\{0,1,\ldots,2\xi_\varphi-1\}$ and then split Γ_ℓ into F_ℓ and A_ℓ in such a way that

$$\#\Gamma_\ell/2 = \#F_\ell = \#A_\ell \text{ and } x < y \text{ if } (x,y) \in F_\ell \times A_\ell.$$

For example, $F_\beta = \{0\}*\prod_2^{\beta}\{0,1,\ldots,2\xi_\varphi-1\}$, $A_\beta = \{1\}*\prod_2^{\beta}\{0,1,\ldots,2\xi_\varphi-1\}$ and

$$\Gamma_\beta = \{0,1\} * \prod_2^{\beta}\{0,1,\ldots,2\xi_\varphi-1\}. \tag{27}$$

Suppose, for $\ell \in \mathbb{N}$, Γ_ℓ has been defined. Let F_ℓ and A_ℓ be subsets of Γ_ℓ defined so

$$\#\Gamma_\ell/2 = \#F_\ell = \#A_\ell \text{ and } x < y \text{ for } (x,y) \in F_\ell \times A_\ell.$$

Define $\Gamma_{\ell+1} = F_\ell * \{0, 1, \ldots, 2\xi_{\varphi(\ell+1)} - 1\}$. Once A_ℓ is defined the extensions of its elements are for $\wp > \ell$

$$\Omega_{\ell,\wp} = A_\ell * \{0, 1, \ldots, 2\zeta_{\varphi(\ell+1)} - 1\} * \prod_{\ell+2}^{\wp} \{0, 1, \ldots, \zeta_\varphi - 1\}. \tag{28}$$

Each nonnegative integer can be written uniquely as

$$\ell = \mu + (\psi\beta + 1)\gamma + \sum_{1}^{\psi-1} (i\beta + 1) n_i,$$

where $\psi \in \mathbb{N}$, $\mu \in \{0, 1, \ldots, \psi\beta\}$ and $\gamma \in \{0, 1, \ldots, n_\psi - 1\}$. With this notation for ℓ,

$$\Gamma_\ell = \prod_{1}^{\gamma+\varrho_{\psi-1}} \{0\} * \{0, 1, \ldots, 2\xi_\psi 2^{-\mu} - 1\} * \prod_{\gamma+2+\varrho_{\psi-1}}^{\ell} \{0, 1, \ldots, 2\xi_\varphi - 1\}.$$

Note that $\varphi(1+\gamma+\varrho_{\psi-1}) = \psi$. For $\ell \in \{0, 1, \ldots, \beta\}$, $\prod_1^{\gamma+\varrho_{\psi-1}}\{0\} = \prod_1^0\{0\} = \emptyset$. See (27). The splitting of Γ_ℓ occurs in the $\gamma + 1 + \varrho_{\psi-1}$ set, i. e.,

$$F_\ell = \prod_{1}^{\gamma+\varrho_{\psi-1}} \{0\} * \{0, 1, \ldots, \xi_\psi 2^{-\mu} - 1\} * \prod_{\gamma+2+\varrho_{\psi-1}}^{\ell} \{0, 1, \ldots, 2\xi_\varphi - 1\}$$

and

$$A_\ell = \prod_{1}^{\gamma+\varrho_{\psi-1}} \{0\} * \{\xi_\psi 2^{-\mu}, \ldots, 2\xi_\psi 2^{-\mu} - 1\} * \prod_{\gamma+2+\varrho_{\psi-1}}^{\ell} \{0, 1, \ldots, 2\xi_\varphi - 1\}. \tag{29}$$

If $\mu = \psi\beta$, $\Gamma_{\ell+1} = \prod_1^{\gamma+1+\varrho_{\psi-1}}\{0\} * \prod_{\gamma+2+\varrho_{\psi-1}}^{\ell+1}\{0, 1, \ldots, 2\xi_\varphi - 1\}$. For $\ell \in \{0\} \cup \mathbb{N}$, define $\Omega_\ell = \Omega_{\ell,\ell+1} * \prod_{\ell+2}^{\infty}\{0, 1, \ldots, \zeta_\varphi - 1\}$.

Since the ratio of the cardinality of $\Gamma_{\ell+1}$ to that of Γ_ℓ is $\xi_{\varphi(\ell+1)}$ and $\#\Gamma_1 = \xi_1$, $2\#A_\ell = \prod_1^\ell \xi_\varphi$. We can calculate the cardinality of $\Omega_{\ell,\wp}$ from (28). Recall that $c(\wp)$ is defined in (3).

Proposition 7.1. *For any positive integer $\wp$ and $\ell < \wp$,*

$$\#\Omega_{\ell,\wp} = \prod_{1}^{\ell} \xi_\varphi \cdot \prod_{\ell+1}^{\wp} \zeta_\varphi \text{ and } c(\wp) = \#\left(F_\wp \cup A_\wp \cup \bigcup_{0\le k<\wp} \Omega_{k,\wp}\right).$$

The intervals $\mathcal{I}_\wp$ are defined to be

$$\Big\{K_x;\ x \in F_\wp \cup A_\wp \cup \bigcup_{j\in\{0,1,\ldots,\wp-1\}} \Omega_{j,\wp}\Big\}. \tag{30}$$

Set $Y_\ell = \bigcap_{\wp=\ell+1}^{\infty} \bigcup_{x\in\Omega_{\ell,\wp}} K_x$ which by (29) is a subset of

$$\begin{aligned}
&\bigcup\Big\{K_w;\ w \in \prod_{1}^{\gamma+\varrho_{\psi-1}} \{0\} * \{\xi_\psi 2^{-\mu}, \ldots, 2\xi_\psi 2^{-\mu} - 1\}\Big\} \\
&= \left[\frac{\xi_\psi 2^{-\mu}}{\delta(1+\gamma+\varrho_{\psi-1})}, \frac{\xi_\psi 2^{-\mu+1}}{\delta(1+\gamma+\varrho_{\psi-1})}\right].
\end{aligned}$$

Set $Y = \{0\} \cup \bigcup_{\ell=0}^{\infty} Y_\ell = \bigcap_{\wp=1}^{\infty} \bigcup_{x\in F_\wp \cup A_\wp \cup \bigcup_{0\le j<\wp} \Omega_{j,\wp}} K_x$.

8 The Ratio $\log \#\Omega_{\ell,\wp} / \log \delta(\wp)$

Lemma 8.1. *Suppose $q \in \mathbb{N}$ and $\ell \in \{\varrho_{q-1}, \varrho_{q-1}+1, \ldots, \varrho_q - 1\}$. Then*

$$\{(\log \#\Omega_{\ell,\varrho_k} + m \log \zeta_{k+1}) / \log \delta(m + \varrho_k);\ k \ge q,\ m \in \{0, 1, \ldots, n_{k+1}\}\}$$

has the cluster point x if and only if $x \in [s, t]$.

We show for $k \ge q$ and $m \in \{0, 1, \ldots, n_{k+1}\}$,

$$s - 2/k < (\log \#\Omega_{\ell,m+\varrho_k}) / \log \delta(m + \varrho_k) < t + 3/k. \tag{31}$$

Since $k \ge q$, for $m \in \{0, 1, \ldots, n_{k+1}\}$,

$$\#\Omega_{\ell,m+\varrho_k} = \prod_{1}^{q-1} \xi_i^{n_i} \cdot \prod_{1+\varrho_{q-1}}^{\ell} \xi_q \cdot \prod_{\ell+1}^{\varrho_q} \zeta_q \cdot \prod_{q+1}^{k} \zeta_i^{n_i} \cdot \zeta_{k+1}^{m}.$$

When $m = n_{k+1}$,

$$\frac{\log \#\Omega_{\ell,\varrho_{k+1}}}{\log \delta(\varrho_{k+1})} - \frac{\log \zeta_{k+1}}{\log \tau_{k+1}} = \frac{\log \#\Omega_{\ell,\varrho_k}}{\log \delta(\varrho_{k+1})} - \frac{\log \zeta_{k+1}}{\log \tau_{k+1}} \frac{\log \delta(\varrho_k)}{\log \delta(\varrho_{k+1})}.$$

This vanishes as k goes to infinity. Since (15) holds and $\#\Omega_{\ell,\varrho_k} \le \delta(\varrho_k)$,

$$\left| \frac{\log \#\Omega_{\ell,\varrho_{k+1}}}{\log \delta\left(\varrho_{k+1}\right)} - \frac{\log \zeta_{k+1}}{\log \tau_{k+1}} \right| < \frac{1}{k+1}. \tag{32}$$

For $m \in \{0, 1, \ldots, n_{k+1}\}$, we claim that

$$\frac{\log \#\Omega_{\ell,m+\varrho_k}}{\log \delta(m+\varrho_k)} \ge \frac{\log \zeta_k^{n_k} + \log \zeta_{k+1}^{m}}{\log \delta(m+\varrho_k)} \ge \min\left\{ \frac{\log \zeta_{k+1}^{n_{k+1}}}{\log \delta(\varrho_{k+1})}, \frac{\log \zeta_k^{n_k}}{\log \delta(\varrho_k)} \right\}. \tag{33}$$

In order to get this substitute in (21) $w = \log \zeta_k^{n_k}$, $x = \log \zeta_{k+1}$, $z = \log \delta(\varrho_k)$ and $y = \log \tau_{k+1}$. Under these new variables, (21) still vanishes as $\overset{k\to\infty}{\longrightarrow} 0$. Either

$$\left\{ \frac{\log \zeta_k^{n_k} + m \log \zeta_{k+1}}{\log \delta(\varrho_k) + m \log \tau_{k+1}};\ m \in \{0, 1, \ldots, n_{k+1}\} \right\}$$

is increasing or it is decreasing. Bounds that are lower than the ones in (33) are

$$\min\left\{ \frac{\log \zeta_{k+1}}{\log \tau_{k+1}}, \frac{\log \zeta_k}{\log \tau_k} \right\} - 1/k > s - 2/k. \tag{34}$$

They come from (22) when ζ replaces ξ and then from (6).

For an upper bound,

$$\begin{aligned} \frac{\log \#\Omega_{\ell,m+\varrho_k}}{\log \delta(m+\varrho_k)} \le & \frac{\sum_1^{k-1} n_i \log \xi_i}{\log \varrho_k} \\ & + \max_{m\in\{0,1,\ldots,n_{k+1}\}} \left\{ \frac{\log \zeta_k^{n_k} + m \log \zeta_{k+1}}{\log \delta(\varrho_k) + m \log \tau_{k+1}} \right\}. \end{aligned} \tag{35}$$

Using (15) twice in (35) and then (22) with ζ replacing ξ,

$$\frac{\log \#\Omega_{\ell,m+\varrho_k}}{\log \delta(m+\varrho_k)} < \frac{2}{k} + \max\left\{ \frac{\log \zeta_{k+1}}{\log \tau_{k+1}}, \frac{\log \zeta_k}{\log \tau_k} \right\}.$$

By (7),

$$\frac{\log \#\Omega_{\ell,m+\varrho_k}}{\log \delta(m+\varrho_k)} < t + \frac{3}{k}. \tag{36}$$

As in the calculation of (26), the elements of $\{(\log \#\Omega_{\ell,\wp})/\log \delta(\wp);\ \wp \in \mathbb{N}\}$ which are in $[s,t]$ form a dense subset of $[s,t]$.

By combining (34) and (36), we get

$$\delta(m+\varrho_{k+1})^{-t-3/k} < (\#\Omega_{\ell,m+\varrho_{k+1}})^{-1} < \delta(m+\varrho_{k+1})^{-s+2/k}. \tag{37}$$

By (6) and (32), if $2|k$,

$$\delta(\varrho_{k+1})^{-s-2/(k+1)} < (\#\Omega_{\ell,\varrho_{k+1}})^{-1} < \delta(\varrho_{k+1})^{-s+2/(k+1)}. \tag{38}$$

By (7) and (32), if $2|k+1$,

$$\delta(\varrho_{k+1})^{-t-2/(k+1)} < (\#\Omega_{\ell,\varrho_{k+1}})^{-1} < \delta(\varrho_{k+1})^{-t+2/(k+1)}. \tag{39}$$

9 Verification of the Dimensions

Instead of calculating $(\log N_r(Y))/\log r$ directly $(\log c(\wp))/\log\delta(\wp)$ can be used.

Lemma 9.1. $\sup_{\delta(\wp+1)^{-1}\le r\le\delta(\wp)^{-1}}\left\{\left|\frac{\log c(\wp)}{\log\delta(\wp)}+\frac{\log N_r(Y)}{\log r}\right|\right\}\overset{\wp\to\infty}{\longrightarrow}0.$

For $\ell\in\{0,1,\ldots,\wp-1\}$ and $x\in\Omega_{\ell,\wp}$, the intervals

$$\{K_{x*w};\ w\in\{0,1,\ldots,\zeta_{\varphi(\wp+1)}-1\}\}$$

are contiguous elements of $\mathcal{I}_{\wp+1}$. For $x\in F_\wp\cup A_\wp$, there are either $2\xi_{\varphi(\wp+1)}$ or $2\zeta_{\varphi(\wp+1)}$ contiguous intervals in $\mathcal{I}_{\wp+1}$ that are also in K_x. $N_r(Y)$ is nondecreasing with decreasing r. If $r\in(\delta(\wp+1)^{-1},\delta(\wp)^{-1}]$,

$$c(\wp)\le N_{\delta(\wp)^{-1}}(Y)\le N_r(Y)\le N_{\delta(\wp+1)^{-1}}(Y)\le c(\wp)(2\xi_{\varphi(\wp+1)}+2).$$

We get lower and upper bounds,

$$\frac{\log c(\wp)}{\log\delta(\wp+1)}\le\frac{\log N_r(Y)}{-\log r}\le\frac{\log c(\wp)+\log(2+\tau_{\varphi(\wp+1)})}{\log\delta(\wp)}$$

and $|-(\log N_r(Y))/\log r-(\log c(\wp))/\log\delta(\wp)|$

$$\le\max\left\{\frac{\log(2+\tau_{\varphi(\wp+1)})}{\log\delta(\wp)},\frac{\log c(\wp)}{\log\delta(\wp+1)}\frac{\log\tau_{\varphi(\wp+1)}}{\log\delta(\wp)}\right\}. \tag{40}$$

The first element in (40) vanishes with increasing $\wp$ due to (13). The second vanishes due to (4), (13) and the fact that $\xi_j<\tau_j$.

Theorem 9.1. *$u=\underline{\dim}_BY$ and $v=\overline{\dim}_BY$.*

By Theorem 6.2, the cluster points of $\{(\log c(\wp))/\log\delta(\wp);\ \wp\in\mathbb{N}\}$ make up the interval $[u,v]$. The inequalities in (24) and (25), show

$$\liminf\frac{\log c(\varrho_{k+1})}{\log\delta(\varrho_{k+1})}=u\text{ and }\limsup\frac{\log c(\varrho_{k+1})}{\log\delta(\varrho_{k+1})}=v.$$

Theorem 9.2. *Suppose* $q \in \mathbb{N}$ *and* $\ell \in \{\varrho_{q-1}, \varrho_{q-1}+1, \ldots, \varrho_q - 1\}$. *Then* $\underline{\dim}_B Y_\ell = s$ *and* $\overline{\dim}_B Y_\ell = t$.

For $\wp \geq \varrho_q$ and $r \in (\delta(\wp+1)^{-1}, \delta(\wp)^{-1}]$, we have $\#\Omega_{\ell,\wp} \leq N_r(Y_\ell) \leq \#\Omega_{\ell,\wp} \cdot (2\zeta_{\varphi(\wp+1)} + 2)$ and

$$\frac{\log \#\Omega_{\ell,\wp}}{\log \delta(\wp+1)} \leq \frac{\log N_r(Y_\ell)}{-\log r} \leq \frac{\log \#\Omega_{\ell,\wp} + \log(2\zeta_{\varphi(\wp+1)} + 2)}{\log \delta(\wp)}.$$

As in Lemma 9.1,

$$\sup_{\delta(\wp+1)^{-1} \leq r \leq \delta(\wp)^{-1}} \{|(\log \#\Omega_{\ell,\wp}) / \log \delta(\wp) + (\log N_r(Y_\ell)) / \log r|\} \overset{\wp \to \infty}{\longrightarrow} 0.$$

By (31), both $\underline{\dim}_B Y_\ell$ and $\overline{\dim}_B Y_\ell$ must be in $[s,t]$. The inequalities in (38) and (39) confirm that the lower box dimension must be s and the upper box dimension must be t.

9.1 Hausdorff Dimensions

The product topology for $\prod_1^\infty \{0, 1, \ldots, \tau_\varphi - 1\}$ has as a basis the sets $\{\Lambda_w;\ w \in \bigcup_{\wp=1}^\infty \prod_1^\wp \{0, 1, \ldots, \tau_\varphi - 1\}\}$. In this topology, Ω_ℓ is compact.

Lemma 9.2. *If* $\mathcal{T} \subseteq \bigcup_{\wp=1}^\infty \Omega_{\ell,\wp}$ *and* $\{\Lambda_w;\ w \in \mathcal{T}\}$ *covers* Ω_ℓ,

$$1 \leq \sum_{w \in \mathcal{T}} (\#\Omega_{\ell,\wp})^{-1}. \tag{41}$$

For distinct w and x, $\Lambda_w \cap \Lambda_x \neq \emptyset \Rightarrow \Lambda_w \subset \Lambda_x$ or $\Lambda_x \subset \Lambda_w$; i. e., either x is an extension of w or w is an extension of x. For $x \in \Omega_\ell$, $\{\Lambda_w;\ w \in \mathcal{T},\ x \in \Lambda_w\}$ has a minimal element $\Lambda_{f(x)}$ in the sense that $f(x) \in \mathcal{T}$, $x \in \Lambda_{f(x)}$ and

$$\bigcup_{w \in \mathcal{T},\ x \in \Lambda_w} \Lambda_w = \Lambda_{f(x)}.$$

The set $\{\Lambda_{f(x)};\ x \in \Omega_\ell\}$ must be a finite cover of Ω_ℓ and its elements are pairwise disjoint.

Theorem 9.3. *If* $q \in \mathbb{N}$ *and* $\ell \in \{\varrho_{q-1}, \varrho_{q-1}+1, \ldots, \varrho_q - 1\}$, $s = \dim_{\mathcal{H}} Y_\ell$.

Suppose $s/2 > \varepsilon > 0$ and $k_0 = \max\{1 + [2v/\varepsilon u], q, 2 + [2/\varepsilon]\}$. If $k \geq k_0$, $\varepsilon > 2/(k-1)$. For such k, convert part of (37) into

$$(\#\Omega_{\ell, m+\varrho_k})^{-1} < \delta(m + \varrho_k)^{-s+\varepsilon}. \tag{42}$$

Pick $h \leq \delta(\varrho_{k_0})^{-1}$. Let $\mathcal{R}$ be a cover of Y_ℓ with the diameter of each $E \in \mathcal{R}$ less than h. In order to find a lower bound for $\mathcal{H}^{s-2\varepsilon}Y_\ell$, we look for a lower bound for $(\mathrm{diam}E)^{s-2\varepsilon}$. As a way of approximating the diameter of E find the positive integer $g(E)$ for which $\delta(g(E)+1)^{-1} < \mathrm{diam}\ E \leq \delta(g(E))^{-1}$. Associated with E is a cover of the sequences $Q^{-1}(E)$. Let

$$\Psi_E = \{w \in \Omega_{\ell,g(E)+1};\ Q(\Lambda_w) \cap E \neq \emptyset\}.$$

The intervals $\{Q(\Lambda_w);\ w \in \Psi_E\}$ cover E and $\Omega_\ell = \bigcup_{E\in\mathcal{R}} \bigcup_{w\in\Psi_E} \Lambda_w$. Recall that $2\xi_j < \tau_j$ for any j. There is at least one, but at most three intervals in

$$\{Q(\Lambda_w);\ w \in \Omega_{\ell,g(E)},\ Q(\Lambda_w) \cap E \neq \emptyset\}.$$

If there is only one, $\#\Psi_E \leq 2\zeta_{\varphi(g(E)+1)}$. If there are at least two, $\#\Psi_E \leq 2 + 2\zeta_{\varphi(g(E)+1)}$. This forces the first inequality in

$$\#\Psi_E/2 \leq 1 + \tau_{\varphi(g(E)+1)}/2 < \tau_{\varphi(g(E)+1)} < \delta(g(E))^{\varepsilon}. \tag{43}$$

The second comes from $\tau_1 > 2^{2/(1-u)} > 2$. There is a positive integer k so that $g(E) \geq \varrho_k \geq \varrho_{k_0}$. Since $2v/uk < \varepsilon$, inequality (13) forces the last inequality in (43). We have

$$(\mathrm{diam}\ E)^{s-2\varepsilon} > \delta(g(E)+1)^{-s+\varepsilon}\delta(g(E)+1)^{\varepsilon} > \delta(g(E)+1)^{-s+\varepsilon}\#\Psi_E/2.$$

Finally $\dim_{\mathcal{H}} Y_\ell \geq s$, because using (42),

$$2\mathcal{H}^{s-2\varepsilon}Y_\ell \geq \inf\{\sum_{E\in\mathcal{R}} \sum_{w\in\Psi_E} (\#\Omega_{\ell,g(E)+1})^{-1};\ Y_\ell \subseteq \cup\mathcal{R},\ \mathrm{diam}\ E \leq \delta(\varrho_{k_0})^{-1}\} \geq 1.$$

On the other hand $\dim_{\mathcal{H}} Y_\ell \leq \underline{\dim}_B Y_\ell \leq s$.

9.2 Packing Dimensions

The following helps define some packings. Given $k \in \mathbb{N}$ and $m \in \{0,1\}$, let

$$\mathcal{A}_{k,m} = \{w \in \Omega_{\ell,\varrho_k};\ 2|(w(\varrho_k)+m)\}.$$

If x and w are distinct sequences in $\mathcal{A}_{k,m}$ and if $x' \in \Lambda_x$ and $w' \in \Lambda_w$, then

$$\emptyset = B_{\delta(\varrho_k)^{-1}/4}(Q(w')) \cap B_{\delta(\varrho_k)^{-1}/4}(Q(x')).$$

Let $\mathcal{A}_{k,m}|_E = \{w \in \mathcal{A}_{k,m};\ Q(\Lambda_w) \cap E \neq \emptyset\}$.

Theorem 9.4. *If* $q \in \mathbb{N}$ *and* $\ell \in \{\varrho_{q-1}, \varrho_{q-1}+1, \ldots, \varrho_q - 1\}$, $t = \dim_{\mathcal{P}} Y_\ell$.

Under the assumption $t - \dim_{\mathcal{P}} Y_\ell > \varepsilon > 0$, $\mathcal{P}^{t-\varepsilon} Y_\ell = 0$ there must be a countable set $\{J_i;\ i \in \mathcal{M}\}$ for which $Y_\ell = \bigcup_{i \in \mathcal{M}} J_i$ and $\sum_{i \in \mathcal{M}} P_0^{t-\varepsilon} J_i < 2^{t-\varepsilon-2}$. Let $d : \mathcal{M} \to \mathbb{N}$ be such that $2 | d(i)$ and $d(i) > \max\{1 + [2/\varepsilon], q\}$. By (39), $(\#\Omega_{\ell, \varrho_{d(i)}})^{-1} < \delta(\varrho_{d(i)})^{-t+2/d(i)} < \delta(\varrho_{d(i)})^{-t+\varepsilon}$. For $w \in \mathcal{A}_{d(i),0}|_{J_i}$, let I_w be a closed ball with radius $\delta(\varrho_{d(i)})^{-1}/4$ and center in $Q(\Lambda_w) \cap J_i$. Then $\left\{I_w;\ w \in \mathcal{A}_{d(i),0}|_{J_i}\right\}$ is a $\delta(\varrho_{d(i)})^{-1}/2$-packing of J_i. The same holds for $\mathcal{A}_{d(i),1}|_{J_i}$ and

$$2P^{t-\varepsilon}_{\delta(\varrho_{d(i)})^{-1}/2} J_i \geq (\#\mathcal{A}_{d(i),0}|_{J_i} + \#\mathcal{A}_{d(i),1}|_{J_i})(\delta(\varrho_{d(i)})/2)^{-t+\varepsilon}.$$

A cover of Ω_ℓ is $\bigcup_{i \in \mathcal{M}} \{\Lambda_w;\ w \in \mathcal{A}_{d(i),0}|_{J_i} \cup \mathcal{A}_{d(i),1}|_{J_i}\}$, (41) holds and

$$1 \leq \sum_{\mathcal{M}} \#(\mathcal{A}_{d(i),0}|_{J_i} \cup \mathcal{A}_{d(i),1}|_{J_i})(\#\Omega_{\ell, \varrho_{d(i)}})^{-1}.$$

Each $d(i)$ is arbitrarily small and it can be assumed that

$$\sum_{i \in \mathcal{M}} P^{t-\varepsilon}_{\delta(\varrho_{d(i)})^{-1}/2} J_i < 2^{t-\varepsilon-2}.$$

This creates a contradiction. The packing dimension of Y_ℓ is not less than t.

Finally we claim that if $\varepsilon > 0$, $\mathcal{P}^{t+\varepsilon} Y_\ell = 0$. Since Y_ℓ is a cover of itself, $P_0^{t+\varepsilon} Y_\ell \geq \mathcal{P}^{t+\varepsilon} Y_\ell$. By combining Proposition 4.2 and Theorem 9.2, we have $P_0^{t+\varepsilon} Y_\ell = 0$.

References

[1] G. A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.

[2] K. J. Falconer, *Fractal Geometry*, Wiley, New York, 1990.

[3] X. Saint Raymond and C. Tricot, *Packing regularity of sets in n-space*, Math. Proc. Camb. Phil. Soc. **103** (1988), 133–145.

[4] S. J. Taylor and C. Tricot, *Packing measure, and its evaluation for a brownian path*, Trans. Amer. Math. Soc. **288** (1985), 679–699.

[5] C. Tricot, *Two definitions of fractional dimensions*, Math. Proc. Camb. Phil. Soc. **91** (1982), 57–74.