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INTEGRATION BY PARTS AND OTHER THEOREMS FOR R³S-INTEGRALS

Abstract

This paper is a continuation of [3], in which was introduced the Refinement-Ross-Riemann-Stieltjes (R^3S) Integral, and in which some of its advantages were exhibited. After a brief summary of [3], this paper proves an integration by parts theorem which shows incidentally that if f is R^3S-integrable with respect to g then g is R^3S-integrable with respect to f. Theorems on term-by-term integration of sequences analogous to the Helly-Bray Theorem are next proved, in a context of Wiener's functions of bounded generalized variation as developed by L. C. Young and me. In a similar context I prove also a theorem resembling the classical theorem of Riesz representing linear functionals by Stieltjes integrals.

10 Introduction

The Refinement-Ross-Riemann-Stieltjes (R^3S) Integral was introduced in [3], and some of its fundamental properties were established there. Its definition is repeated in §11. It extends the Ross-Riemann-Stieltjes (R^2S) Integral [2, 6, 7] which succeeded in overcoming, in an elementary way, some disadvantages of the classical Riemann-Stieltjes (RS) Integral, notably its failure to exist when the integrand and the integrator functions have a common point of discontinuity.

This paper is a continuation of [3]. The numbering of new theorems and lemmas is from 23 to 34, following on the numbering in [3]. Similarly the numbering of new formulae is from (31) to (56), and of sections from 10 to 15. This perversion is intended to facilitate reference to appropriate places in [3];

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however is is hoped that the summary provided in §11 will minimize the need for such reference.

The R^2S-integral in [6, 7] is confined to increasing integrators, but is simply extended in [2, 3] to integrators of bounded variation. The R^3S-integral is shown in [3] to be a further extension, in which the integrator may have unbounded variation at the expense of heavier restriction on the integrand (but less heavy than bounded variation). The R^3S-integral also possesses a certain symmetry between integrator and integrand; this shows up prominently in §12 on integration by parts, particularly in Theorem 25.

In §13 on R^3S-integration of sequences, Theorem 28 resembles the Helly-Bray Theorem [8, p. 31], but in a more general context and Theorem 29 goes further in that direction.

Theorem 34 in §15 is an analogue of the famous theorem of F. Riesz representing a linear functional by a Stieltjes integral.

11 Background

In [3] it is evident that a major stimulus for studying the R^3S-integral is the following existence theorem for non-absolutely convergent integrals, stated and proved in [3].

Theorem 21. *If $p^{-1}+q^{-1}>1$, $p\geq 1$, $q\geq 1$, $f\in W_p$ and $g\in W_q$, then the R^3S-integral of f with respect to g exists.*

Here W_p, named Wiener [9], is the class of complex-valued functions f on a compact interval $[a,b]$ whose pth power variation $V_p(f)=V_p(f;a,b)$ is finite; where

$$V_p(f;a,b)=\sup\Big(\sum_{n=1}^{l}\big|f(x_n)-f(x_{n-1})\big|^p\Big)^{\frac{1}{p}}, \tag{14}$$

the upper bound being taken for all partitions $a=x_0<x_1<\ldots<x_l=b$. W_1 is the ordinary class of functions of bounded variation; as p increases W_p expands, and the expansion is proper.

The earliest version of this theorem was due to Young and Love, in [10]; it was somewhat hampered by working with the classical *RS*-integral, which fails to exist if the two functions involved have a common discontinuity.

For $p\geq 1$ all functions in W_p are bounded and simply discontinuous; that is, they have simple discontinuities only.

Lemma 13. *If $p\geq 1$, f and g are complex-valued functions on $[a,b]$, and k is a complex constant, then*

$$V_p(f+g)\leq V_p(f)+V_p(g) \quad \textit{and} \quad V_p(kf)=|k|V_p(f)\,.$$

Lemma 14. *If $q \geq p \geq 1$ and*

$$V_\infty(f; a, b) = \sup\big\{|f(x) - f(y)| \,:\, a \leq x < y \leq b\big\}\,,$$

then

$$V_\infty(f) \leq V_q(f) \leq V_p(f)\,.$$

Definition of the $\mathbf{R^3}$S-integral. (repeated from [3]) is as follows.

Let g be a *simply discontinuous* complex valued function on a compact interval $[a, b]$. Let P be a *partition* $a = x_0 < x_1 < \ldots < x_l = b$, and let P^* be P together with any *associated points* ξ_n such that $x_{n-1} < \xi_n < x_n$ for $n = 1, 2, \ldots, l$. Let

$$\left.\begin{aligned} \Delta_n = g(x_n-) - g(x_{n-1}+)\,, \quad \delta_n = g(x_n+) - g(x_n-)\,,\\ g(a-) = g(a)\,, \quad g(b+) = g(b)\,. \end{aligned}\right\} \tag{7}$$

For a complex-valued function f on $[a, b]$ define an *approximative sum*

$$S(P^*) = S(f, g, P^*) = \sum_{n=1}^{l} f(\xi_n)\Delta_n + \sum_{n=0}^{l} f(x_n)\delta_n\,. \tag{8}$$

The last summation may be called the *jump sum*.

Suppose that there is a complex number I with the property that, for each $\epsilon > 0$ there is a partition $P(\epsilon)$ such that

$$\big|S(P^*) - I\big| < \epsilon \quad \text{whenever} \quad P \supset P(\epsilon)\,, \tag{9}$$

that is, whenever P is a *refinement* of $P(\epsilon)$ and P^* is associated with P. It is easily seen that I is unique; I is then called the R^3S*-integral* of f with respect to g on $[a, b]$,

$$I = (R^3S)\int_a^b f\,dg\,, \tag{10}$$

and f is said to be R^3S*-integrable* with respect to g on $[a, b]$, or briefly, $f \in R^3S(g)$.

Certain other approximative sums are useful when f, as well as g, is simply discontinuous. These are

$$\left.\begin{aligned} S(P^+) &= \sum_{n=1}^{l} f(x_{n-1}+)\Delta_n + \sum_{n=0}^{l} f(x_n)\delta_n\,,\\ S(P^-) &= \sum_{n=1}^{l} f(x_n-)\Delta_n + \sum_{n=0}^{l} f(x_n)\delta_n\,. \end{aligned}\right\} \tag{20}$$

12 R^3S-Integration by Parts

Substantial leads towards this were given by Young [10], Hewitt [1] and Ross [6, 7]. The latter two consider only functions of bounded variation, indeed mostly increasing functions. As might be expected from Theorem 21, integration by parts extends to a wide range of functions of unbounded variation.

A function f is said to be *normalized* if, for all x concerned, it is simply discontinuous and $f(x) = \frac{1}{2}\{f(x+) + f(x-)\}$.

Lemma 23. *Let f and g be simply discontinuous in $[a, b]$, let f be normalized in (a, b) and let E be a dense subset of (a, b). In order that f should be R^3S-integrable with respect to g on $[a, b]$, with integral I, it is necessary and sufficient that for each $\epsilon > 0$ there be a partition $P(\epsilon)$ such that $|I - S(P^*)| < \epsilon$ whenever $P \supset P(\epsilon)$ and every ξ_n in P^* is in E.*

Proof. The necessity is obvious. For the sufficiency, suppose that the condition holds. Let $J(P)$ denote the jump sum on P; that is, the last summation in (8). Suppose that the ξ_n are restricted only by the requirement that $x_{n-1} < \xi_n < x_n$, as in (8). Then there are sequences

$$\{s_{n,r}\}_{r=1}^{\infty} \subset E \cap (x_{n-1}, x_n) \quad \text{and} \quad \{t_{n,r}\}_{r=1}^{\infty} \subset E \cap (x_{n-1}, x_n)$$

such that $s_{n,r} \uparrow \xi_n$ and $t_{n,r} \downarrow \xi_n$ as $r \to \infty$. For P^* with these ξ_n as the associated points,

$$\begin{aligned} S(P^*) &= \sum_{n=1}^{l} f(\xi_n)\Delta_n + J(P) = \tfrac{1}{2}\sum_{n=1}^{l}\{f(\xi_n-) + f(\xi_n+)\}\Delta_n + J(P) \\ &= \tfrac{1}{2}\lim_{r\to\infty}\sum_{n=1}^{l} f(s_{n,r})\Delta_n + \tfrac{1}{2}\lim_{r\to\infty}\sum_{n=1}^{l} f(t_{n,r})\Delta_n + J(P)\,, \end{aligned}$$

$$\begin{aligned} I - S(P^*) =& \tfrac{1}{2}\lim_{r\to\infty}\Big\{I - \sum_{n=1}^{l} f(s_{n,r})\Delta_n - J(P)\Big\} \\ &+ \tfrac{1}{2}\lim_{r\to\infty}\Big\{I - \sum_{n=1}^{l} f(t_{n,r})\Delta_n - J(P)\Big\}\,; \end{aligned}$$

so that $|I - S(P^*)| \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$ whenever $P \supset P(\epsilon)$. This proves Lemma 23. □

Lemma 24. *If f and g are simply discontinuous in $[a, b]$ and normalized in (a, b), P is any partition of $[a, b]$, $S(P^{\pm})$ are the sums defined in (20) and $T(P^{\pm})$ are the results of interchanging f and g in $S(P^{\pm})$, then*

$$\tfrac{1}{2}\{S(P^+) + S(P^-)\} + \tfrac{1}{2}\{T(P^+) + T(P^-)\} = B - A$$

where A and B are as in (33) *(in Theorem 25 below).*

PROOF. Let $J(P)$ be the jump sum for f and g on P; that is, the last summation in (8) and (20). Observing (7),

$$\begin{aligned} J(P) &= f(a)\{g(a+)-g(a)\} + \sum_{n=1}^{l-1} f(x_n)\{g(x_n+)-g(x_n-)\} \\ &\quad + f(b)\{g(b)-g(b-)\} \\ &= f(a-)\{g(a+)-g(a-)\} + f(b+)\{g(b+)-g(b-)\} \\ &\quad + \tfrac{1}{2}\sum_{n=1}^{l-1}\{f(x_n+)+f(x_n-)\}\{g(x_n+)-g(x_n-)\} \\ &= -\tfrac{1}{2}\{f(a+)-f(a-)\}\{g(a+)-g(a-)\} \\ &\quad + \tfrac{1}{2}\{f(b+)-f(b-)\}\{g(b+)-g(b-)\} \\ &\quad + \tfrac{1}{2}\sum_{n=0}^{l}\{f(x_n+)-f(x_n-)\}\{g(x_n+)-g(x_n-)\}\,. \end{aligned} \tag{31}$$

Let $K(P)$ be the result of interchanging f and g in $J(P)$. Then

$$\begin{aligned} J(P)+K(P) &= -\{f(a+)-f(a-)\}\{g(a+)-g(a-)\} \\ &\quad + \{f(b+)-f(b-)\}\{g(b+)-g(b-)\} \\ &\quad + \sum_{n=0}^{l}\{f(x_n+)g(x_n+)-f(x_n-)g(x_n-)\}\,. \end{aligned} \tag{32}$$

By (20),

$$\tfrac{1}{2}\{S(P^+)+S(P^-)\} = \tfrac{1}{2}\sum_{n=1}^{l}\{f(x_n-)+f(x_{n-1}+)\}\{g(x_n-)-g(x_{n-1}+)\} + J(P)\,,$$

$$\tfrac{1}{2}\{T(P^+)+T(P^-)\} = \tfrac{1}{2}\sum_{n=1}^{l}\{f(x_n-)-f(x_{n-1}+)\}\{g(x_n-)+g(x_{n-1}+)\} + K(P)\,.$$

The sum of these two right sides is, using (32) and (7),

$$\begin{aligned}
&\sum_{n=1}^{l}\big\{f(x_n-)g(x_n-)-f(x_{n-1}+)g(x_{n-1}+)\big\}\\
&+\sum_{n=0}^{l}\big\{f(x_n+)g(x_n+)-f(x_n-)g(x_n-)\big\}\\
&-\big\{f(a+)-f(a-)\big\}\big\{g(a+)-g(a-)\big\}+\big\{f(b+)-f(b-)\big\}\big\{g(b+)-g(b-)\big\}\\
=\;& f(x_l+)g(x_l+)-f(x_0-)g(x_0-)\\
&-\big\{f(a+)-f(a)\big\}\big\{g(a+)-g(a)\big\}+\big\{f(b)-f(b-)\big\}\big\{g(b)-g(b-)\big\}\\
=\;& B-A\,.
\end{aligned}$$

□

Theorem 25. *If f and g are simply discontinuous in $[a,b]$ and normalized in (a,b), and $g\in R^3S(f)$, then $f\in R^3S(g)$ and*

$$\int_a^b f\,dg+\int_a^b g\,df=B-A\,,$$

where $A=f(a)g(a)+\big\{f(a+)-f(a)\big\}\big\{g(a+)-g(a)\big\}$

and $B=f(b)g(b)+\big\{f(b)-f(b-)\big\}\big\{g(b)-g(b-)\big\}\,.$ (33)

Remarks. Observe that f and g are not required to be in Wiener classes. The familiar form of integration by parts, with the right side $B-A$ replaced by $f(b)g(b)-f(a)g(a)$, occurs if one of f and g is continuous at a and one of f and g is continuous at b.

Proof. (i) Let I be the R^3S-integral of g with respect to f on $[a,b]$. Let P and P^* be as in §11. By (9) there is a partition $P(\epsilon)$ such that

$$\big|I-T(P^*)\big|<\epsilon\quad\text{whenever}\quad P\supset P(\epsilon)\,. \tag{34}$$

Here

$$T(P^*)=\sum_{n=1}^{l}g(\xi_n)\big\{f(x_n-)-f(x_{n-1}+)\big\}+K(P) \tag{35}$$

where $x_{n-1}<\xi_n<x_n$ and $K(P)$ is the jump sum $J(P)$ with f and g interchanged, so that $K(P)=\sum_{n=0}^{l}g(x_n)\big\{f(x_n+)-f(x_n-)\big\}$. Making $\xi_n\to x_{n-1}+$, and separately $\xi_n\to x_n-$,

$$T(P^*)\to\sum_{n=1}^{l}g(x_{n-1}+)\big\{f(x_n-)-f(x_{n-1}+)\big\}+K(P)=T(P^+)\,,$$

$$T(P^*) \to \sum_{n=1}^{l} g(x_n-)\big\{f(x_n-) - f(x_{n-1}+)\big\} + K(P) = T(P^-)$$

respectively, in keeping with the notation in (20). By (34), $\big|I - T(P^{\pm})\big| \le \epsilon$ whenever $P \supset P(\epsilon)$, and so

$$\Big|I - \tfrac{1}{2}\big\{T(P^+) + T(P^-)\big\}\Big| \le \epsilon \quad \text{whenever} \quad P \supset P(\epsilon)\,. \tag{36}$$

(ii) Suppose now that l is even. Let $\sum_o$ denote summation over odd n and $\sum_e$ summation over even n. In (35) make $\xi_n \to x_{n-1}+$ for odd n, $\xi_n \to x_n-$ for even n; these give

$$T(P^*) \to \sum_{n=1}^{l}{}_{o}\, g(x_{n-1}+)\big\{f(x_n-) - f(x_{n-1}+)\big\}$$
$$+ \sum_{n=1}^{l}{}_{e}\, g(x_n-)\big\{f(x_n-) - f(x_{n-1}+)\big\} + K(P)\,.$$

Again, in (35) make $\xi_n \to x_n-$ for odd n, $\xi_n \to x_{n-1}+$ for even n. Then

$$T(P^*) \to \sum_{n=1}^{l}{}_{o}\, g(x_n-)\big\{f(x_n-) - f(x_{n-1}+)\big\}$$
$$+ \sum_{n=1}^{l}{}_{e}\, g(x_{n-1}+)\big\{f(x_n-) - f(x_{n-1}+)\big\} + K(P)\,.$$

Subtracting these limits gives, because of (34),

$$\Big|\sum_{n=1}^{l}{}_{o} \big\{f(x_n-) - f(x_{n-1}+)\big\}\big\{g(x_n-) - g(x_{n-1}+)\big\}$$
$$- \sum_{n=1}^{l}{}_{e} \big\{f(x_n-) - f(x_{n-1}+)\big\}\big\{g(x_n-) - g(x_{n-1}+)\big\}\Big| \le 2\epsilon \tag{37}$$

whenever $P \supset P(\epsilon)$.

(iii) Let P and P^* be as in (i); that is, as in §11, and let Q be the partition

$$a = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \ldots < x_{l-1} < \xi_l < x_l = b\,;$$

that is, Q consists of all the points of P^*. Observe that (36) and (37) involve only the x_n, not the ξ_n; this will enable them to be used with P replaced by Q, as will be done shortly.

Let all the ξ_n be points of continuity of f; such points are of course dense since f is simply discontinuous. By (8) and (20)

$$\begin{aligned}
S(P^*) - \tfrac{1}{2}\big\{S(Q^+) + S(Q^-)\big\} \\
&= \sum_{n=1}^{l} f(\xi_n)\big\{g(x_n-) - g(x_{n-1}+)\big\} + J(P) - J(Q) \\
&\quad - \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(x_{n-1}+) + f(\xi_n-)\big\}\big\{g(\xi_n-) - g(x_{n-1}+)\big\} \\
&\quad - \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(\xi_n+) + f(x_n-)\big\}\big\{g(x_n-) - g(\xi_n+)\big\} \\
&= \sum_{n=1}^{l} f(\xi_n)\big\{g(x_n-) - g(x_{n-1}+) - g(\xi_n+) + g(\xi_n-)\big\} \\
&\quad - \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(\xi_n) + f(x_{n-1}+)\big\}\big\{g(\xi_n-) - g(x_{n-1}+)\big\} \\
&\quad - \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(x_n-) + f(\xi_n)\big\}\big\{g(x_n-) - g(\xi_n+)\big\} \\
&= \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(\xi_n) - f(x_{n-1}+)\big\}\big\{g(\xi_n-) - g(x_{n-1}+)\big\} \\
&\quad + \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(\xi_n) - f(x_n-)\big\}\big\{g(x_n-) - g(\xi_n+)\big\} \\
&= \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(\xi_n-) - f(x_{n-1}+)\big\}\big\{g(\xi_n-) - g(x_{n-1}+)\big\} \\
&\quad - \tfrac{1}{2}\sum_{n=1}^{l}\big\{f(x_n-) - f(\xi_n+)\big\}\big\{g(x_n-) - g(\xi_n+)\big\}. \qquad (38)
\end{aligned}$$

(iv) Let A and B be as in (33). Then

$$\begin{aligned}\left|S(P^*) - (B - A - I)\right| &\le \left|S(P^*) - \tfrac{1}{2}\left\{S(Q^+) + S(Q^-)\right\}\right| \\ &+ \left|\tfrac{1}{2}\left\{S(Q^+) + S(Q^-)\right\} + \tfrac{1}{2}\left\{T(Q^+) + T(Q^-)\right\} - (B - A)\right| \\ &+ \left|I - \tfrac{1}{2}\left\{T(Q^+) + T(Q^-)\right\}\right|. \qquad (39)\end{aligned}$$

The middle line on the right of (39) is zero, by Lemma 24 with P replaced by Q, a change which does not affect $B - A$. The last line on the right is, by (36), at most ϵ if $Q \supset P(\epsilon)$ and this is indeed so whenever $P \supset P(\epsilon)$, because then $Q \supset P \supset P(\epsilon)$.

Now Q partitions $[a, b]$ into an even number, $2l$, of subintervals. A suitable change of notation for the points of Q would turn (38) into one half the contents of the modulus signs in (37) and so $\left|S(P^*) - \tfrac{1}{2}\left\{S(Q^+) + S(Q^-)\right\}\right| \le \epsilon$ whenever $Q \supset P(\epsilon)$, and therefore whenever $P \supset P(\epsilon)$. Thus (39) gives

$$\left|S(P^*) - (B - A - I)\right| \le 2\epsilon \quad \text{whenever} \quad P \supset P(\epsilon)\,.$$

This inequality has been obtained under the assumption that the associated points ξ_n are all in the dense set of points of continuity of f. By Lemma 23, f is R^3S-integrable with respect to g, with integral equal to $B - A - I$, completing the proof of Theorem 25. □

13 Limits of R³S-Integrals

In §11 only the part of Theorem 21 relevant there is quoted. The rest of that theorem [3, p. 308] is now needed; it is as follows.

Theorem 21 continued. *The integral satisfies the inequality*

$$\left|\int_a^b f\,dg - C\right| \le 2\zeta(p,q)V_p(f; a+, b-)V_q(g; a+, b-)\,,$$

where $\zeta(p, q)$ is independent of f, g, a and b, and

$$C = f(b)\left\{g(b) - g(b-)\right\} + f(a+)\left\{g(b-) - g(a+)\right\} + f(a)\left\{g(a+) - g(a)\right\}.$$

The inequality also holds with $f(a+)$ (in C) replaced by $f(b-)$.

The following theorem [3, p. 310] is also needed.

Theorem 22. *If g is simply discontinuous in $[a,b]$, $a<c<b$ and either side of the equation*

$$\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg$$

exists in the R^3S-sense, then so does the other side and the equation holds.

This theorem does not require f and g to be in Wiener classes.

Lemma 26. *If $p^{-1}+q^{-1}>1$, $p\geq 1$, $q\geq 1$, $f\in W_p$ and $g\in W_q$ on $[a,b]$, and $\zeta(p,q)$ is the constant in Theorems 17 and 21, then*

$$\Big|\int_a^b \big\{f(x)-f(a+)\big\}\,dg(x)\Big| \leq \big\{2\zeta(p,q)+1\big\}V_p(f;a,b)V_q(g;a,b)$$

and

$$\Big|\int_a^b \big\{f(x)-f(a)\big\}\,dg(x)\Big| \leq \big\{2\zeta(p,q)+2\big\}V_p(f;a,b)V_q(g;a,b)\,.$$

These inequalities also hold with replacement of $f(a+)$ or $f(a)$ on the left by $f(b-)$ or $f(b)$ respectively.

Proof. In Theorem 21 replace $f(x)$ by $f(x)-f(a+)$; then C in that theorem becomes C', say, with

$$|C'| \leq \big|f(a)-f(a+)\big|\,\big|g(a+)-g(a)\big| + \big|f(b)-f(a+)\big|\,\big|g(b)-g(b-)\big|\,.$$

By Jensen's extension of Hölder's Inequality,

$$\begin{aligned}|C'| &\leq \Big\{\big|f(a)-f(a+)\big|^p + \big|f(a+)-f(b)\big|^p\Big\}^{\frac{1}{p}}\Big\{\big|g(a)-g(a+)\big|^q + \big|g(b-)-g(b)\big|^q\Big\}^{\frac{1}{q}}\\ &\leq V_p(f;a,b)V_q(g;a,b)\,.\end{aligned}$$

The first inequality in Lemma 26 now follows easily from "Theorem 21 continued".

The second inequality can be obtained similarly, or deduced from the first by "adding" the trivial inequality

$$\begin{aligned}\Big|\int_a^b \big\{f(a+)-f(a)\big\}\,dg(x)\Big| &= \big|f(a+)-f(a)\big|\,\big|g(b)-g(a)\big|\\ &\leq V_p(f;a,b)V_q(g;a,b)\,.\end{aligned}$$

The other two inequalities are proved in a similar manner. □

Lemma 27. *If $r > p \ge 1$ and $f \in W_p(a,b)$ then, given $\epsilon > 0$, there is a step function s such that $V_r(f - s; a, b) < \epsilon$.*

PROOF. This is the main part of [5, p. 7, Lemma 2]. □

Theorem 28. *Let $p^{-1} + q^{-1} > 1$, $p \ge 1$, $q \ge 1$, $f \in W_p$ and $g_n \in W_q$ on $[a,b]$. Let $V_q(g_n; a, b)$ be a bounded function of the positive integer n, and $g_n(x) \to g(x)$ as $n \to \infty$ for each $x \in [a,b]$. If also $g_n(x\pm) \to g(x\pm)$ at each discontinuity x of f, and at the end-points $x = a$ and b, then*

$$\int_a^b f\,dg_n \to \int_a^b f\,dg \qquad \text{as } n \to \infty\,.$$

PROOF. (i) For any partition $a = x_0 < x_1 < \ldots < x_l = b$, denoted by P,

$$\sum_{i=1}^{l} \big|g(x_i) - g(x_{i-1})\big|^q = \lim_{n\to\infty} \sum_{i=1}^{l} \big|g_n(x_i) - g_n(x_{i-1})\big|^q \le \varlimsup_{n\to\infty} V_q(g_n; a, b)^q\,.$$

So $g \in W_q(a,b)$, and all the R^3S-integrals exist by Theorem 21. Further $V_q(g; a, b) \le \sup V_q(g_n; a, b)$.

(ii) If f is constant throughout (a,b), (7), (8) and (9) show easily that $f \in R^3S(g)$ and

$$\int_a^b f\,dg = f(a)\big\{g(a+) - g(a)\big\} + f(b)\big\{g(b) - g(b-)\big\} + f(b-)\big\{g(b-) - g(a+)\big\}.$$

If f is a step function on $[a,b]$ with discontinuities only at the points of P, the above equation and Theorem 22 give

$$\int_a^b f\,dg = \sum_{i=0}^{l} f(x_i)\big\{g(x_i+) - g(x_i-)\big\} + \sum_{i=1}^{l} f(x_i-)\big\{g(x_i-) - g(x_{i-1}+)\big\}\,, \tag{40}$$

remembering that $g(x_0-) = g(a-) = g(a)$ and $g(x_l+) = g(b+) = g(b)$. Similarly, replacing g by g_n,

$$\int_a^b f\,dg_n = \sum_{i=0}^{l} f(x_i)\big\{g_n(x_i+) - g_n(x_i-)\big\} + \sum_{i=1}^{l} f(x_i-)\big\{g_n(x_i-) - g_n(x_{i-1}+)\big\}\,.$$

Making $n \to \infty$, Theorem 28 is proved for all step functions f.

(iii) Suppose that f is no longer a step function. Fix r such that $1 - q^{-1} < r^{-1} < p^{-1}$. Let $M = 1 + 2\zeta(r,q)$ and $N = 1 + \sup V_q(g_n; a, b)$. Given $\epsilon > 0$, let $\epsilon' = \epsilon/3MN$; then by Lemma 27 there is a step function s such that $V_r(f - s; a, b) < \epsilon'$. Since addition of a constant to s does not alter this inequality, $s(a+)$ may be supposed equal to $f(a+)$.

Let $h = f - s$. Then $h \in W_p \subset W_r$, using Lemma 14 and $h(a+) = 0$. By Lemma 26 with p, f and g replaced by r, h and g_n respectively,

$$\begin{aligned}\Big|\int_a^b h\, dg_n\Big| &= \Big|\int_a^b \big\{h(x) - h(a+)\big\}\, dg_n(x)\Big| \\ &\le \big\{2\zeta(r,q) + 1\big\} V_r(h; a, b) V_q(g_n; a, b) \\ &\le M\epsilon' N = \tfrac{1}{3}\epsilon\,.\end{aligned}$$

By (i) the same inequality holds when g_n is replaced by g. Using linearity,

$$\begin{aligned}\Big|\int_a^b f\, dg_n - \int_a^b f\, dg\Big| &\le \Big|\int_a^b h\, dg_n - \int_a^b h\, dg\Big| + \Big|\int_a^b s\, dg_n - \int_a^b s\, dg\Big| \\ &\le \tfrac{2}{3}\epsilon + \Big|\int_a^b s\, dg_n - \int_a^b s\, dg\Big| < \epsilon\end{aligned}$$

for all n sufficiently large, since by (ii) the theorem holds when f is the step function s. □

Remarks. Theorem 28 resembles the Helly-Bray Theorem [8, p. 31] in which f is continuous and the g_n are of uniformly bounded variation. Theorem 29 (below) is a generalization of Theorem 28. The hypothesis that $g_n(x\pm) \to g(x\pm)$ at the discontinuities of f is essential for both these theorems; this is shown by the following example.

Let $a < c < b$ and $c_n \downarrow c$. Let $f(c) \neq 1$, and

$$\begin{aligned} f(x) &= 0 \quad \text{if } a \le x < c\,, & f(x) &= 1 \quad \text{if } c < x \le b\,,\\ g_n(x) &= 0 \quad \text{if } a \le x < c_n\,, & g_n(x) &= 1 \quad \text{if } c_n \le x \le b\,,\\ g(x) &= 0 \quad \text{if } a \le x \le c\,, & g(x) &= 1 \quad \text{if } c < x \le b\,.\end{aligned}$$

Then $V_q(g_n; a, b) = 1$, and $g_n(x) \to g(x)$ as $n \to \infty$ because

$$g(x) - g_n(x) = \begin{cases} 1 & \text{if } c < x < c_n,\\ 0 & \text{otherwise;}\end{cases}$$

it follows that $g(c+) - g_n(c+) = 1 \neq 0$. The conclusion of Theorem 28 does not hold, because by (7), (8) and (9)

$$\int_a^b f\,dg_n = f(c_n) \to 1 \quad \text{but} \quad \int_a^b f\,dg = f(c) \neq 1\,.$$

Theorem 29. *Let $p^{-1} + q^{-1} > 1$, $p \geq 1$, $q \geq 1$, $f_n \in W_p$ and $g_n \in W_q$ on $[a,b]$. Let $V_p(f_n - f; a, b) \to 0$ as $n \to \infty$ and $V_q(g_n; a, b)$ be bounded. If*

$$\begin{aligned} f_n(x) &\to f(x) && \text{for one } x \in [a,b]\,, \\ g_n(x) &\to g(x) && \text{for all } x \in [a,b]\,, \\ g_n(x\pm) &\to g(x\pm) && \text{at each discontinuity } x \text{ of } f \end{aligned}$$

and also at $x = a$ and b, then

$$\int_a^b f_n\,dg_n \to \int_a^b f\,dg \qquad \text{as } n \to \infty.$$

PROOF. By Lemma 13, $V_p(f) \leq V_p(f - f_n) + V_p(f_n) < \infty$; so $f \in W_p$. Thus the discontinuities of f are simple and enumerable.

The hypotheses imply that $f_n(x) \to f(x)$ for *all* $x \in [a,b]$. For let c be the one value of x at which this happens by hypothesis, and let $h_n(x) = f_n(x) - f(x)$. Then, for each $x \in [a,b]$,

$$\big|h_n(x)\big| \leq \big|h_n(x) - h_n(c)\big| + \big|h_n(c)\big| \leq V_p(h_n; a, b) + \big|h_n(c)\big| \to 0$$

as $n \to \infty$; in particular $h_n(a) \to 0$. Also $h_n \in W_p$.

Using Lemma 26,

$$\begin{aligned} \Big|\int_a^b h_n\,dg_n\Big| &\leq \Big|\int_a^b \big\{h_n(x) - h_n(a)\big\}\,dg_n(x)\Big| + \Big|\int_a^b h_n(a)\,dg_n(x)\Big| \\ &\leq \big\{2\zeta(p,q) + 2\big\}V_p(h_n)V_q(g_n) + \big|h_n(a)\big|V_q(g_n) \to 0\,. \end{aligned}$$

Theorem 28 now gives, as $n \to \infty$

$$\int_a^b f_n\,dg_n = \int_a^b h_n\,dg_n + \int_a^b f\,dg_n \to \int_a^b f\,dg\,,$$

as required. □

14 Preparations for the Representation Theorem

Such a theorem was given in [4, pp. 248 and 255] for linear functionals on a certain subspace of W_p. The subspace excluded all discontinuous (and also some other) functions. Attempts to admit some discontinuous functions were made in [5, pp. 8 and 33]. No proper representation of a linear functional, but only an inequality for it, was achieved in one of these attempts [5, p. 8, Theorem 1] and the other [5, p. 33, Theorem 20] had little more success. The latter attempt expressed the linear functional as a refinement version of the (classical) RS-integral, for a class of (possibly) discontinuous functions in W_p but it too was hampered by the common discontinuity trouble.

In what follows I present such a theorem using the R^3S-integral. It is confined to the subspace W_p^* (defined at (45) in §15) of W_p, just as the theorems mentioned above are but discontinuities are immaterial, and it is a proper representation theorem, not just an inequality. It can be regarded as a converse to Theorem 30 (below), a theorem which is little more than a restatement, made for motivational purposes, of Theorem 21 and Lemma 26.

For $p \geq 1$ and $f \in W_p$ on $[a, b]$, define

$$\|f\| = |f(a)| + V_p(f; a, b)\,; \tag{41}$$

this is known (and easily verified) to be a norm on W_p. Also define, for $a \leq x \leq b$, the Heaviside functions $\overline{x}$ and $\underline{x}$ (different from one used in [5]) as follows.

$$\begin{aligned} \overline{x}(t) &= 1 \quad \text{if} \ \ t \leq x\,, \qquad & \overline{x}(t) &= 0 \quad \text{if} \ \ t > x\,, \\ \underline{x}(t) &= 1 \quad \text{if} \ \ t < x\,, \qquad & \underline{x}(t) &= 0 \quad \text{if} \ \ t \geq x\,. \end{aligned} \tag{42}$$

It follows that $\underline{x}(x) = 0 < 1 = \overline{x}(x)$.

It is convenient here to use the notation of Banach and Riesz for linear functionals, rather than modern notation.

Theorem 30. *If $p^{-1} + q^{-1} > 1$, $p \geq 1$, $q \geq 1$, $f \in W_p$ and $g \in W_q$ on $[a, b]$, then $f \in R^3S(g)$. The R^3S-integral $L(f) = \int_a^b f\,dg$ has the following properties, for fixed g.*

- *L is a bounded linear functional on the space W_p normed by (41).*
- *If, for $\overline{x}$ and $\underline{x}$ as in (42), $\overline{g}$ and $\underline{g}$ are the functions*

  $$\overline{g}(x) = L(\overline{x}) \quad \textit{and} \quad \underline{g}(x) = L(\underline{x})\,,$$

  then $\overline{g}$ and $\underline{g}$ are simply discontinuous in $[a, b]$ and

  $$\overline{g}(x) = \overline{g}(x+) = \underline{g}(x+)\,, \qquad \underline{g}(x) = \underline{g}(x-) = \overline{g}(x-)\,.$$

Proof. (i) Linearity of L has been taken for granted throughout this paper. For boundedness on W_p, Lemma 26 gives, writing K for $2\zeta(p,q)+2$ so that $K>1$,

$$\begin{aligned}\big|L(f)\big| &\le \Big|\int_a^b f(a)\,dg(x)\Big| + KV_p(f;a,b)V_q(g;a,b)\\ &= \big|f(a)\big|\,\big|g(b)-g(a)\big| + KV_p(f;a,b)V_q(g;a,b)\\ &\le KV_q(g;a,b)\big\{|f(a)|+V_p(f;a,b)\big\} = M\|f\|\end{aligned}$$

where M denotes the constant $KV_q(g;a,b)$.

(ii) Since $\overline{x}$ and $\underline{x}$ are step functions, (40) gives, after some algebra,

$$\overline{g}(x) = \int_a^b \overline{x}\,dg = g(x+)-g(a) \quad \text{and} \quad \underline{g}(x) = \int_a^b \underline{x}\,dg = g(x-)-g(a)\,.$$

Let $a<x<y<\frac{1}{2}(x+b)<b$. Then $a<x<y<2y-x<b$ and $g\in W_q(x,b)$; so, keeping x fixed,

$$\big|g(y+)-g(x+)\big| \le V_q(g;x+,2y-x)\to 0 \quad \text{as} \quad y\to x+\ .$$

Hence $g(y+)\to g(x+)$ as $y\to x+$. Thus

$$\overline{g}(x+) = \lim_{y\to x+}\overline{g}(y) = \lim_{y\to x+}\big\{g(y+)-g(a)\big\} = g(x+)-g(a) = \overline{g}(x)$$

which gives one of the required relations. For another,

$$\begin{aligned}\overline{g}(x+)-\underline{g}(x+) &= \lim_{y\to x+}\big\{\overline{g}(y)-\underline{g}(y)\big\} = \lim_{y\to x+}\Big(\int_a^b \overline{y}\,dg - \int_a^b \underline{y}\,dg\Big)\\ &= \lim_{y\to x+}\int_a^b (\overline{y}-\underline{y})\,dg = \lim_{y\to x+}\big\{g(y+)-g(y-)\big\}\,.\end{aligned}$$

If $a<x<y<\frac{1}{2}(x+b)$, then $y<2y-x$ and, as above,

$$\big|g(y+)-g(y-)\big| \le V_q(g;x+,2y-x)\to 0 \quad \text{as} \quad y\to x+\ .$$

Thus $\overline{g}(x+)-\underline{g}(x+)=0$, another of the required relations.

The other two relations can be proved similarly. □

Definitions. Following [5, §12, p. 29], for $p\ge 1$ and $f\in W_p(a,b)$, let

$$V_p^*(f) = V_p^*(f;a,b) = \inf_P\Big(\sum_{n=1}^{l} V_p\big(f;x_{n-1},x_n\big)^p\Big)^{\frac{1}{p}}\,, \tag{43}$$

the lower bound being taken for all partitions P of $[a,b]$. Also let

$$\mathfrak{S}_p(f) = \mathfrak{S}_p(f;a,b) = \\ = \left(\sum_n \left\{ \left|f(s_n+) - f(s_n)\right|^p + \left|f(s_n) - f(s_n-)\right|^p \right\}\right)^{\frac{1}{p}}, \tag{44}$$

where the summation is taken over the enumerable set of discontinuities s_n of f, and $f(a-)$ and $f(b+)$ are understood to be $f(a)$ and $f(b)$ respectively. It is easily shown that $\mathfrak{S}_p(f) \le V_p^*(f) \le V_p(f)$.

Define W_p^* to be the set of functions f in W_p for which

$$\mathfrak{S}_p(f) = V_p^*(f), \tag{45}$$

and let W_{p-} be the union of all W_r with $1 \le r < p$. It is shown in [5, §12, p. 32] that, for $p > 1$,

$$W_{p-} \subset W_p^* \subset W_p. \tag{46}$$

It can also be shown that W_p^* is a closed subspace of W_p, by means of theorems on approximation to functions in W_p^* by step functions [5, Theorems 18 and 19], one of which is quoted below (see Lemma 33).

The definition of $S_p(\mathrm{a};1,k)$ in (17) is also needed. Repeating it here, it is the pth root of

$$S_p(\mathrm{a};1,k)^p = \max \sum_{r=1}^{l} \left| \sum_{i=h(r-1)+1}^{h(r)} a_i \right|^p, \tag{47}$$

where the complex numbers a_i are components of the vector $\mathrm{a} = (a_i)$, and the maximum is taken for all integer sequences

$$0 = h(0) < h(1) < h(2) < \ldots < h(l) = k.$$

This means that the sequence $a_1, a_2, \ldots, a_k$ is partitioned into sums in all possible ways preserving order, and then the pth powers of moduli of the sums added together.

A minor extension of (47) is to define $S_p(\mathrm{a};2,k)$ in the same way except that the value of $h(0)$ is changed from 0 to 1. It is evident that

$$S_p(\mathrm{a};2,k) \le S_p(\mathrm{a};1,k). \tag{48}$$

Lemma 31. *If $p^{-1} + q^{-1} = 1$, $p > 1$, $q > 1$, $A > 0$, $B > 0$, k is a positive integer and* b *is a complex vector (b_i) such that (see (47))*

$$S_q(\mathrm{b}) = S_q(\mathrm{b};1,k) \ge 3,$$

then there is a vector $\mathrm{a} = (a_i)$ *(real if* b *is real) such that*

$$S_p(\mathrm{a}; 1, k) \le A \quad \textit{and} \quad \Big| \sum_{0<i\le j\le k} a_i b_j \Big| \ge AB/2^{1+1/q}\,.$$

This is the lemma in [4, p. 249]. It is obviously analogous to a form of the classical "converse of Hölder's inequality" and it is, in a similar sense, a converse of Theorem 17. It has some minor extensions, not needed here, which allow $p^{-1} + q^{-1} < 1$ and omission of the modulus signs in the final inequality. In the sixth line of [4, p. 249], a_K should be a_N.

Lemma 32. *If* a_n *are complex,* $a_1 + a_2 + \ldots + a_m + a_{m+1} = 0$, $\mathrm{a} = (a_n)$, $p > 0$ *and* S_p *is defined as in* (47) *and* (48)*, then*

$$S_p(\mathrm{a}; 2, m+1) \le 2^{1/p} S_p(\mathrm{a}; 1, m)\,.$$

PROOF. By definition $S_p(\mathrm{a}; 2, m+1)^p$ is the greatest of the sums

$$\begin{gathered} T_m + |a_{m+1}|^p\,,\ T_{m-1} + \big|a_m + a_{m+1}\big|^p\,,\ T_{m-2} + \big|a_{m-1} + a_m + a_{m+1}\big|^p\,, \\ \ldots,\ T_2 + \big|a_3 + a_4 + \ldots + a_{m+1}\big|^p\,,\ \big|a_2 + a_3 + \ldots + a_{m+1}\big|^p\,, \end{gathered} \tag{49}$$

where T_n stands for any sum like the inner sums in (47) but restricted to $a_2, a_3, \ldots, a_n$. In (49) all the sums which involve a_{m+1} have been separated out and written explicitly. Now for $2 \le n \le m$

$$T_n \le T_m \le S_p(\mathrm{a}; 2, m)^p \le S_p(\mathrm{a}; 1, m)^p$$

and consequently $S_p(\mathrm{a}; 2, m+1)^p$ is at most equal to

$$\begin{aligned} & S_p(\mathrm{a}; 1, m)^p + \max\Big\{ \big|a_{m+1}\big|^p,\ \big|a_m + a_{m+1}\big|^p,\ \big|a_{m-1} + a_m + a_{m+1}\big|^p, \\ & \qquad\qquad \ldots,\ \big|a_3 + \ldots + a_{m+1}\big|^p,\ \big|a_2 + a_3 + \ldots + a_{m+1}\big|^p \Big\} \\ &= S_p(\mathrm{a}; 1, m)^p + \max\Big\{ \big|a_1 + a_2 + \ldots + a_m\big|^p,\ \big|a_1 + a_2 + \ldots + a_{m-1}\big|^p, \\ & \qquad\qquad \big|a_1 + a_2 + \ldots + a_{m-2}\big|^p,\ \ldots,\ \big|a_1 + a_2\big|^p,\ \big|a_1\big|^p \Big\} \\ &\le S_p(\mathrm{a}; 1, m)^p \quad + \quad S_p(\mathrm{a}; 1, m)^p\,. \end{aligned}$$

The stated inequality follows from this. □

Lemma 33. *If $p>1$ and $f\in W_p^*$ then, given $\epsilon>0$ there is a step function s such that*

$$V_p(f-s)<\epsilon\,. \tag{50}$$

Further, there is a partition $P(\epsilon)$ such that, for all $P\supset P(\epsilon)$ and all P^* associated with P as in §11, the step function s given by

$$s(x_n)=f(x_n)\quad\text{and}\quad s(x)=f(\xi_n)\quad\text{for}\quad x_{n-1}<x<x_n\,,$$

satisfies (50)

This is [5, p. 30, Theorem 18] with one difference, that the associated points ξ_n of P^* are confined to the *open* interval (x_{n-1},x_n); this simply omits a little of what is proved in [5]. There is a converse [5, p. 32, Theorem 19], but it is not needed here.

15 A Riesz-type Representation Theorem

Theorem 34. *If $p^{-1}+q^{-1}=1$, $p>1$, L is a bounded linear functional on $W_p^*(a,b)$ normed as in* (41), *$\overline{x}$ and $\underline{x}$ are the Heaviside functions in* (42), *and the functions $\overline{g}$ and $\underline{g}$ defined by*

$$\overline{g}(x)=L(\overline{x})\qquad\text{and}\qquad\underline{g}(x)=L(\underline{x}) \tag{51}$$

satisfy the equations

$$\left.\begin{aligned}&\overline{g}(x)=\tfrac12\{\overline{g}(x+)+\underline{g}(x+)\}\qquad\text{for}\ \ a\le x<b\\ \text{and}\qquad&\\ &\underline{g}(x)=\tfrac12\{\overline{g}(x-)+\underline{g}(x-)\}\qquad\text{for}\ \ a<x\le b\,,\end{aligned}\right\} \tag{52}$$

then there is $g\in W_q$ such that $L(f)=(R^3S)\int_a^b f\,dg$ for all $f\in W_p^*$.

Proof. I prove first that $\overline{g}$ and $\underline{g}$ are in W_q. This ensures that they are simply discontinuous, so that equations (52) have meaning.

(i) Suppose that $\overline{g}\notin W_q$. Let M be a bound of L on W_p^*. There is a partition P such that

$$\sum_{n=1}^{l}\left|\overline{g}(x_n)-\overline{g}(x_{n-1})\right|^q>2^{3q+1}M^q\,.$$

Taking b_n, A and B in Lemma 31 as $\overline{g}(x_n)-\overline{g}(x_{n-1})$, 1 and $2^{3+1/q}M$ respectively, there is $\mathrm{a}=(a_n)$ such that $S_p(\mathrm{a};1,l)\le 1$ and

$$\left|\sum_{0<i\le j\le l}a_i\{\overline{g}(x_j)-\overline{g}(x_{j-1})\}\right|\ge 4M\,.$$

Writing $A_j = \sum_{i=1}^{j} a_i$ this gives, since each $\overline{x}_j \in W_1 \subset W_p^*$ by (46),

$$\begin{aligned} 4M &\le \Big| \sum_{j=1}^{l} A_j \big\{ \overline{g}(x_j) - \overline{g}(x_{j-1}) \big\} \Big| \\ &= \Big| L\Big(\sum_{j=1}^{l} A_j (\overline{x}_j - \overline{x}_{j-1}) \Big) \Big| \le M \Big\| \sum_{j=1}^{l} A_j \big(\overline{x}_j - \overline{x}_{j-1} \big) \Big\| ; \end{aligned} \tag{53}$$

this leads to the contradiction

$$4 \le V_p \Big(\sum_{j=1}^{l} A_j \big(\overline{x}_j - \overline{x}_{j-1} \big) \Big) = S_p(\mathrm{a}; 1, l) \le 1 \,. \tag{54}$$

The middle equality in (54) holds because $\overline{x}_j - \overline{x}_{j-1}$ is the characteristic function of $(x_{j-1}, x_j]$, so that the linear combination of these in (53) is the step function which has jumps a_j at x_{j-1} for $j = 1, 2, \ldots, l$. The contradiction (54) proves that $\overline{g} \in W_q$.

(ii) A proof that $\underline{g} \in W_q$ is exactly like (i) as far as (53), the function occurring in (53) being replaced, on account of (51), by

$$\sum_{j=1}^{l} A_j \big(\underline{x}_j - \underline{x}_{j-1} \big) \,.$$

Now $\underline{x}_j - \underline{x}_{j-1}$ is the characteristic function of $[x_{j-1}, x_j)$, so this linear combination is the step function with jumps a_2 at x_1, a_3 at x_2, $\ldots$, a_j at x_{j-1}. It has no jump at x_0, but as the other end-point x_l is approached it jumps from A_l to 0. Denoting this last jump $-A_l$ by a_{l+1},

$$a_1 + a_2 + \ldots + a_l + a_{l+1} = 0 \,.$$

Instead of (53) and (54) I now have, using (41) and Lemma 32,

$$\begin{aligned} 4 &\le |a_1| + V_p \Big(\sum_{j=1}^{l} A_j \big(\underline{x}_j - \underline{x}_{j-1} \big) \Big) \\ &= |a_1| + S_p(\mathrm{a}; 2, l+1) \\ &\le |a_1| + 2^{1/p} S_p(\mathrm{a}; 1, l) \\ &\le \Big(1 + 2^{1/p} \Big) S_p(\mathrm{a}; 1, l) \\ &\le 2^{1/p} + 2^{1/p} = 2^{1+1/p} < 4 \,. \end{aligned}$$

This contradiction shows that $\underline{g} \in W_q$.

(iii) Given $f \in W_p^*$ and $\epsilon > 0$, Lemma 33 provides a partition $P(\epsilon/M)$ such that $V_p(f-s) < \epsilon/M$ whenever s is the step function

$$s = \sum_{n=0}^{l} f(x_n)\big(\overline{x}_n - \underline{x}_n\big) + \sum_{n=1}^{l} f(\xi_n)\big(\underline{x}_n - \overline{x}_{n-1}\big)\,,$$

$x_{n-1} < \xi_n < x_n$ for each n, and the partition P is a refinement of $P(\epsilon/M)$. By (51),

$$L(s) = \sum_{n=0}^{l} f(x_n)\big\{\overline{g}(x_n) - \underline{g}(x_n)\big\} + \sum_{n=1}^{l} f(\xi_n)\big\{\underline{g}(x_n) - \underline{g}(x_{n-1})\big\}\,. \tag{55}$$

Define g by $g(a) = \underline{g}(a)$, $g(b) = \overline{g}(b)$, and

$$g(x) = \tfrac{1}{2}\big\{\overline{g}(x) + \underline{g}(x)\big\} \quad \text{for} \quad a < x < b\,. \tag{56}$$

Then $g \in W_q$ by (i) and (ii); and by (52)

$$\begin{aligned} g(x+) &= \overline{g}(x) \qquad && \text{for} \quad a \le x < b\,, \\ g(x-) &= \underline{g}(x) \qquad && \text{for} \quad a < x \le b\,. \end{aligned}$$

These with (55) give

$$\begin{aligned} L(s) =& f(a)\big\{g(a+) - g(a)\big\} + f(b)\big\{g(b) - g(b-)\big\} \\ &+ \sum_{n=1}^{l-1} f(x_n)\big\{g(x_n+) - g(x_n-)\big\} + \sum_{n=1}^{l} f(\xi_n)\big\{g(x_n-) - g(x_{n-1}+)\big\} \end{aligned}$$

which is equal to $S(P^*)$ in the notation of (7) and (8). Thus

$$\big|L(f) - S(P^*)\big| = \big|L(f-s)\big| \le M\|f-s\| = MV_p(f-s) < \epsilon$$

whenever $P \supset P(\epsilon/M)$ and P^* is associated with P. So by (9) and (10), $f \in R^3S(g)$ and $L(f)$ is the value of the R^3S-integral. □

Remark. By (56) and (51),

$$g(x) = L\Big\{\tfrac{1}{2}(\overline{x} + \underline{x})\Big\} = L(h) \quad \text{for} \quad a \le x \le b\,,$$

where h is the Heaviside function

$$h(t) = 1 \;\text{ for } t < x\,, \qquad h(x) = \tfrac{1}{2}\,, \qquad h(t) = 0 \;\text{ for } t > x\,.$$

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