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Robert E. Zink, Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, U.S.A. email: zink@math.purdue.edu

ON THE HAAR MEASURES IN TOPOLOGICAL FIELDS

Abstract

By virtue of the uniqueness theorem for the Haar measure on a topological group, a simple argument is sufficient to show that the Haar measures on a locally–compact topological field, corresponding to the additive and multiplicative structures of the field, are absolutely continuous with respect to one another.

1 Prologue.

In his consideration of the interplay between measure theory and topology in a locally–compact topological group, Halmos [3] has shown that a relatively–invariant measure on the Borel σ–ring of subsets of the group and the Haar measure on the group are connected by an integral relationship. Since the additively–invariant Haar measure on a locally–compact topological field proves to be multiplicatively, relatively invariant, one is led to suspect that the additive and multiplicative Haar measures on the field should be related in the manner suggested by the foregoing result. Indeed, Halmos has observed that this is the case in the ordinary real field, and the resolution of this conjecture in the general setting involves only the application of some of the nifty ideas supplied by Halmos, and by Weiss and Zierler [5], and the time–honored techniques of measure theory.

2 Exposition.

Let X be a proper, locally–compact topological field, let $X^+ = (X; +)$ and $X^* = (X^*; \cdot)$ be the associated additive and multiplicative subgroups, let μ

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and ν be the Haar measures corresponding to X^+ and X^*, and let $\mathcal{S}$ denote the σ–ring of Borel subsets of X. Since a proper field topology is nontrivial, as well as nondiscrete, X is a Hausdorff space.

Lemma 1. *Let $x \in X$, and let μ_x be the measure given by the relation*

$$\mu_x(E) = \mu(xE), \forall\ E \in \mathcal{S}.$$

Then μ_x is invariant under translations, and thus, by virtue of the uniqueness of the Haar measure, μ_x is a constant multiple of μ.

PROOF. For each $b \in X$, one has

$$\begin{aligned} \mu_x(b+E) &= \mu(x(b+E)) = \mu(xb+xE) \\ &= \mu(xE) = \mu_x(E),\ \forall\ E \in \mathcal{S}; \end{aligned}$$

hence,

$$\mu_x = \phi(x)\mu.$$

□

In a lovely treatment of the classification problem for locally–compact division rings, Weiss and Zierler [5] have shown that this classification can be determined in short order by approaching the problem from a measure–theoretic point of view. Ab initio they observe that the real–valued function ϕ is everywhere continuous, and from this fundamental property follows the absolute continuity of μ and ν with respect to one another.

Lemma 2. *The function*

$$\phi \colon X \to \mathbb{R}$$

satisfies the following conditions:
(i) $\phi(x) \geq 0$, $\forall\ x \in X$; $\phi(x) = 0$ iff $x = 0$;
(ii) $\phi(xy) = \phi(x)\phi(y)$, $\forall\ (x, y) \in X \times X$;
(iii) There exists a positive number M such that

$$\phi(1+a) \leq M,\ \forall\ a \in X \ni \phi(a) \leq 1;$$

(iv) ϕ is everywhere continuous.

The validity of the first two of these propositions is evident, and demonstrations of the latter two can be found in [5]. From (iii) follows also the condition

$$\phi(x+y) \leq \phi(x) + \phi(y),\ \forall\ (x, y) \in X \times X,$$

so that ϕ is a valuation for X. (In the ordinary real field, for example, ϕ is simply the absolute value function.)

Lemma 3. *One has, for each f, nonnegative and measurable $(\mathcal{S})$,*

$$\int_{aE} f(x)d\mu(x) = \int_E f(ax)d\mu_a(x),\ \forall E \in \mathcal{S},\ \forall\dashv \in \mathcal{X},$$

where $\mu_a = \phi(a)\mu$.

Proof. (i) For $F \in \mathcal{S}$, $\int_{aE} \chi_F(x)d\mu(x) = \mu(aE \cap F)$, and

$$\begin{aligned}
\int_E \chi_F(ax)d\mu_a(x) &= \int_E \chi_{a^{-1}F}(x)d\mu_a(x) = \mu_a(E \cap a^{-1}F) \\
&= \mu(a(E \cap a^{-1}F)) = \mu(aE \cap F).
\end{aligned}$$

(ii) For each nonnegative, simple function $f = \sum_{j=1}^n c_j\chi_{F_j}$,

$$\begin{aligned}
\int_{aE} f(x)d\mu(x) &= \sum_{j=1}^n c_j \int_{aE} \chi_{F_j} d\mu(x) = \sum_{j=1}^n c_j\mu(aE \cap F_j) \\
&= \sum_{j=1}^n c_j\mu(a(E \cap a^{-1}F_j) = \sum_{j=1}^n c_j\mu_a(E \cap a^{-1}F_j) \\
&= \sum_{j=1}^n c_j \int_E \chi_{a^{-1}F_j}(x)d\mu_a(x) = \sum_{j=1}^n c_j \int_E \chi_{F_j}(ax)d\mu_a(x) \\
&= \int_E \sum_{j=1}^n c_j\chi_{F_j}(ax)d\mu_a(x) = \int_E f(ax)d\mu_a(x).
\end{aligned}$$

(iii) For f nonnegative and measurable $(\mathcal{S})$, let $\{f_n\}_{n=1}^\infty$ be a nondecreasing sequence of simple functions for which $f = \lim_n f_n$. Then

$$\begin{aligned}
\int_{aE} f(x)d\mu(x) &= \lim_n \int_{aE} f_n(x)d\mu(x) \\
&= \lim_n \int_E f_n(ax)d\mu_a(x) = \int_E f(ax)d\mu_a(x).
\end{aligned}$$

□

Lemma 4. *For each Borel set E, the restriction of ϕ to the domain E is measurable $(\mathcal{S})$.*

Proof. If E be compact, then for each $c \in \mathbb{R}$, $\{x \colon \phi(x) \leq c\} \cap E$ is a closed subset of E and is, thus, compact and so measurable. In the general case, there is a countable family $\{C_n \colon n \in \}$ of compact sets such that $E \subset \bigcup_{n=1}^\infty C_n[[3]; 24]$. Hence, for each $c \in \mathbb{R}$, $\{x \colon \phi(x) \leq c\} \cap \bigcup_{n=1}^\infty C_n \in \mathcal{S}$, and thus also $\{x \colon \phi(x) \leq c\} \cap E \in \mathcal{S}$ as well. □

Theorem 5. *The set function θ, well defined by*

$$\theta(E) = \int_E \frac{1}{\phi(x)} \, d\mu(x), \ \forall \ E \in \mathcal{S},$$

is "the" multiplicative Haar measure.

PROOF. For each $E \in \mathcal{S}$ and $a \in X^*$, one has

$$\begin{aligned} \theta(aE) &= \int_{aE} \frac{1}{\phi(x)} \, d\mu(x) = \int_E \frac{1}{\phi(ax)} \, d\mu_a(x) \\ &= \int_E \frac{1}{\phi(ax)} \, \phi(a) d\mu(x) = \int_E \frac{1}{\phi(x)} \, d\mu(x) = \theta(E); \end{aligned}$$

thus, θ is multiplicatively invariant and so must be a reasonable facsimile of ν. □

Now let λ be the measure given by

$$\lambda(E) = \int_E \phi(x) d\theta(x), \quad \forall \ E \in \mathcal{S}.$$

Lemma 6. *For every f nonnegative and measurable ($\mathcal{S}$),*

$$\int_E f(x) d\lambda(x) = \int_E f(x)\phi(x) d\theta(x),$$

and

$$\int_E f(x) d\theta(x) = \int_E f(x) \, \frac{1}{\phi(x)} \, d\mu(x).$$

PROOF. Here, again, the standard technique, as employed above, suffices. □

Corollary 7. $\lambda = \mu$.

PROOF. For each $E \in \mathcal{S}$,

$$\lambda(E) = \int_E \phi(x) d\theta(x) = \int_E \phi(x) \, \frac{1}{\phi(x)} \, d\mu(x) = \mu(E).$$

□

3 Epilogue

An unexpected corollary of this brief study of the Haar measures in a topological field is a serendipitous consequence of the fundamental work of Weiss and Zierler.

Theorem 8. *Every proper, locally–compact topological field is σ–compact.*

Proof. Let X be a topological field of this nature. It suffices to show that, for every natural number t, $S_t =: \{x\colon \phi(x) \leq t\}$ is compact, since $X = \bigcup_{t=1}^{\infty} S_t$. This fact, in turn, is essentially contained in one of the demonstrations given in [5], an adaptation of which follows.

Let C be a compact neighborhood of 0, and let V be a neighborhood of 0 such that $VC \subset C$, and let a be an element of $V \cap C$ for which $0 < \phi(a) < 1$.

Suppose that, for some n, $a^n S_t \subset C$. Then $S_t \subset a^{-n}C$. Now S_t is closed and $a^{-n}C$ is compact, and so S_t must be compact.

Suppose, on the other hand, that the inclusion $a^n S_t \subset C$ holds for no $n \in \mathbb{N}$. Then, for each n, there is an $s_n \in S_t$ such that $a^n s_n \notin C$. Since each $a^k \in C$ and $\phi(a) < 1$, $\lim_k \phi(a^k) = 0$ yields $\lim_k a^k = 0$. Thus, for n fixed, $a^k s_n \in C$, for all sufficiently large k, by virtue of the continuity of multiplication. From the well–ordering principle follows the existence of $k_n \geq n$ such that $a^{k_n} s_n \notin C$ but $a^{k_n+1} s_n \in C$. But then each $a^{k_n} s_n$ lies in the compact set $a^{-1}C$, so that $\{a^{k_n} s_n\colon n \in \mathbb{N}\}$ has a cluster point, c, in $a^{-1}C$. Since $\phi(a^{k_n} s_n) = (\phi(a))^{k_n} \phi(s_n) \leq t(\phi(a))^n$, $\lim_n \phi(a^{k_n} s_n) = 0$. From the continuity of ϕ follows also $\phi(c) = 0$, and thus $c = 0$. Since no $a^{k_n} s_n \in C$, this is impossible, and so one must have $a^n S_t \subset C$, for some $n \in \mathbb{N}$. □

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