ON THE HAUSDORFF MEASURE OF A CLASS OF SELF-SIMILAR SETS

Abstract

We develop a new combinatorial method to estimate Hausdorff measures of various self-similar sets. This method can be applied to the evaluation of Hausdorff measures which are induced by various Hausdorff functions including power functions. Moreover, a few examples for evaluations of the lower and upper bounds of Hausdorff measures of uniform Cantor sets are introduced.

1 Introduction

Throughout the paper, we use $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{R}$ to denote the set of all non-negative integers, of all positive integers and of all real numbers, respectively. $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space, and $D$ will be some fixed closed subset of $\mathbb{R}^n$. By $d(A)$ we denote the diameter of any subset $A$ of $\mathbb{R}^n$, and the cardinal number of a set $C$ will be denoted by $\#C$.

A monotonically increasing function $h : [0, \infty) \to [0, \infty)$ is called a Hausdorff function if and only if $h(t) > 0$ for $t > 0$, $h(0) = 0$ and $h$ is continuous from the right. It is well-known that every Hausdorff function $h$ induces a corresponding Hausdorff measure $\mu^h$ as follows

$$\mu^h(C) = \lim_{\delta \to 0} \inf \sum_i h(d(U_i)),$$

where we take the infimum over all $\delta$-coverings $\{U_i\}_i$ of $C$ (see [10]).
A mapping $S : D \to D$ is called a similarity on $D$ if there is a constant $c$ ($0 < c < 1$) such that $|S(x) - S(y)| = c|x - y|$ for all $x, y \in D$. The constant $c$ is called the (similarity) ratio of $S$.

Throughout the paper, suppose $m (> 1)$ is a fixed integer. Let $\{S_1, \ldots, S_m\}$ be a family of similarities on $D$. We say that a subset $F$ of $D$ is self-similar under $S_1, \ldots, S_m$ if

$$F = \bigcup_{i=1}^{m} S_i(F).$$

**Definition 1.** A family $\{S_1, \ldots, S_m\}$ of similarities on $D$ is said to be disjoint if $S_i(D) \cap S_j(D) = \emptyset$ for all $i \neq j$.

Hutchinson [6] proved that for every disjoint family $\{S_1, \ldots, S_m\}$ of similarities on $D$ there exists a unique non-empty compact set which is self-similar under the $S_i$'s. Many self-similar sets are well-known, e.g., Cantor sets, Cantor dusts, the Sierpiński gasket, the von Koch curve, etc.

For the time being, let $\{S_1, \ldots, S_m\}$ be any disjoint family of similarities on $D$ with the ratios $c_1, \ldots, c_m$, respectively. Let $\ell \in \mathbb{N}$ be fixed. By $I_\ell$ we denote the family of all finite sequences $(a_i)_{i=1}^{\ell}$ satisfying $a_i \in \{1, \ldots, m\}$, i.e.,

$$I_\ell = \{(a_i)_{i=1}^{\ell} : a_i \in \{1, \ldots, m\}\}.$$

For every finite sequence $a = (a_i) \in I_\ell$, let $S_a = S_{a_1} \circ S_{a_2} \circ \cdots \circ S_{a_\ell}$ with the convention

$$S_a(x) = S_{a_1}(S_{a_2}(\cdots(S_{a_\ell}(x))\cdots)).$$

We denote by $\mathcal{H}$ the class of all Hausdorff functions and by $\mathcal{H}_m^c$ the class of all Hausdorff functions satisfying

$$h(ct) = \frac{1}{m} h(t)$$

for all sufficiently small $t > 0$.

In this paper, a new combinatorial method to estimate Hausdorff measures (which are induced not only by power functions but also by other Hausdorff functions different from power functions) of self-similar sets are investigated (see Theorem 8), and the method is applied to the estimation of Hausdorff measures of uniform Cantor sets.
2 Preliminaries

Definition 2. Suppose \( \{S_1, \ldots, S_m\} \) is a disjoint family of similarities on \( D \). Let \( C \) be an arbitrary subset of \( \mathbb{R}^n \). For any non-negative integer \( i \) let

\[
\alpha_i(C) = \lim_{\ell \to \infty} \frac{\# \{ a \in \mathcal{I}_{i+\ell} : S_a(D) \subset C \}}{m^\ell}.
\]

Lemma 1. Let \( \{S_1, \ldots, S_m\} \) be a disjoint family of similarities on \( D \), and let \( C \) be an arbitrary subset of \( D \). For each positive integer \( i \)

(a) \( \alpha_i(C) = m\alpha_{i-1}(C) \),

(b) \( \alpha_i(C) = \alpha_{i+1}(S_j(C)) \quad (j = 1, \ldots, m) \),

(c) \( \alpha_i(C) = \alpha_i \left( \bigcup_{j=1}^m S_j(C) \right) \).

Proof. (a) It follows from Definition 2 that

\[
\alpha_i(C) = \lim_{\ell \to \infty} \frac{m\# \{ a \in \mathcal{I}_{i-1+\ell+1} : S_a(D) \subset C \}}{m^\ell} = m\alpha_{i-1}(C).
\]

(b) Let \( A, B \) be arbitrary subsets of \( D \) and \( j \in \{1, \ldots, m\} \) fixed. If \( x \in A \setminus B \) and \( S_j(A) \subset S_j(B) \), then there exists some \( y \in B \) such that \( S_j(y) = S_j(x) \) contrary to the injectivity of \( S_j \). So \( S_j(A) \not\subset S_j(B) \) if \( A \not\subset B \). On the other hand, it is obvious that \( S_j(A) \subset S_j(B) \) if \( A \subset B \). Hence, we obtain that \( S_j(A) \subset S_j(B) \) if and only if \( A \subset B \). So, for all \( j \in \{1, \ldots, m\} \), we have

\[
\alpha_i(C) = \lim_{\ell \to \infty} \frac{\# \{ a \in \mathcal{I}_{i+\ell} : S_j \circ S_a(D) \subset S_j(C) \}}{m^\ell}. \]

Consequently if \( S_a(D) \subset C \), then \( S_j \circ S_a(D) \subset S_j(C) \subset S_j(D) \) and \( S_k \circ S_a(D) \subset S_k(C) \subset S_k(D) \) for all \( j, k \in \{1, \ldots, m\} \). If \( j \neq k \), then \( S_k \circ S_a(D) \not\subset S_j(C) \), since \( \{S_1, \ldots, S_m\} \) is disjoint. Thus, combining this fact with the above equality, we get

\[
\alpha_i(C) = \lim_{\ell \to \infty} \frac{\# \{ a \in \mathcal{I}_{i+1+\ell} : S_a(D) \subset S_j(C) \}}{m^\ell} = \alpha_{i+1}(S_j(C)).
\]

(c) Since \( \{S_1, \ldots, S_m\} \) is disjoint, by (a) and (b), we obtain

\[
\alpha_i \left( \bigcup_{j=1}^m S_j(C) \right) = \sum_{j=1}^m \alpha_i(S_j(C)) = \sum_{j=1}^m \alpha_{i-1}(C) = m\alpha_{i-1}(C) = \alpha_i(C). \quad \square
\]
3 Method of Substitution

Let \( D \neq \emptyset \) be a closed subset of \( \mathbb{R}^n \) and let \( \{ S_1, \ldots, S_m \} \) be a disjoint family of similarities on \( D \) with common ratio \( c \) (0 < c < 1). Suppose \( D_o \) is a subset of \( D \) with non-empty interior such that \( c^{i_o+1}d(D) < d(D_o) \leq c^i d(D) \) for a fixed non-negative integer \( i_o \). Let \( D_i \) be a subset of \( D \) for which \( c^{i+1}d(D) < d(D_i) \leq c^i d(D) \), and suppose that there exist an integer \( i_o(\geq i_s) \) and a finite sequence \( a \in I_{i_o} \) satisfying \( S_a(D) \subset D_o \). (The last hypothesis guarantees that \( \alpha_0(D_o) > 0 \).

We now introduce a new method to estimate Hausdorff measures of self-similar sets.

(a) Let \( \varepsilon > 0 \) be given such that \( \varepsilon/(1 - \alpha_0(D_o)) \) is sufficiently small (cf. the proof of Lemma 2 below). According to Definition 2, it is possible to choose a positive integer \( i_0 \) such that

\[
m^{i_0}(\alpha_{i_0}(D_o) - \varepsilon) < n_o = \#\{ a \in I_{i_0+i_o} : S_a(D) \subset D_o \} < m^{i_0}(\alpha_{i_0}(D_o) + \varepsilon).
\]

Let \( I^1, \ldots, I^n_o \) be an enumeration of the set \( \{ S_a(D) : S_a(D) \subset D_o ; a \in \{ a \} \} \). For every \( j = 1, \ldots, n_o \) there is exactly one \( a \in I_{i_0+i_o} \) with \( I^j_0 = S_a(D) \). Divide each \( I^j_0 \) (\( j = 1, \ldots, n_o \)) into \( V^j_0 = S_a(D) \) (where \( a \in I_{i_0+i_o} \) with \( I^j_0 = S_a(D) \)) and \( R^j_0 = I^j_0 \setminus V^j_0 \).

We further describe the process of our method by induction on \( \ell = 0, 1, 2, \ldots \).

(b) Assume that we have already chosen a sufficiently large integer \( i_\ell \) (\( \ell \geq 1 \)) such that

\[
m^{i_\ell} \sum_{j=1}^{n_{\ell-1}} \left( \alpha_{i_\ell+i_o+i_{\ell-1}}(R^j_{\ell-1}) - \varepsilon \right) < n_\ell
\]

\[
< m^{i_\ell} \sum_{j=1}^{n_{\ell-1}} \left( \alpha_{i_\ell+i_o+i_{\ell-1}}(R^j_{\ell-1}) + \varepsilon \right)
\]

where

\[n_\ell = \#\{ a \in I_{i_\ell+i_o+i_1 + \ldots + i_{\ell-1}} : \text{there exists some } j \in \{1, \ldots, n_{\ell-1}\}
\]

\[\text{with } S_a(D) \subset R^j_{\ell-1} \} .\]

Then, let \( I^1_\ell, \ldots, I^n_\ell \) be an enumeration of the set

\[\{ S_a(D) : S_a(D) \subset \bigcup_{j=1}^{n_{\ell-1}} R^j_{\ell-1} ; a \in I_{i_\ell+i_o+i_1 + \ldots + i_{\ell}} \} .\]
For every $j = 1, \ldots, n_\ell$ there exists a unique $a \in I_{i, +i_0 + \ldots + i_\ell}$ such that $I_\ell^j = S_a(D)$. Divide every $I_\ell^j$ ($j = 1, \ldots, n_\ell$) into $V_\ell^j = S_a(D_o)$ (where $a \in I_{i, +i_0 + \ldots + i_\ell}$ with $I_\ell^j = S_a(D)$) and $R_\ell^j = I_\ell^j \setminus V_\ell^j$. By Definition 2, we can again choose a large integer $i_{\ell+1}$ such that

$$ m^{i_{\ell+1}} \sum_{j=1}^{n_\ell} (\alpha_{i, +i_0 + \ldots + i_\ell}(R_\ell^j) - \varepsilon) < n_{\ell+1} < m^{i_{\ell+1}} \sum_{j=1}^{n_\ell} (\alpha_{i, +i_0 + \ldots + i_\ell}(R_\ell^j) + \varepsilon),$$

where

$$ n_{\ell+1} = \# \{a \in I_{i, +i_0 + \ldots + i_{\ell+1}} : \text{there exists some } j \in \{1, \ldots, n_\ell\} \text{ with } S_a(D) \subset R_\ell^j \}.$$

(c) Repeat the process (b) for $\ell + 1$.

**Definition 3.** Let $h \in \mathcal{H}$. Suppose $\{S_1, \ldots, S_m\}$ is a disjoint family of similarities on $D$ with common ratio $c$. The above process is called $D_o$-substitution of $D_o$ with respect to the sequence $(i_\ell)$. Every $V_\ell^j$ ($\ell = 0, 1, 2, \ldots; j = 1, \ldots, n_\ell$) is called an element of $D_o$-substitution of $D_o$. The $D_o$-substitution of $D_o$ is said to be efficient if

$$ \sigma(D_o; D_o) = \lim_{\varepsilon \to 0} \sum_{V \text{ is an element of } D_o} h(d(V)) < h(d(D_o)).$$

**Remark 1.** Obviously, $\{V_\ell^j\}$ which was obtained from the above process covers almost all of the self-similar set $F$ under $S_1, \ldots, S_m$. Indeed, for every $h \in \mathcal{H}$ we can select a covering of $F$ consisting of $\{V_\ell^j\}$ and $\{E_j\}$ such that the values of $\sum h(d(E_j))$ and $\sum \alpha_{i,}(E_j)$ are as small as desired by taking the values of $i_\ell$'s sufficiently large in the above process.

**Lemma 2.** For every positive integer $\ell$

$$ m^{i_0 + \ldots + i_\ell} (\alpha_{i,}(D_o) - \varepsilon)(1 - \alpha_0(D_o) - \varepsilon)^\ell < n_\ell < m^{i_0 + \ldots + i_\ell} (\alpha_{i,}(D_o) + \varepsilon)(1 - \alpha_0(D_o) + \varepsilon)^\ell.$$

**Proof.** By Lemma 1 (b) we have, for every positive integer $\ell$,

$$ \alpha_{i, +i_0 + \ldots + i_{\ell-1}}(R_\ell^j) = \alpha_{i, +i_0 + \ldots + i_{\ell-1}}(I_{\ell-1}^j \setminus V_\ell^j) $$

$$ = \alpha_{i, +i_0 + \ldots + i_{\ell-1}}(S_a(D) \setminus S_a(D_o)) $$

$$ = \alpha_{i, +i_0 + \ldots + i_{\ell-1}}(S_a(D)) - \alpha_{i, +i_0 + \ldots + i_{\ell-1}}(S_a(D_o)) $$

$$ = 1 - \alpha_0(D_o) $$
where \( a \in I_{i_{\ast}+i_{0}+\cdots+i_{\ell-1}} \) with \( S_a(D) = I_{\ell-1} \). Combining this result with the inequalities for \( n_{\ell}' \)’s in the above process (b) and by induction on \( \ell \), we complete the proof.

**Theorem 3.** Let \( \{S_1, \ldots, S_m\} \) be a disjoint family of similarities on \( D \) with common ratio \( c \). Suppose \( h \in H_m^c \). If \( i_{\ast} \) is so large that the relation (2) holds for all \( 0 < t \leq c^{i_{\ast}} d(D) \), then we have \( \sigma(D_o; D_v) = \alpha_{i_{\ast}}(D_v) h(d(D_v)) / \alpha_{i_{\ast}}(D_o) \).

**Proof.** It follows from Definition 3, Lemma 2 and Lemma 1 (a) that

\[
\sigma(D_o; D_v) = \lim_{\varepsilon \to 0} \sum_{\ell=0}^{\infty} \sum_{j=1}^{n_{\ell}} h(d(V_{\ell}^j)) = \lim_{\varepsilon \to 0} \sum_{\ell=0}^{\infty} n_{\ell} h(c^{i_{\ast}+i_{0}+\cdots+i_{\ell}} d(D_v)) = \lim_{\varepsilon \to 0} \sum_{\ell=0}^{\infty} \frac{n_{\ell}}{m^{i_{\ast}}} h(d(D_v)) = \frac{\alpha_{i_{\ast}}(D_v)}{\alpha_{i_{\ast}}(D_o)} h(d(D_v)) = \alpha_{m(i_{\ast})} h(d(D_v)) / \alpha_{m(i_{\ast})} h(d(D_v)).
\]

**Definition 4.** Suppose \( \{S_1, \ldots, S_m\} \) is a disjoint family of similarities on \( D \) with common ratio \( c \). Let \( i \) be a non-negative integer. Define

\[
\Phi_i(D_1) = \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \subset D_1; c^{i+1} d(D) < d(I) \leq c^i d(D) \right\}
\]

for all subsets \( D_1 \) of \( D \) with \( d(D_1) \geq c^i d(D) \), where we follow the convention that if \( a > 0 \), then \( a/0 := \infty \).

In the definition of \( \Phi_i \), the infimum has to be taken over all ‘test’ sets \( I \) whose diameters lie between \( c^{i+1} d(D) \) and \( c^i d(D) \). Therefore, the \( \Phi_i \) can be defined on the only sets \( D_1 \) with \( d(D_1) \geq c^i d(D) \).

**Definition 5.** For any compact subsets \( A, B \) of \( D \) the distance \( \rho(A, B) \) between \( A \) and \( B \) is defined by \( \rho(A, B) = \min\{ |x - y| : x \in A; y \in B \} \). Suppose \( \{S_1, \ldots, S_m\} \) is a disjoint family of similarities on \( D \) with common ratio \( c \). Let \( \delta = \min\{ \rho(S_i(D), S_j(D)) : i \neq j \} \) and \( \tau = \min\{ i \in \mathbb{N}_0 : c^{i+1} d(D) \leq \delta \} \). The constant \( \tau \) is called the index of the self-similar set under \( S_1, \ldots, S_m \).
Lemma 4. Let \( \tau \) be the index of the self-similar set under \( S_1, \ldots, S_m \) with common ratio \( c \). Assume that \( i \geq \tau \) and \( a \in I_{i-\tau} \) are fixed. Let \( h \in \mathcal{H} \). Then 
\[ \Phi_i(S_a(D)) = \Phi_i(D), \] 
where we set \( S_a(D) = D \) for \( a \in I_0 \).

Proof. If \( i = \tau \), the statement of the lemma is obvious. Now, let \( i > \tau \), then we have by the fact that \( S_1, \ldots, S_m \) are similarities with common ratio \( c \) and by Definition 5

\[
\min\{\rho(S_b(D), S_{b'}(D)) : b, b' \in I_{i-\tau}; b \neq b'\} = c^{i-\tau-1} \min\{\rho(S_j(D), S_k(D)) : j \neq k\} 
\geq c^{i-\tau-1} c^{\tau+1} d(D) 
= c^i d(D). 
\]

For any subset \( A \) of \( D \) let
\[
C_1(A) = \{I \subset D : I \subset A; c^{i+1} d(D) < d(I) \leq c^i d(D)\}, \\
C_2(A) = \{I \subset D : I \cap A \neq \emptyset; I \not\subset A; c^{i+1} d(D) < d(I) \leq c^i d(D)\}
\]
and
\[
C_3(A) = \{I \subset D : I \cap A = \emptyset; c^{i+1} d(D) < d(I) \leq c^i d(D)\}.
\]

Then we have
\[
C_1(D) = \bigcup_{b \in I_{i-\tau}} C_1(S_b(D)) \cup \bigcup_{b \in I_{i-\tau}} C_2(S_b(D)) \cup C_3 \left( \bigcup_{b \in I_{i-\tau}} S_b(D) \right).
\]

Since the structure in \( S_b(D) \) is congruent to that in \( S_a(D) \) and (3) implies that if \( I \in C_2(S_b(D)) \), then \( \#(I \cap S_{b'}(D)) \leq 1 \) for any \( b' \in I_{i-\tau} \) with \( b' \neq b \), the above equality implies that
\[
\sup\{\alpha_i(I) : I \in C_1(D)\} = \sup\{\alpha_i(I) : I \in C_1(S_a(D))\}.
\]

Further, since the structure in \( S_b(D) \) is congruent to that in \( S_a(D) \), we see that
\[
\inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \in C_1(S_b(D)) \right\} = \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \in C_1(S_a(D)) \right\}
\]
for any \( b \in I_{i-\tau} \). As it was already stated, (3) implies that if \( I \in C_2(S_b(D)) \), then \( \#(I \cap S_{b'}(D)) \leq 1 \) and hence \( \alpha_i(I \cap S_{b'}(D)) = 0 \) for each \( b' \in I_{i-\tau} \) with \( b' \neq b \). Hence, we get
\[
\inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \in C_2(S_b(D)) \right\} \geq \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \in C_1(S_a(D)) \right\}.
\]
Trivially, we have
\[ \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \in C_d \left( \bigcup_{b \in I_{i-\tau}} S_b(D) \right) \right\} = \infty. \]

Therefore, we may conclude that
\[ \Phi_i(D) = \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \in C_1(D) \right\} = \Phi_i(S_a(D)). \]

**Lemma 5.** Suppose \( \{S_1, \ldots, S_m\} \) is a disjoint family of similarities on \( D \) with common ratio \( c \). Let \( h \in \mathcal{H}_m^c \). Suppose \( i_* \) is given such that (2) holds for all \( 0 < t \leq c^{i_*} d(D) \). Assume that \( D_1, D_2 \) are subsets of \( D \) such that
\[ c^{i_*+1} d(D) < d(D_1) \leq c^{i_*} d(D) ; \; c^{i_*+1} d(D) < d(D_2) \leq c^{i_*} d(D) ; \; \frac{h(d(D_1))}{\alpha_i(D_1)} < \frac{h(d(D_2))}{\alpha_i(D_2)}. \]

(a) There exists an efficient \( D_1 \)-substitution of \( D_2 \).

(b) There exists no efficient \( D_2 \)-substitution of \( D_1 \).

**Proof.** (a) Using Theorem 3 we obtain
\[ \sigma(D_2; D_1) = \alpha_i(D_2) \frac{h(d(D_1))}{\alpha_i(D_1)} < \alpha_i(D_2) \frac{h(d(D_2))}{\alpha_i(D_2)} = h(d(D_2)). \]

(b) As in the proof of (a), it is easy to see
\[ \sigma(D_1; D_2) = \alpha_i(D_1) \frac{h(d(D_2))}{\alpha_i(D_2)} > \alpha_i(D_1) \frac{h(d(D_1))}{\alpha_i(D_1)} = h(d(D_1)) \]
by using Theorem 3 again.

**Lemma 6.** Let \( \tau \) be the index of the self-similar set under \( S_1, \ldots, S_m \) with common ratio \( c \). Let \( h \in \mathcal{H}_m^c \). Suppose \( i_*(\geq \tau) \) is a given positive integer such that (2) holds for all \( 0 < t \leq c^{i_\tau} d(D) \). Then we have \( m^i \Phi_i(D) = m^{i+1} \Phi_{i+1}(D) \).
Proof. By Lemma 1 (a) and (b) we obtain, for any \( j \in \{1, \ldots, m\}, \)

\[
\Phi_i(D) = \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \subset D; c^{i+1}d(D) < d(I) \leq c^i d(D) \right\}
\]

\[
= m \cdot \inf \left\{ \frac{h(d(S_j(I)))}{\alpha_{i+1}(S_j(I))} : I \subset D; c^{i+1}d(D) < d(I) \leq c^i d(D) \right\}
\]

\[
\geq m \cdot \inf \left\{ \frac{h(d(J))}{\alpha_{i+1}(J)} : J \subset D; c^{i+2}d(D) < d(J) \leq c^{i+1}d(D) \right\}
\]

\[
= m \Phi_{i+1}(D),
\]

where the second equality holds because, by (2) and our hypothesis for \( i \), we have

\[
h(d(S_j(I))) = (1/m)h(d(I)) \quad \text{and} \quad \alpha_{i+1}(S_j(I)) = \alpha_i(I).
\]

On the other hand, let \( I \) be a subset of \( D \) with \( c^{i+2}d(D) < d(I) \leq c^{i+1}d(D) \).

Case I. Assume that there exists a finite sequence \( a = (a_1, \ldots, a_{i+1}) \in \mathcal{I}_{i+1} \) such that \( I \subsetneq S_a(D) \). Then we may choose a subset \( I' \) of \( S_{a_2} \circ \cdots \circ S_{a_{i+1}}(D) \) such that \( I = S_a(I') \). Hence, using the properties of \( S_j \) and Lemma 1 (b) we have \( d(I) = cd(I') \) and \( \alpha_{i+1}(I) = \alpha_i(I') \) and so, using the properties of \( h \)

\[
\frac{h(d(I))}{\alpha_i(I)} = \frac{1}{m} \frac{h(d(I'))}{\alpha_i(I')}.
\] (5)

Case II. Now assume that every \( S_a(D), a \in \mathcal{I}_{i+1}, \) does not include \( I \). In view of (3) we have

\[
\min \{ \rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{i+1}; b \neq b' \} \geq c^{i+1}d(D) \geq d(I).
\]

Therefore, we may merely consider the case where there exists a unique \( a \in \mathcal{I}_{i+1} \) such that the interior of the intersection of \( I \) and \( S_a(D) \) is non-empty. Due to the properties of the self-similar sets there exists a set \( I_1 \subset \mathbb{R}^n \) similar to \( I \) with \( \alpha_1(I_1) = \alpha_i(I_1 \cap D) = \alpha_{i+1}(I) \) and \( cd(I_1) = d(I) \). Thus, we may choose a subset \( I' \) of \( D \) including \( I_1 \cap D \) and satisfying \( \alpha_i(I') \geq \alpha_{i+1}(I) \) and \( cd(I') = d(I) \). Since

\[
d(I_1) = c^{-1}d(I) \leq c^i d(D) < d(D),
\]

we may select an appropriate subset \( I_2 \) of \( D \) such that \( d((I_1 \cap D) \cup I_2) = d(I_1) \).

If we put \( I' = (I_1 \cap D) \cup I_2 \), then \( I' \) satisfies the desired properties. Hence, using the properties of \( h \)

\[
\frac{h(d(I))}{\alpha_i(I)} \geq \frac{1}{m} \frac{h(d(I'))}{\alpha_i(I')}.
\] (6)
Finally, by (5) and (6)

\[
\Phi_{i+1}(D) = \inf \left\{ \frac{h(d(I))}{\alpha_{i+1}(I)} : I \subset D; \ c^{i+2}d(D) < d(I) \leq c^{i+1}d(D) \right\}
\]

\[
\geq \frac{1}{m} \cdot \inf \left\{ \frac{h(d(I'))}{\alpha_i(I')} : I' \subset D; \ c^{i+1}d(D) < d(I') \leq c^i d(D) \right\}
\]

\[
= \frac{1}{m} \cdot \Phi_i(D).
\]

The assertion of the lemma follows from (4) and the last inequality.

Lemma 7. Let \( F \) be the self-similar set under \( S_1, \ldots, S_m \) with common ratio \( c \) and let \( \tau \) be the index of \( F \). Let \( h \in \mathcal{H}_m^c \). Suppose \( i (\geq 2\tau) \) is a given integer such that (2) holds for all \( 0 < t \leq c^{i-2\tau}d(D) \). Then for any \( a \in \mathcal{T}_{\tau-i} \)

\[
\Phi_{i-\tau}(D) = \inf \left\{ \sum_j h(d(U_j)) : \{U_j\} \text{ is a } c^i d(D)-\text{covering of } F \cap S_a(D) \right\}
\]

Proof. Let \( \varepsilon > 0 \) be arbitrarily small, and let \( \mathcal{U}_i \) be the set of all \( c^i d(D) \)-coverings of \( F \cap S_a(D) \). By Definition 4, we can choose a subset \( D' \) of \( D \) such that

\[
c^{i-\tau+1}d(D) < d(D') \leq c^{-\tau}d(D)
\]

and

\[
\Phi_{i-\tau}(D) \leq \frac{h(d(D'))}{\alpha_{i-\tau}(D')} \leq \Phi_{i-\tau}(D) + \varepsilon. \tag{7}
\]

Let \( \mathcal{U}_i^c \) be the set of all \( c^i d(D) \)-coverings of \( F \cap S_a(D) \) consisting of \( \{U_j\} \) and \( \{V_j\} \) with the properties

(i) if \( c^{k+1}d(D) < d(U_j) \leq c^k d(D) \), then there exists some \( b \in \mathcal{T}_{k-i+\tau} \) such that

\[
\frac{h(d(U_j))}{\alpha_k(U_j)} \leq \frac{h(d(S_b(D')))}{\alpha_k(S_b(D'))} \tag{8}
\]

(ii) \( \{V_j\} \) satisfies

\[
\sum_j h(d(V_j)) < \varepsilon; \quad \sum j \alpha_{i-\tau}(V_j) < \varepsilon. \tag{9}
\]

Now assume \( \{B_j\} \in \mathcal{U}_i \) such that there exists a \( j_0 \) satisfying

\[
\frac{h(d(B_{j_0}))}{\alpha_k(B_{j_0})} > \frac{h(d(S_b(D')))}{\alpha_k(S_b(D'))}
\]
for some \( b \in I_{k-i+i^*} \) with \( c^{k+1}d(D) < d(B_{j_0}) \leq c^k d(D) \). Then by taking \( i^* = k \) in Lemma 5 (a), there exists an efficient \( S_b(D') \)-substitution \( \{B'_j\} \) of \( B_{j_0} \cap D \).

In particular, we may choose a \( c^\ell d(D) \)-covering \( \{E_j\} \) of \( (F \cap S_b(D) \cap B_{j_0}) \cup B'_j \) such that \( \sum h(d(E_j)) < \varepsilon/2^j \) and \( \sum \alpha_{i-r}(E_j) < \varepsilon/2^j \) (cf. Remark 1). We now replace the \( B_{j_0} \) in the covering \( \{B_j\} \) by \( \{B'_j\} \cup \{E_j\} \).

As described above, we may ultimately transform the given covering \( \{B_j\} \in \mathcal{U}_i \) into some \( \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^* \). Therefore, we have

\[
\inf\left\{ \sum_j h(d(U_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^* \right\} \\
\leq \inf\left\{ \sum_j h(d(U_j)) : \{U_j\} \in \mathcal{U}_i \right\} \\
\leq \inf\left\{ \sum_j h(d(U_j)) + \sum_j h(d(V_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i \right\}.
\]

The first inequality in (10) follows from the above consideration, and the fact \( \mathcal{U}_i \subset \mathcal{U}_i^* \) implies the second inequality in (10).

Now, let \( \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^* \) be given such that there exists a sequence \( (k_j) \), \( k_j \geq i \), satisfying

\[
\sum h(d(U_j)) < \varepsilon/2^j \text{ and } \sum \alpha_{i-r}(U_j) < \varepsilon/2^j.
\]

By (8), Lemma 1 (a) and (7) in order, we have for some \( b \in I_{k_j-i+i^*} \)

\[
h(d(U_j)) \leq \alpha_{k_j}(U_j) \frac{h(d(S_b(D')))}{\alpha_{k_j}(S_b(D'))} \\
= \alpha_{i-r}(U_j) \frac{h(d(S_b(D')))}{\alpha_{i-r}(S_b(D'))} \\
= \alpha_{i-r}(U_j) \frac{h(d(D'))}{\alpha_{i-r}(D')} \\
\leq \alpha_{i-r}(U_j) (\Phi_{i-r}(D) + \varepsilon).
\]

Since \( \inf\{\sum \alpha_{i-r}(U_j) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i \} \leq 1 \), condition (9), together with the above inequality, implies

\[
\inf\left\{ \sum_j h(d(U_j)) + \sum_j h(d(V_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i \right\} \\
\leq \inf\left\{ \sum_j \alpha_{i-r}(U_j) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i \right\} \cdot (\Phi_{i-r}(D) + \varepsilon) + \varepsilon \\
\leq \Phi_{i-r}(D) + 2\varepsilon.
\]
Since $\varepsilon > 0$ may be sufficiently small, we obtain
\[ \inf \left\{ \sum_j h(d(U_j)) + \sum_j h(d(V_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon \right\} \leq \Phi_i - \tau(D). \tag{11} \]

On the other hand, if $\{U_j\} \cup \{V_j\}$ belongs to $\mathcal{U}_i^\varepsilon$, then for some $k$ with $c^{k+1}d(D) < d(U_j) \leq c^k d(D)$
\[ \frac{h(d(U_j))}{\alpha_k(U_j)} \geq \Phi_k(D). \tag{12} \]

Thus, it follows from (12), Lemma 1 (a) and Lemma 6 that
\[ h(d(U_j)) \geq \alpha_k(U_j)\Phi_k(D) = \alpha_{i-\tau}(U_j)\Phi_{i-\tau}(D). \]

Hence, as $\sum \alpha_{i-\tau}(U_j) \geq 1 - \varepsilon$ by (9), we get
\[ \sum_j h(d(U_j)) \geq \Phi_{i-\tau}(D) \sum_j \alpha_{i-\tau}(U_j) \geq \Phi_{i-\tau}(D)(1 - \varepsilon) \]
and so
\[ \inf \left\{ \sum_j h(d(U_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon \right\} \geq \Phi_{i-\tau}(D). \tag{13} \]

The assertion of the lemma follows from (10), (11) and (13).

Now, we shall prove the main result of this paper.

**Theorem 8.** Suppose $F$ is the self-similar set under $S_1, \ldots, S_m$ with common ratio $c$ and let $\tau$ be the index of $F$. Let $h \in \mathcal{H}^c_{m^*}$. Suppose $i (\geq \tau)$ is a positive integer such that (2) holds for all $0 < t \leq c^{i-\tau}d(D)$. Then for any $a \in \mathcal{I}_{i-\tau}$ \( \mu^h(F) = m^i \Phi_i(S_a(D)). \)

**Proof.** Let $j (> 2\tau)$ be a sufficiently large integer for which (2) holds for all $0 < t \leq c^{j-2\tau}d(D)$. By (3), we get
\[ \min \{\rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{j-\tau}; b \neq b'\} \geq c^jd(D). \]

Hence, by Lemma 7, we have
\[ \inf \left\{ \sum_k h(d(U_k)) : \{U_k\} \text{ is a } c^j d(D)-\text{covering of } F \right\} \]
\[ = \sum_{b \in \mathcal{I}_{j-\tau}} \inf \left\{ \sum_k h(d(U_k)) : \{U_k\} \text{ is a } c^j d(D)-\text{covering of } F \cap S_b(D) \right\} \]
\[ = m^{j-\tau} \Phi_{j-\tau}(D). \]
Thus, it follows from Lemma 6 and Lemma 4 that

\[ \mu^h(F) = \lim_{j \to \infty} \inf \left\{ \sum_k h(d(U_k)) : \{U_k\} \text{ is a c}^j d(D)-\text{covering of } F \right\} \]

\[ = \lim_{j \to \infty} m^{j-\tau} \Phi_{j-\tau}(D) \]

\[ = m^i \Phi_i(D) \]

\[ = m^i \Phi_i(S_a(D)). \]

4 Applications

First, we introduce the definition of the uniform Cantor set. Let \( m \geq 2 \) be a given integer, and we choose positive real numbers \( c \) and \( d \) such that \( mc + (m-1)d = 1 \). In this section, let \( D = [0,1] \) and define the similarities \( S_i : D \to D \) \((i = 1, \ldots, m)\) by \( S_i(x) = (i-1)(c+d) + cx \). Then \( c \) is the common ratio of the similarities \( S_1, \ldots, S_m \) and the family of those similarities is disjoint. The self-similar set \( F \) under the \( S_i \)'s is called a uniform Cantor set.

Let \( C \) be a compact subset of \( \mathbb{R} \) and let \( t > 0 \) be given. The \( t \)-entropy of \( C \) is defined by

\[ E(C,t) = \min \{ n \in \mathbb{N} : \{U_1, \ldots, U_n\} \text{ is a } t\text{-covering of } C \}. \]

Mycielski [8] and Kahnert [7] have considered the Hausdorff function

\[ h_C(t) = \begin{cases} \frac{1}{E(C,t)} & \text{for } t > 0, \\ 0 & \text{for } t = 0 \end{cases} \]

to construct an invariant Hausdorff measure \( \mu^{hc} \). From now on, we write \( \mu^C \) instead of \( \mu^{hc} \). For some interesting properties of such a measure. See [3, 4, 8, 9].

It is easy to see \( \mu^D(F) = 0 \). Obviously, the Cantor set \( F \) can be covered by \( m^n \) intervals of the length \( c^n \) for every positive integer \( n \). Since \( h_D(c^n) \leq c^n \), (1) implies \( \mu^D(F) \leq \lim_{n \to \infty} m^n c^n = 0 \).

Analogously, we can easily prove \( \mu^F(D) = \infty \). Let \( \{U_i\} \) be a \( c^n \)-covering of \( D \). Then \( \sum d(U_i) \geq 1 \). Let \( n_i \geq n \) be an integer with \( c^{n_i+1} < d(U_i) \leq c^{n_i} \) for every \( i \). Then

\[ h_F(d(U_i)) \geq m^{-(n_i+1)} = c^{n_i} m^{-1}(mc)^{-n_i} > d(U_i)m^{-1}(mc)^{-n} \]
and hence
\[ \sum_i h_F(d(U_i)) \geq m^{-1}(mc)^{-n} \sum_i d(U_i) \geq m^{-1}(mc)^{-n}. \]

Therefore, (1) implies \( \mu^F(D) \geq \lim_{n \to \infty} m^{-1}(mc)^{-n} = \infty. \)

Similarly, we can show \( \mu^D(D) = 1 \) and \( \mu^F(F) \leq 1 \), but it is virtually impossible to determine the exact value of \( \mu^F(F) \) by using formula (1). However, we can use Theorem 8 to evaluate the lower and upper bounds for the set \( \{ \mu^F(F) : F \text{ is a uniform Cantor set} \} \) as well as the exact values of \( \mu^F(F) \) for many special cases. Moreover, Theorem 8 might provide us with a possibility to evaluate the value of \( \mu^F(F) \) within a given error.

**Remark 2.** Let \( C \) be a compact subset of \( \mathbb{R} \) with a positive Lebesgue measure. Then we can also prove that \( \mu^C(C) = 1. \)

In the following lemma, we prove \( h_F \in \mathcal{H}_m^c. \)

**Lemma 9.** \( h_F \in \mathcal{H}_m^c. \)

**Proof.** Let \( n = E(F,t) \). Suppose \( \{U_i\}_{i=1,...,n} \) is a \( t \)-covering of \( F \). Then \( \{S_j(U_i)\}_{i=1,...,n} \) is a \( ct \)-covering of \( F \cap S_j(D) \) for any \( j = 1,\ldots,m \). Hence, we obtain \( E(F,ct) \leq mE(F,t). \)

On the other hand, let \( n = E(F \cap S_j(D),ct) \) and suppose \( \{U_i\}_{i=1,...,n} \) is a \( ct \)-covering of \( F \cap S_j(D) \) for some \( j = 1,\ldots,m \). Then \( \{S_j^{-1}(U_i)\}_{i=1,...,n} \) is a \( t \)-covering of \( F \). Hence, if \( 0 < t < d/c \), then since \( ct < d \)

\[ E(F,t) \leq E(F \cap S_j(D),ct) = \frac{1}{m} E(F,ct). \]

Let \( \tau \) be the index of \( F \). According to Theorem 8 and Lemma 9, the value of \( \mu^F(F) \) can be evaluated by the formula

\[ \mu^F(F) = m^i \cdot \inf \left\{ \frac{h_F(d(I))}{\alpha_2(I)} : I \subset [0,c^{i-\tau}]; \ c^{i+1} < d(I) \leq c^i \right\} \quad (14) \]

where \( i (\geq \tau) \) is a sufficiently large integer.

**Theorem 10.** For any uniform Cantor set \( F \) \( 1/2 \leq \mu^F(F) < 1. \)

**Proof.** (a) First, assume that \( I (\subset [0,c^{i-\tau}]) \) is an interval with

\[ (k+1)c^{i+1} + kc^i d < d(I) \leq (k+1)(c^{i+1} + c^i d) \]
for some $k \in \{0, 1, \ldots, m - 2\}$. By considering the structure of $F$ and the fact $E(S_a(D), d(I)) \leq [(m + k)/(k + 1)]$ ($a \in \mathcal{I}_i$), where $[x]$ denotes the greatest integer which does not exceed $x$, we conclude

$$E(F, d(I)) \leq m^i \left[ \frac{m + k}{k + 1} \right] \text{ and } \alpha_i(I) \leq \frac{k + 1}{m}.$$  

Hence, the fact $m \leq (k + 1)[(m + k)/(k + 1)] \leq m + k \leq 2m - 2$ implies

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} \geq \frac{m}{(k + 1)[(m + k)/(k + 1)]} \geq \frac{1}{2}.$$  

Now, assume that $I (\subset [0, c_i - \tau])$ is an interval with

$$k(c_i + 1 + c_i d) < d(I) \leq (k + 1)c_i + 1 + kc_i d$$

for some $k \in \{1, 2, \ldots, m - 1\}$. As in the previous case, we obtain

$$E(F, d(I)) \leq m^i \left[ \frac{m + k - 1}{k} \right], \quad \alpha_i(I) \leq \frac{k + 1}{m},$$

and hence the inequality

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} \geq \frac{m}{(k + 1)[(m + k - 1)/k]} \geq \frac{1}{2}$$

follows from the fact $\frac{m + k - 1}{k} \leq \frac{2m}{k + 1}$ which can be easily proved under the assumption $m \geq k + 1$.

Altogether, Theorem 8, together with (15) and (16), implies $\mu_F(F) \geq 1/2$.

(b) Consider an interval $I = [0, c_i - c_i^2]$. By (3) we have

$$\min \{\rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{i-\tau}; \ b \neq b'\} \geq c_i.$$  

Therefore, if $a \in \mathcal{I}_{i-\tau}$, then

$$E(F, d(I)) = m^{i-\tau} E(F \cap S_a(D), d(I)) \geq m^{i-\tau} (m^\tau + 1)$$

and

$$\alpha_i(I) = \frac{m^i - 1}{m^\tau}.$$  

Hence, by (14), (17) and (18), we obtain

$$\mu_F(F) \leq m^i \frac{h_F(d(I))}{\alpha_i(I)} \leq \frac{m^\tau}{m^\tau + 1} \frac{m^i}{m^i - 1} < 1$$

because $i$ can be arbitrarily large.

\[ \square \]

As we can see in the following theorem, $1/2$ is the best possible estimation of the lower bound for $\mu_F(F)$ in Theorem 10.
Theorem 11. Let $\tau$ be the index of the uniform Cantor set $F$.

(a) If $\tau = 0$, then $\mu^F(F) = 1/2$.

(b) $\mu^F(F) \to 1$ as $\tau \to \infty$.

Proof. (a) Let $i, k$ be positive integers. Choose an interval $I = [0, c^i - c^{i+k}]$. Clearly, we obtain $h_F(d(I)) = \frac{1}{2m^\tau}$, $\alpha_i(I) = 1 - \frac{1}{m^\tau}$ and hence

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} = \frac{1}{2} \frac{m^k}{m^i - 1}. \tag{19}$$

By letting $k \to \infty$ in (19) and considering (14) and Theorem 10, we conclude $\mu^F(F) = 1/2$.

(b) Let $\tau$ and $i (> \tau)$ be sufficiently large integers. Suppose $I (\subset [0, c^i-\tau])$ is an interval for which there exists an $n \in \mathbb{N} \cup \{\infty\}$ such that $\ell_n(k_j) - c^{i+n} - 1d \leq d(I) \leq \ell_n(k_j)$ with

$$\ell_n(k_j) = \sum_{j=1}^\infty k_j (c^{i+j} + c^{i+j-1}d)$$

where $k_j \in \{0, 1, \ldots, m-1\}$, $k_k \neq 0$ and $k_j = 0$ for any $j > n$ (for the case of $n \in \mathbb{N}$) and we follow the convention $c^{i+n} - 1d$ for any $j > n$ (for the case of $n \in \mathbb{N}$) and we follow the convention $c^{i+n} = 0$ (for $n = \infty$). In view of (3) we have

$$E(F, d(I)) \leq m^i \tau \left( \frac{c^{i-\tau}}{\ell_n(k_j) - c^{i+n} - 1d} + 1 \right) \leq m^i \tau \frac{c^{i-\tau} + 1}{k_1 c + k_2 c^2 + \cdots};$$

$$\alpha_i(I) \leq \frac{k_1}{m} + \frac{k_2}{m^2} + \cdots$$

and hence

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} \geq \frac{(mc)^\tau (k_1 c + k_2 c^2 + \cdots)}{1 + c^{i-\tau}} \left( \frac{k_1}{m} + \frac{k_2}{m^2} + \cdots \right)^{-1} \geq \frac{(mc)^\tau}{1 + c^{i-\tau}} g(\tau)$$

where $g(\tau) \to 1$ as $\tau \to \infty$ ($c \to 1/m$ as $\tau \to \infty$). By (14) we get

$$\mu^F(F) \geq \frac{(mc)^\tau}{1 + c^{i-\tau}} g(\tau). \tag{20}$$

From $mc + (m - 1)d = 1$ and $c^{i+1} \leq d < c^i$ (see Definition 5) it follows that $(mc)^\tau \to 1$ as $\tau \to \infty$. Consequently, by Theorem 10 and (20), we conclude that $\mu^F(F) \to 1$ as $\tau \to \infty$. □

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References


