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ADJOINT CLASSES OF GENERALIZED-STIELTJES INTEGRABLE FUNCTIONS

Abstract

The present paper gives a pair of adjoint classes of Riemann-Stieltjes integrable functions and two pairs of adjoint classes of generalized Stieltjes integrable functions.

1 Introduction

Kenneth A. Ross [7] gives a recommended definition of the Riemann-Stieltjes integral that has some advantages over the classical definition. In the classical theory, the following results are known about the adjoint classes of Riemann-Stieltjes integrable functions.

(1) $C[a,b]$ and $BV[a,b]$ are adjoint w.r.t. $(R\text{-}S)\int_a^b d\,dg$ [5].

(2) $R[a,b]$ and $AC[a,b]$ are adjoint w.r.t. $(R\text{-}S)\int_a^b f\,dg$ [2].

It seems reasonable then to ask what the analogous results in the generalized theory should be. The aim of the present paper is to answer this question and give the corresponding results. For this purpose, however, we shall first deal with another pair of adjoint classes of Riemann-Stieltjes integrable functions with the classical definition. So Section 2 is devoted to this pursuit.

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2 The Classes $\tilde{\mathbf{C}}[\mathbf{a},\mathbf{b}]$ and $\mathbf{BV}\cap\mathbf{C}[\mathbf{a},\mathbf{b}]$

Definition. A function f is said to belong to $\tilde{C}[a,b]$, if f is bounded and has only a countable number of discontinuities on $[a,b]$.

Theorem 2.1. *If $f\in\tilde{C}[a,b]$ and $g\in BV\cap C[a,b]$, then $(R\text{-}S)\int_a^b f\,dg$ exists.*

PROOF. Suppose g is increasing and continuous on $[a,b]$. By [7], to show that $(R\text{-}S)\int_a^b f\,dg$ exists, it suffices to show that given $\epsilon>0$, there is a partition of $[a,b]$:

$$\mathbb{P}=\{a=x_0<\cdots<x_n=b\}$$

such that

$$\sum_{i=1}^{n}\omega_f\big[x_{i-1},x_i\big]\cdot\big[g(x_i)-g(x_{i-1})\big]<\epsilon\,,$$

where $\omega_f[x_{i-1},x_i]$ is the oscillation of f on $[x_{i-1},x_i]$.

Let D be the set of discontinuities of f on $[a,b]$. Then $[a,b]=D\cup([a,b]\setminus D)$. Since $f\in\tilde{C}[a,b]$, D is countable. Suppose $D=\{y_k\,:\,k=1,2,\ldots\}\subset[a,b]$. Then, given $\epsilon>0$, for each y_k in D there is an open interval $Q_k=(y_k-\delta_k,y_k+\delta_k)$ such that $\omega_g\big(Q_k\cap[a,b]\big)<\dfrac{\epsilon}{2^k}$. Set $I_1=\{Q_k\,:\,k=1,\ldots\}$, which is an open covering of D. If $x\in[a,b]\setminus D$, then there is an open interval P_x containing x such that $\omega_f\big(P_x\cap[a,b]\big)<\epsilon$. Set $I_2=\{P_x\,:\,x\in[a,b]\setminus D\}$, which is an open covering of $[a,b]\setminus D$. Therefore $I=I_1\cup I_2$ is an open covering of $[a,b]$. By the Heine-Borel Theorem there is a finite collection $\{O_1,\ldots,O_n\}$ of open intervals in I such that $[a,b]\subset\cup_{i=1}^{n}O_i$. It is not hard to see that one can find a partition $\mathbb{P}=\{a=x_0<\cdots<x_m=b\}$ of $[a,b]$ such that each subinterval $[x_{i-1},x_i]$ is contained in some interval of $\{O_1,\ldots,O_n\}$. Thus we write

$$\sum_{i=1}^{m}\omega_f\big[x_{i-1},x_i\big]\cdot\big[g(x_i)-g(x_{i-1})\big]$$

$$=\sum_{i'}\omega_f\big[x_{i'-1},x_{i'}\big]\cdot\big[g(x_{i'})-g(x_{i'-1})\big]+\sum_{i''}\omega_f\big[x_{i''-1},x_{i''}\big]\cdot\big[g(x_{i''})-g(x_{i''-1})\big]$$

such that each $[x_{i'-1},x_{i'}]\subset O_{k'}\in I_1$ and each $[x_{i''-1},x_{i''}]\subset O_{k''}\in I_2$. Then, the above sum is less than

$$\begin{aligned}&\omega_f[a,b]\cdot\sum_{k=1}^{\infty}\omega_g\big(Q_k\cap[a,b]\big)+\epsilon\cdot\sum_{i''}\big[g(x_{i''})-g(x_{i''-1})\big]\\&<\omega_f[a,b]\cdot\epsilon\Big(\tfrac{1}{2}+\cdots+\tfrac{1}{2^k}+\cdots\Big)+\epsilon\cdot\big[g(b)-g(a)\big]\\&=\Big(\omega_f[a,b]+\big[g(b)-g(a)\big]\Big)\cdot\epsilon\,.\end{aligned}$$

Consequently, the integral $(R\text{-}S)\int_a^b f\,dg$ exists. □

The following Theorem is a converse result to Theorem 2.1.

Theorem 2.2. *Let f be defined on $[a,b]$. If $f \notin \tilde{C}[a,b]$, then there exists a function $g \in BV \cap C[a,b]$ such that the integral $(R\text{-}S)\int_a^b f\,dg$ does not exist.*

Proof. If $f \notin R[a,b]$, then taking $g \equiv x$ on $[a,b]$ proves the conclusion. So, we now assume $f \in R[a,b]$. Let D be the set of discontinuities of f on $[a,b]$. Then,

$$D = \cup_{k=1}^{\infty}\Big\{x \in [a,b] \,:\, \omega_f(x) \geq \tfrac{1}{k}\Big\},$$

where $\omega_f(x)$ is the oscillation of f at x. Since $f \notin \tilde{C}[a,b]$, D is uncountable. Hence, there is a number $k_0 \geq 1$ such that $D_1 = \{x \in [a,b] \,:\, \omega_f(x) \geq 1/k_0\}$ is also uncountable. Since the function $\omega_f(x)$ is upper semicontinuous, D_1 is a closed subset of $[a,b]$. By a Theorem in [6], there is a perfect subset D_2 of D_1. Set $c = \inf\{x \,:\, x \in D_2\}$ and $d = \sup\{x \,:\, x \in D_2\}$. We have $a \leq c < d \leq b$ and $c, d \in D_2$. Let $(c,d) \setminus D_2 = \cup_{k=1}^{\infty}(x_k, y_k)$ and put $I = \big\{(x_k, y_k) \,:\, k = 1, \ldots\big\}$. It follows that $D_2 = [c,d] \setminus \cup_{k=1}^{\infty}(x_k, y_k)$. That D_2 is perfect and that $m(D_2) = 0$ (because $f \in R[a,b]$) imply that the collection I of intervals has the following properties.

(i) Any two intervals in I are disjoint and have no common endpoints. Neither c nor d is the endpoint of any intervals in I.

(ii) $\sum_{k=1}^{\infty}(y_k - x_k) = d - c\,.$

With these properties of I, we can define a function g_1 on $[c,d]$ such that g_1 is increasing and continuous with $g_1'(x) = 0$ in $(c,d) \setminus D_2$. In order to do this, we now classify the intervals in the following fashion. Take an interval, denoted by $\big(x_1^{(1)}, y_1^{(1)}\big)$, from I and define $I_1 = \big\{(x_1^{(1)}, y_1^{(1)})\big\}$. Then, choose two intervals, denoted by $\big(x_1^{(2)}, y_1^{(2)}\big)$ and $\big(x_2^{(2)}, y_2^{(2)}\big)$, from I such that $y_1^{(2)} < x_1^{(1)}$, $y_1^{(1)} < x_2^{(2)}$ and define $I_2 = \big\{(x_i^{(2)}, y_i^{(2)}) \,:\, i = 1, 2\big\}$. In general, suppose $I_1, I_2, \ldots, I_h$ are defined, respectively. We then define

$$I_{h+1} = \Big\{\big(x_i^{(h+1)}, y_i^{(h+1)}\big) \in I \,:\, i = 1, \ldots, 2^h\Big\}$$

such that for each i $(1 \leq i \leq 2^h - 1)$, there is exactly one interval $(x_j^{(k)}, y_j^{(k)})$ of

$$\Big\{\big(x_j^{(k)}, y_j^{(k)}\big) \in I_k \,:\, 1 \leq k \leq h\,;\; 1 \leq j \leq 2^{k-1}\Big\}$$

satisfying $y_i^{(h+1)} < x_j^{(k)} < y_j^{(k)} < x_{i+1}^{(h+1)}$. Thus, $I = \cup_{k=1}^{\infty} I_k$. Now, in the same way as the Cantor function is defined in terms of the Cantor set [4], we define a function g_1 on $[c,d]$ as follows. For $h \geq 1$ and $i = 1, \ldots, 2^h$ define

$$g_1(x) = \frac{2i-1}{2^h} \text{ on } \left[x_i^{(h)}, y_i^{(h)}\right].$$

Whence, on the dense subset $[c,d] \setminus D_2$ the function g_1 is increasing and its range is dense in $[0,1]$. Therefore, the domain of g_1 can be extended to $[c,d]$ so that g_1 is increasing and continuous on $[c,d]$ with range $[0,1]$. Since g_1 is locally constant on the open subset $[c,d] \setminus D_2$, which has measure $d-c$, it follows that $g_1'(x) = 0$ *a.e.* in $[c,d]$. If we define $g \equiv g_1$ on $[c,d]$, $g \equiv 0$ on $[a,c)$ and $g \equiv 1$ on $(d,b]$, then g is increasing and continuous on $[a,b]$ and $g' = 0$ *a.e.* in $[a,b]$ such that $(R\text{-}S) \int_a^b f\,dg$ does not exist. In fact, since g is increasing and continuous on $[c,d]$ with $g(c) = 0$, $g(d) = 1$ and $g' = 0$ *a.e.* in $[c,d]$, by Lemma 2 in [2], for arbitrary $\delta > 0$ and $\epsilon = 1/2$, there is a finite collection $\{[s_n, t_n]\}_{n=1,\ldots,N}$ of nonoverlapping intervals on $[c,d]$ such that

$$\sum_{n=1}^{N} \big(t_n - s_n\big) < \delta\,,\ \ g(t_n) - g(s_n) > 0$$

for $n = 1, \ldots, N$ and

$$\sum_{n=1}^{N} \big[g(t_n) - g(s_n)\big] \geq \big[g(d) - g(c)\big] - \epsilon = \frac{1}{2}\,.$$

Then, it follows from the definition of g that each interval $[s_n, t_n]$ must contain the points of D_2. Hence $\omega_f[s_n, t_n] \geq 1/k_0$ for $n = 1, \ldots, N$, since $\omega_f(x) \geq 1/k_0$ at each $x \in D_2$. Thus, for each partition $\mathbb{P}$ of $[c,d]$ with $\text{mesh}(\mathbb{P}) < \delta$ and including each $[s_n, t_n]$ we have

$$U_g(f,\mathbb{P}) - L_g(f,\mathbb{P}) \geq \sum_{n=1}^{N} \omega_f[s_n,t_n]\,[g(t_n) - g(s_n)] \geq 1/2k_0 > 0\,,$$

where $U_g(f,\mathbb{P})$ and $L_g(f,\mathbb{P})$ are defined as in [7]. This means $(R\text{-}S) \int_c^d f\,dg$ does not exist, and neither does $(R\text{-}S) \int_a^b f\,dg$. □

Theorem 2.1 and Theorem 2.2 imply the following assertion.

Corollary. *$\tilde{C}[a,b]$ and $BV \cap C[a,b]$ are adjoint with respect to the integral $(R\text{-}S) \int_a^b f\,dg$.*

Proof. Since $BV[a,b]$ and $C[a,b]$ are adjoint w.r.t. $(R\text{-}S)\int_a^b f\,dg$ and $(BV\cup C)\subset\tilde{C}$, and so $(R\text{-}S)\int_a^b f\,dg$ exists for all $f\in\tilde{C}[a,b]$ implies $g\in BV\cap C[a,b]$. Hence combining the above fact with Theorem 2.1 and Theorem 2.2, we complete the proof of the Corollary. □

Now, we return to generalized-Stieltjes integrable functions.

3 Adjoint Classes of Generalized-Stieltjes Integrable Functions

In order to approach the topic of this section, we need to make a slight modification in the recommended definition of the Riemann-Stieltjes integral in [7]. To define an integral $RS\int_a^b f\,dF$ with recommended definition, Kenneth A. Ross assumes that f is bounded and F is nondecreasing on $[a,b]$. We now consider F to be a regulated function [1]. A function F is said to be regulated on $[a,b]$, if it has only discontinuities of the first kind on $[a,b]$. We denote the class of regulated functions on $[a,b]$ by $RL[a,b]$. Every regulated function is bounded and its discontinuities are countable (cf. [3]). By Lemma 3 in [3], the following important fact is obtained. If $F\in RL[a,b]$, then given $\delta>0$ there exists a partition $\mathbb{P}$ of $[a,b]$ such that F-mesh$(\mathbb{P})<\delta$, which is defined as in [7]. Let f be bounded on $[a,b]$ and $F\in RL[a,b]$, we then define a generalized-Stieltjes integral $(g\text{-}S)\int_a^b f\,dF$ in the same way as the R-S integral with the recommended definition is defined in [7]. Also, we introduce a class of functions denoted by $\overline{BV}[a,b]$, as follows.

Definition. Let $g\in RL[a,b]$, and let $\mathbb{P}=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. Put

$$V_g(\mathbb{P})=\sum_{i=1}^{n}\big|g(x_i-0)-g(x_{i-1}+0)\big|+\sum_{i=0}^{n}\big|g(x_i+0)-g(x_i-0)\big|$$

where $g(a-0)=g(a)$ and $g(b+0)=g(b)$ and set $V_g=\sup V_g(\mathbb{P})$, where we take the suprema over all possible partitions of $[a,b]$. If $V_g<\infty$, we then write $g\in\overline{BV}[a,b]$.

Remark 1. If $g\in\overline{BV}[a,b]$, then g can be written as $g=g_1+\phi+\psi$, such that $g_1\in BV\cap C[a,b]$; $\phi(x)=\sum_{n=1}^{\infty}c_n\theta(x-u_n)$, where $\{u_n\}$ consists of all discontinuities of g on $[a,b]$ and $c_n=g(u_n+0)-g(u_n-0)$,

$$\theta(x)=\begin{cases}0 & \text{if } x\le 0\\ 1 & \text{if } x>0\end{cases}$$

and $\psi(x-0)=\psi(x+0)=0$ in (a,b), $\psi(a+0)=\psi(b-0)=0$.

PROOF. Write $g(x)=g(x-0)+g(x)-g(x-0)$ for x in $(a,b]$ and define $g(a-0)=g(a)$. Then, the function $\psi(x)=g(x)-g(x-0)$ satisfies the requirements of Remark 1. Since $g\in\overline{BV}[a,b]$ implies $\sum|c_n|<\infty$, and so $g(x-0)$ can be written as

$$g(x-0)=g(x-0)-\sum_{u_n<x}c_n+\sum_{u_n<x}c_n\,.$$

Now, let $g_1(x)=g(x-0)-\sum_{u_n<x}c_n$ and

$$\phi(x)=\sum_{u_n<x}c_n=\sum_{u_n<x}c_n\theta(x-u_n)=\sum_{n=1}^{\infty}c_n\theta(x-u_n)$$

on $[a,b]$. Then, it is not hard to see that $g_1\in C[a,b]$. Also, since $g\in\overline{BV}[a,b]$ implies $g(x-0)\in BV[a,b]$, and

$$\sum_{n=1}^{\infty}c_n\theta(x-u_n)=\sum_{n=1}^{\infty}c_n^+\theta(x-u_n)-\sum_{n=1}^{\infty}c_n^-\theta(x-u_n)$$

with

$$c_n^+=\frac{|c_n|+c_n}{2}\text{ and }c_n^-=\frac{|c_n|-c_n}{2}\,,$$

implies that $\phi\in BV[a,b]$, and so $g_1\in BV[a,b]$. Hence, $g_1\in BV\cap C[a,b]$. □

Remark 2. For every bounded function f on $[a,b]$ and any function of the form $\varphi(x)=\sum_{n=1}^{\infty}c_n\theta(x-u_n)$ with $\sum|c_n|<\infty$ and $\{u_n\}\subset[a,b]$, we have

$$(g\text{-}S)\int_a^b f\,d\phi=\sum_{n=1}^{\infty}c_nf(u_n)\,.$$

PROOF. From the proof of Remark 1, we have

$$\phi(x)=\sum c_n^+\theta(x-u_n)-\sum c_n^-\theta(x-u_n)\,.$$

Then, by using a Theorem in [7], it follows that for every bounded function f on $[a,b]$

$$(g\text{-}S)\int_a^b f\,d\phi=\sum_{n=1}^{\infty}c_nf(u_n)\,.$$ □

Theorem 3.1. *If $f \in \tilde{C}[a,b]$ and $g \in \overline{BV}[a,b]$, then $(g\text{-}S)\int_a^b f\,dg$ exists.*

Proof. Since $g \in \overline{BV}[a,b]$, by Remark 1, g can be written as $g = g_1 + \phi + \psi$ such that $g_1 \in BV \cap C[a,b]$, $\phi = \sum c_n \theta(x - u_n)$ as in Remark 2 and ψ with the property that one-sided limits are equal to 0 at every point of $[a,b]$. Let $f \in \tilde{C}[a,b]$. Thus, since $(g\text{-}S)\int_a^b f\,d\psi = 0$ by definition, $(g\text{-}S)\int_a^b f\,d\phi = \sum_{n=1}^{\infty} c_n f(u_n)$ by Remark 2 and

$$(g\text{-}S)\int_a^b f\,dg_1 = (R\text{-}S)\int_a^b f\,dg_1$$

by Theorem 2.1 and [7], and so the $(g\text{-}S)\int_a^b f\,dg$ exists. □

Theorem 3.2. *$\tilde{C}[a,b]$ and $\overline{BV}[a,b]$ are adjoint with respect to $(g\text{-}S)\int_a^b f\,dg$.*

Proof. Let f be bounded on $[a,b]$ and let $(g\text{-}S)\int_a^b f\,dg$ exist for all $g \in \overline{BV}[a,b]$. But $(BV \cap C[a,b]) \subset \overline{BV}[a,b]$, and so the fact that $(g\text{-}S)\int_a^b f\,dg$ exists for all $g \in \overline{BV}[a,b]$ implies $(g\text{-}S)\int_a^b f\,dg$ exists for all $g \in BV \cap C[a,b]$. It is not hard to see that if $(g\text{-}S)\int_a^b f\,dg$ exists and $g \in BV \cap C[a,b]$, then $(R\text{-}S)\int_a^b f\,dg$ exists and equals $(g\text{-}S)\int_a^b f\,dg$ (cf. [7]). Thus, by Theorem 2.2, we see that $f \in \tilde{C}[a,b]$.

Now, let $g \in RL[a,b]$ and let $(g\text{-}S)\int_a^b f\,dg$ exist for all $f \in \tilde{C}[a,b]$. Let $\{u_n\}$ be a sequence of all the discontinuities of g on $[a,b]$. Suppose $\sum_{n=1}^{\infty}|c_n| = \infty$, where $c_n = g(u_n+0) - g(u_n-0)$ and as before $g(a-0) = g(a)$ and $g(b+0) = g(b)$. Set $x = (a+b)/2$. Then at least one of the series $\sum_{u_n<x}|c_n|$ or $\sum_{u_n>x}|c_n|$, is divergent. Say $\sum_{u_n>x}|c_n| = \infty$. Then, set $x_1 = (x+b)/2$ and at least one of the series $\sum_{x<u_n<x_1}|c_n|$ or $\sum_{u_n>x_1}|c_n|$ is divergent. If this procedure is repeated infinitely, there must exist a point x^* on $[a,b]$ such that for any neighborhood $O(x^*,\eta)$ of x^* with arbitrary small $\eta > 0$, the series $\sum_{u_n}|c_n| = \infty$, where the sum is taken over all u_n in $O(x^*,\eta)$. For convenience, let $x^* = b$. In this case, we can find an increasing subsequence of $\{u_n\}$, denoted by $\{u(n_k)\}$, such that $u(n_k) \to b$ as $k \to \infty$ and $\sum_{k=1}^{\infty}|c(n_k)| = \infty$. Now, we define a function on $[a,b]$ by

$$f(x) = \begin{cases} \text{sign}\,\{c(n_k)\} & \text{if } x = u(n_k) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f \in \tilde{C}[a,b]$ but $(g\text{-}S)\int_a^b f\,dg$ does not exist, since, if a partition, $\mathbb{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$, of $[a,b]$ contains $\{u(n_k) : k = 1,\ldots,K\}$, then we can choose a proper point ξ_i in each interval (t_{i-1}, t_i) such that $f(\xi_i) = 0$, and so the Stieltjes sum corresponding to $\mathbb{P}$

$$\begin{aligned} S_g(f, \mathbb{P}) &= \sum_{i=0}^{n} f(t_i)\big[g(t_i + 0) - g(t_i - 0)\big] + \sum_{i=1}^{n} f(\xi_i)\big[g(t_i - 0) - g(t_{i-1} + 0)\big] \\ &= \sum_{k=1}^{K} |c(n_k)| \to \infty (\text{ as } K \to \infty). \end{aligned}$$

This contradicts the hypothesis. So, we must have $\sum_{n=1}^{\infty} |c_n| < \infty$. From the proof of Remark 1, we see that $\sum_{n=1}^{\infty} |c_n| < \infty$ implies $g = g_1 + \phi + \psi$ with $g_1 \in C[a, b]$, $\phi = \sum c_n \theta(x - u_n)$ and $\psi(x + 0) = \psi(x - 0) = 0$ at every point of $[a, b]$. By Remark 2, it follows that $(g\text{-}S) \int_a^b f\, dg_1$ exists for all $f \in \tilde{C}[a, b]$. Since $C[a, b] \subset \tilde{C}[a, b]$ and

$$(g\text{-}S) \int_a^b f\, dg_1 = (R\text{-}S) \int_a^b f\, dg_1$$

when $(g\text{-}S) \int_a^b f\, dg_1$ exists and either of f and g_1 is continuous on $[a, b]$, and so $g_1 \in BV[a, b]$ for $C[a, b]$ and $BV[a, b]$ are adjoint with respect to the $(R\text{-}S) \int_a^b f\, dg$. Hence $g \in \overline{BV}[a, b]$.

Consequently, combining the above results with Theorem 3.1, we see that $\tilde{C}[a, b]$ and $\overline{BV}[a, b]$ are adjoint with respect to the $(g\text{-}S) \int_a^b f\, dg$. □

The next result deals with another pair of adjoint classes of the generalized-Stieltjes integrable function. One of the classes is defined next.

Definition. Let $g \in \overline{BV}[a, b]$ and let $g = g_1 + \phi + \psi$, where g_1, ϕ and ψ are defined as in Remark 1. If $g_1 \in AC[a, b]$, then we say $g \in \overline{AC}[a, b]$.

Theorem 3.3. *$R[a, b]$ and $\overline{AC}[a, b]$ are adjoint with respect to $(g\text{-}S) \int_a^b f\, dg$.*

PROOF. (1) Let $f \in R[a, b]$ and $g \in \overline{AC}[a, b]$. It follows by Remark 2 and [2] that $(g\text{-}S) \int_a^b f\, dg$ exists.
(2) Let f be bounded on $[a, b]$ and $(g\text{-}S) \int_a^b f\, dg$ exist for all $g \in \overline{AC}[a, b]$. Then, it follows that

$$(g\text{-}S) \int_a^b f\, dg = (R\text{-}S) \int_a^b f\, dg$$

exists for all $g \in AC[a, b]$, since $AC \subset \overline{AC}$. Then, $f \in R[a, b]$, since $R[a, b]$ and $AC[a, b]$ are adjoint w.r.t. the $(R\text{-}S) \int_a^b f\, dg$.

(3) Let $g \in RL[a,b]$ and $(g\text{-}S)\int_a^b f\,dg$ exists for all $f \in R[a,b]$. Since $\tilde{C}[a,b] \subset R[a,b]$ and by the proof of Theorem 3.2, $g \in \overline{BV}[a,b]$ and so by Remark 1, $g = g_1 + \phi + \psi$ with $g_1 \in BV \cap C[a,b]$, $\phi = \sum c_n \theta(x - u_n)$ and $\psi(x+0) = \psi(x-0) = 0$ on $[a,b]$. By Remark 2, $(g\text{-}S)\int_a^b f\,dg_1$ exists for all $f \in R[a,b]$ and so does $(R\text{-}S)\int_a^b f\,dg_1$, since $g_1 \in BV \cap C[a,b]$. Hence, $g_1 \in AC[a,b]$, since $R[a,b]$ and $AC[a,b]$ are adjoint w.r.t. the $(R\text{-}S)\int_a^b f\,dg$. Thus, $g \in \overline{AC}[a,b]$.

Finally (1), (2) and (3) imply $R[a,b]$ and $\overline{AC}[a,b]$ are adjoint w.r.t. the $(g\text{-}S)\int_a^b f\,dg$. □

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