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Olena Karlova, Department of Mathematical Analysis, Chernivtsi National University, Chernivtsi 58012, Kotsjubyns'koho 2, Ukraine. email: okarlova@ukr.net

EXTENSION OF CONTINUOUS FUNCTIONS TO BAIRE-ONE FUNCTIONS

Abstract

We introduce the notion of B_1-retract and investigate the connection between B_1- and H_1-retracts.

1 Introduction.

Recall that a function $f : X \to Y$ between topological spaces X and Y *belongs to the first Baire class* or *is a Baire-one function*, if it is a pointwise limit of a sequence of continuous functions. A function f *belongs to the first Lebesgue class* or *is a Lebesgue-one function* if $f^{-1}(F)$ is a G_δ-set in X for every closed set F in Y. We shall denote by $B_1(X, Y)$ $(H_1(X, Y))$ the collection of all functions of the first Baire (Lebesgue) class from X to Y.

K. Kuratowski [9, p. 445] proved that every continuous function $f : E \to \mathbb{R}$ on an arbitrary subset E of a metric space X can be extended to a Lebesgue-one function on the whole space. According to Lebesgue-Hausdorff Theorem [9, p. 402] the extension is also a Baire-one function.

O. Kalenda and J. Spurný [7] showed that if E is a Lindelöf hereditarily Baire subspace or E is a Lindelöf G_δ-subspace of a completely regular space X then every Baire-one function $f : E \to \mathbb{R}$ can be extended to a Baire-one function on the whole space.

It was proved in [13] that any Baire-one function with values in a σ-metrizable space with some additional conditions and defined on a Lindelöf G_δ-subspace of a normal space can be extended to a Baire-one function on the whole space.

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The question about the extension of the class of range spaces of extendable functions naturally arises. In [8] the notion of H_1-retract was introduced. A subset E of a topological space X is called *an H_1-retract* if there exists a Lebesgue-one function $r : X \to E$ such that $r(x) = x$ for all $x \in E$. It was shown in [8] that E is an H_1-retract of X iff for any topological space Y and for any continuous function $f : E \to Y$ there exists an extension $g \in H_1(X, Y)$ of f. The following result was established in [8].

Theorem 1.1. *[8, Corollary 3.3] A set E is an H_1-retract of a completely metrizable space X if and only if E is a G_δ-set in X.*

In this paper we introduce the notion of B_1-retract and prove several of its properties. Further, using Theorem 1.1 and the generalization of Lebesgue-Hausdorff Theorem, we find out that B_1-retracts and H_1-retracts are tightly connected in many cases. And in the last section we give two examples which show that even for subsets of the plane $\mathbb{R}^2$ the notions of retract, H_1-retract and B_1-retract are different.

2 B_1-retracts and their properties.

Recall [1] that a subset E of a topological space X is said to be *a retract of* X if there exists a continuous function $r : X \to E$ such that $r(x) = x$ for all $x \in E$. The function r is called *a retraction* of X onto E. It is well-known that a set $E \subseteq X$ is a retract of X if and only if for any topological space Y every continuous function $f : E \to Y$ can be extended to a continuous function $g : X \to Y$.

We call a subset E of a topological space X *a B_1-retract* of X if there exists a Baire-one function $r : X \to E$ such that $r(x) = x$ for all $x \in E$. We call the function r *a B_1-retraction* of X onto E.

Note that a composition of a continuous function and a Baire-one function is a Baire-one function. This fact and the definition of a B_1-retract immediately imply the following proposition.

Proposition 2.1. *Let X be a topological space. A set $E \subseteq X$ is a B_1-retract of X if and only if for any topological space Y every continuous function $f : E \to Y$ can be extended to a Baire-one function $g : X \to Y$.*

A subset A of a topological space X is called *a regular G_δ-set* [12] if there exists a sequence $(G_n)_{n=1}^{\infty}$ of open sets in X such that

$$A = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \overline{G}_n.$$

We say that a topological space X has *a regular G_δ-diagonal* if its diagonal $\Delta = \{(x,x) : x \in X\}$ is a regular G_δ-set in $X \times X$.

Obviously, every regular G_δ-diagonal is closed G_δ-set in $X \times X$. Note that every space with a G_δ-diagonal is Hausdorff.

Proposition 2.2. *Let X be a topological space with a regular G_δ-diagonal and E be a B_1-retract of X. Then E is a G_δ-set in X.*

PROOF. Since the diagonal Δ is regular G_δ-set, it can be represented as $\Delta = \bigcap\limits_{n=1}^{\infty} G_n = \bigcap\limits_{n=1}^{\infty} \overline{G}_n$, where $(G_n)_{n=1}^{\infty}$ is a decreasing sequence of open sets in $X \times X$. Let $r : X \to E$ be a B_1-retraction of X onto E. Consider a function $h : X \to X \times X$, $h(x) = (r(x), x)$. Then $h \in B_1(X, X \times X)$ and $E = h^{-1}(\Delta)$. Let $(h_n)_{n=1}^{\infty}$ be a sequence of continuous functions such that $h_n(x) \to h(x)$ for every $x \in X$. We claim that

$$h^{-1}(\Delta) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} h_n^{-1}(G_m).$$

Indeed, let $x \in h^{-1}(\Delta)$ and $m \in \mathbb{N}$. Then $h(x) \in G_m$. Since $h_n(x) \to h(x)$, there exists a number $n \geq m$ such that $h_n(x) \in G_m$. Now let x belongs to the right side of the equality. Then there exists a sequence $(n_m)_{m=1}^{\infty}$ such that $n_m \geq m$ and $h_{n_m}(x) \in G_m$ for every $m \in \mathbb{N}$. Assume that $h(x) \notin \Delta$. Then there exists a number m_0 such that $h(x) \notin \overline{G}_{m_0}$. Since $h_n(x) \to h(x)$, there exists a number $n_0 \geq m_0$ such that $h_n(x) \notin \overline{G}_{m_0}$ for all $n \geq n_0$. In particular, $h_{n_{n_0}}(x) \notin G_{m_0}$ since $n_{n_0} \geq n_0$. Taking into account that $G_{n_0} \subseteq G_{m_0}$, we conclude $h_{n_{n_0}}(x) \notin G_{n_0}$, a contradiction. Hence, $h(x) \in \Delta$.

Since G_m is an open set in $X \times X$ for every m and h_n is continuous for every n, the set $E = h^{-1}(\Delta)$ is G_δ in X. □

Notice that a B_1-retract, in general, is not a G_δ-set. For example, let X be a space of all functions $x : [0,1] \to [0,1]$ equipped with a topology of pointwise convergence, x be an arbitrary point from X and $E = \{x\}$. Since X is compact and non-metrizable, the diagonal of X is not a G_δ-set [4, p. 264], consequently, it is not a regular G_δ-set, and E is not a G_δ-set in X. But, clearly, E is a retract (and, therefore, E is a B_1-retract) of the space X.

In connection with the previous remark the following open question naturally arises.

Question 2.3. *Do there exist a Hausdorff space X with a G_δ-diagonal, but without a regular G_δ-diagonal, and a B_1-retract of X, which is not a G_δ-set?*

It is well-known that every retract of a connected topological space is also a connected space. The following result states that the same is true for B_1-retracts.

Proposition 2.4. *Let X be a connected topological space and E be a B_1-retract of X. Then E is a connected space.*

PROOF. Let $r : X \to E$ be a B_1-retraction of X onto E and $(r_n)_{n=1}^{\infty}$ be a sequence of continuous functions $r_n : X \to E$ such that $r_n(x) \to r(x)$ for every $x \in X$.

Assume the contrary. Then $E = E_1 \sqcup E_2$, where E_1 and E_2 are open in E non-empty sets. Since r_n is a continuous function and X is a connected space, the set $B_n = r_n(X)$ is connected for every n. Then $B_n \subseteq E_1$ or $B_n \subseteq E_2$ for every $n \in \mathbb{N}$. Choose any $x \in E_1$. Then $r_n(x) \to r(x) = x$. Since E_1 is an open set in E, there exists a number n_1 such that $r_n(x) \in E_1$ for all $n \geq n_1$. Then $r_n(x) \in B_n \cap E_1$, that is, $B_n \subseteq E_1$ for all $n \geq n_1$. Analogously, it can be shown that there exists a number $n_2 \in \mathbb{N}$ such that $B_n \subseteq E_2$ for all $n \geq n_2$. Hence, $B_n \subseteq E_1 \cap E_2$ for all $n \geq \max\{n_1, n_2\}$, a contradiction. □

A topological space Y is called *an equiconnected space* [3] if there exists a continuous function $\gamma : Y \times Y \times [0,1] \to Y$, which for every $y', y'' \in Y$ and $t \in [0,1]$ satisfies the following properties:

(i) $\gamma(y', y'', 0) = y'$,

(ii) $\gamma(y', y'', 1) = y''$,

(iii) $\gamma(y', y', t) = y'$.

We need the following auxiliary fact from [13].

Lemma 2.5. *[13, Lemma 2.1] Let X be a normal space, Y be an equiconnected space, $(F_i)_{i=1}^{n}$ be disjoint closed sets in X and $g_i : X \to Y$ be a continuous function for every $1 \leq i \leq n$. Then there exists a continuous function $g : X \to Y$ such that $g(x) = g_i(x)$ on F_i for every $1 \leq i \leq n$.*

PROOF. The proof is by induction on n. Let $n = 2$. Since F_1 and F_2 are disjoint and closed, by Urysohn's Lemma [4, p. 41] there exists a continuous function $\varphi : X \to [0,1]$ such that $\varphi(x) = 0$ on F_1 and $\varphi(x) = 1$ on F_2. The space Y is equiconnected, therefore there exists a continuous function $\gamma : Y \times Y \times [0,1] \to Y$, which satisfies (i) – (iii). Let

$$g(x) = \gamma(g_1(x), g_2(x), \varphi(x))$$

for every $x \in X$. Clearly, $g : X \to Y$ is continuous. If $x \in F_1$, then $\varphi(x) = 0$ and $g(x) = g_1(x)$. If $x \in F_2$, then $\varphi(x) = 1$ and $g(x) = g_2(x)$.

Assume the lemma is true for all $1 \le k < n$. We will prove it for $k = n$. According to the assumption, there exists a continuous function $\tilde{g} : X \to Y$ such that $\tilde{g}|_{F_i} = g_i$ for every $1 \le i < n$. Since $F = \bigcup\limits_{i=1}^{n-1} F_i$ and F_n are disjoint and closed in X, by the assumption there exists such a continuous function $g : X \to Y$ that $g|_F = \tilde{g}$ and $g|_{F_n} = g_n$. Then $g|_{F_i} = g_i$ for every $1 \le i \le n$. $\square$

We call a subset A of a topological space X *an ambiguous set* if it is simultaneously F_σ and G_δ in X. Recall that a topological space X is called *perfectly normal* if it is normal and every closed subset of X is a G_δ-set.

Theorem 2.6. *Let X be a perfectly normal space, $E \subseteq X$ be an equiconnected space, $E = \bigcup\limits_{n=1}^{\infty} E_n$, and the following conditions hold:*

(1) $E_n \cap E_m = \emptyset$ if $n \ne m$;

(2) E_n is an ambiguous set in E for every $n \in \mathbb{N}$;

(3) E_n is a B_1-retract of X for every $n \in \mathbb{N}$;

(4) E is a G_δ-set in X.

Then E is a B_1-retract of X.

PROOF. From the condition (2) and [9, p. 359] it follows that for every $n \in \mathbb{N}$ there exists an ambiguous set C_n in X such that $C_n \cap E = E_n$. Let $D_1 = C_1$ and $D_n = C_n \setminus \bigcup\limits_{k<n} C_k$ if $n \ge 2$. Then D_n is an ambiguous set for every n. Moreover, D_n are disjoint sets and $D_n \cap E = E_n$ for every $n \in \mathbb{N}$. Since $X \setminus E$ is an F_σ-set in X, there exists a sequence $(F_n)_{n=1}^{\infty}$ of closed subsets of X such that $X \setminus E = \bigcup\limits_{n=1}^{\infty} F_n$. Let $X_1 = F_1 \cup D_1$, and $X_n = (F_n \cup D_n) \setminus (\bigcup\limits_{k<n} (F_k \cup D_k))$ if $n \ge 2$. Obviously, X_n is an ambiguous set in X for every n, $X_n \cap X_m = \emptyset$ if $n \ne m$, and $X = \bigcup\limits_{n=1}^{\infty} X_n$.

We will show that $X_n \cap E = E_n$ for every $n \in \mathbb{N}$. Indeed, if $x \in X_n \cap E$, then

$$x \in (F_n \cup D_n) \cap E = (F_n \cap E) \cup (D_n \cap E) = D_n \cap E = E_n.$$

If $x \in E_n$, then $x \in D_n \cap E$, therefore $x \in D_n$ and $x \notin F_m$ for all m. Moreover, $x \notin D_k$ for all $k < n$, since $D_n \cap D_k = \emptyset$ if $n \ne k$. Hence, $x \in X_n \cap E$.

According to (3), there exists a sequence of B_1-retractions $r_n : X \to E_n$. Let $r(x) = r_n(x)$ if $x \in X_n$. We will show that $r \in B_1(X, E)$.

For every $n \in \mathbb{N}$ there exists a sequence $(r_{n,m})_{m=1}^{\infty}$ of continuous functions $r_{n,m} : X \to E_n$ such that $\lim\limits_{m\to\infty} r_{n,m}(x) = r_n(x)$ for every $x \in X$. Notice that

$\lim_{m\to\infty} r_{n,m}(x) = r(x)$ on X_n. The set X_n is F_σ, therefore, for every n there exists an increasing sequence $(B_{n,m})_{m=1}^\infty$ of closed subsets $B_{n,m}$ of X such that $X_n = \bigcup\limits_{m=1}^{\infty} B_{n,m}$. Let $A_{n,m} = \emptyset$ if $n > m$, and $A_{n,m} = B_{n,m}$ if $n \le m$. Then Lemma 2.5 implies that for every $m \in \mathbb{N}$ there exists a continuous function $g_m : X \to E$ such that $g_m|_{A_{n,m}} = r_{n,m}$ since a family $\{A_{n,m} : n \in \mathbb{N}\}$ is finite for every $m \in \mathbb{N}$.

It remains to prove that $g_m(x) \to r(x)$ on X. Fix $x \in X$. Then $x \in X_n$ for some $n \in \mathbb{N}$. The sequence $(A_{n,m})_{m=1}^\infty$ is increasing, and, in consequence, there exists m_0 such that $x \in A_{n,m}$ for every $m \ge m_0$. Then $g_m(x) = r_{n,m}(x)$ for all $m \ge m_0$. Hence, $\lim_{m\to\infty} g_m(x) = \lim_{m\to\infty} r_{n,m}(x) = r(x)$. Therefore, $r \in B_1(X, Y)$.

It is easy to see that $r(x) = x$ for all $x \in E$. Hence, r is a B_1-retraction of X onto E. $\square$

3 The connection between B_1-retracts and H_1-retracts.

Recall that a family $\mathcal{A}$ of subsets of a topological space X is *discrete* if every point $x \in X$ has a neighbourhood U that intersects at most one of the sets $A \in \mathcal{A}$. A family $\mathcal{A} = (A_i : i \in I)$ of subsets of a topological space X is said to be *strongly discrete* [11] if there is a discrete family $\mathcal{G} = (G_i : i \in I)$ of open sets in X such that $\overline{A}_i \subseteq G_i$ for any $i \in I$. A family $\mathcal{A}$ is *σ-discrete (strongly σ-discrete)* if it can be represented as the union of countably many discrete (strongly discrete) families in X.

A family $\mathcal{B}$ of subsets of topological space X is *a base for a function* $f : X \to Y$ if for any open set V in Y there exists a subfamily $\mathcal{B}_V \subseteq \mathcal{B}$ such that $f^{-1}(V) = \bigcup \mathcal{B}_V$. If $\mathcal{B}$ is (strongly) σ-discrete then it is called *(strongly) σ-discrete base for f* and function $f : X \to Y$ with (strongly) σ-discrete base is called *(strongly) σ-discrete function*. We shall denote by $\Sigma(X, Y)$ $(\Sigma^*(X, Y))$ the set of all (strongly) σ-discrete functions from X to Y.

A topological space X is *collectionwise normal* if X is T_1-space and for each discrete family $(F_i : i \in I)$ of closed sets there exists a discrete family $(G_i : i \in I)$ of open sets such that $F_i \subseteq G_i$ for every $i \in I$. It is easy to see that a space is collectionwise normal if and only if every discrete family of its subsets is strongly discrete.

Note that any function with values in a second countable topological space is strongly σ-discrete. R. Hansell [6] proved that every Lebesgue-one function with a complete metric domain space and a metric range space is σ-discrete. Taking into account that a complete metric space is collectionwise normal, we obtain the strongly σ-discreteness of such a function.

Recall that a topological space X is *arcwise connected* if for any two points x and y from X there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. A space X is called *locally arcwise connected* if for every $x \in X$ and for any its neighbourhood U there exists a neighbourhood V of x such that for each $y \in V$ there is a continuous function $f : [0,1] \to U$ such that $f(0) = x$ and $f(1) = y$.

We shall need the following results of L. Veselý [11] and M. Fosgerau [5] concerning the equality between Baire and Lebesgue classes.

Theorem 3.1. *[11, Theorem 3.7(i)] Let X be a normal space, Y be an arcwise connected and locally arcwise connected metric space. Then*

$$H_1(X,Y) \cap \Sigma^*(X,Y) = B_1(X,Y).$$

Theorem 3.2. *[5, Theorem 2] Let Y be a complete metric space. Then the following conditions are equivalent:*
(i) Y is connected and locally connected;
(ii) $H_1(X,Y) \cap \Sigma(X,Y) = B_1(X,Y)$ for any metric space X.

Theorem 3.3. *Let X be a normal space and E be an arcwise connected and locally arcwise connected metrizable ambiguous subspace of X. If one of the following conditions holds*
(i) E is separable, or
(ii) X is collectionwise normal,
then E is a B_1-retract of X.

PROOF. Fix any point $x^* \in E$ and define

$$r(x) = \begin{cases} x, & \text{if } x \in E, \\ x^*, & \text{if } x \in X \setminus E. \end{cases}$$

We claim that r is an H_1-retraction of X onto E. Indeed, take an arbitrary open set V in E. If $x^* \notin V$, then $r^{-1}(V) = V$. Since E is metrizable, V is an F_σ-set in E. Moreover, E is F_σ in X, therefore, V is F_σ in X. If $x^* \in V$, then $r^{-1}(V) = V \cup (X \setminus E)$. Since V and $X \setminus E$ are F_σ-sets in X, $r^{-1}(V)$ is also an F_σ-set in X.

(i) Since E is a second countable space, a function r is strongly σ-discrete. According to Theorem 3.1, $r \in B_1(X,E)$.

(ii) We show that $r : X \to E$ is strongly σ-discrete. Since E is F_σ in X, there exists an increasing sequence $(F_n)_{n=1}^\infty$ of closed subsets of X such that $E = \bigcup\limits_{n=1}^\infty F_n$. Note that every metrizable space has a σ-discrete base according to Bing's Theorem [4, p. 282], therefore, for every n we can choose a σ-discrete

base $\mathcal{U}_n$ of F_n. Then $\mathcal{U}_n = \bigcup\limits_{m=1}^{\infty} \mathcal{U}_{n,m}$, where $(\mathcal{U}_{n,m})_{m=1}^{\infty}$ is a sequence of discrete families in F_n, $n \in \mathbb{N}$. The set F_n is closed in X, and, consequently, $\mathcal{U}_{n,m}$ is discrete in X, $n, m \in \mathbb{N}$. Since X is collectionwise normal, the family $\mathcal{U}_{n,m}$ is strongly discrete in X. Then the families $\mathcal{U}_n$ and $\mathcal{U} = \bigcup\limits_{n=1}^{\infty} \mathcal{U}_n$ are strongly σ-discrete in X. Let $\mathcal{B} = \mathcal{U} \cup \{X \setminus E\}$. Then $\mathcal{B}$ is a strongly σ-discrete family in X.

We prove that $\mathcal{B}$ is a base for r. Let U be an open set in E. Then $U = \bigcup\limits_{n=1}^{\infty} (U \cap F_n)$. Since $U \cap F_n$ is an open set in F_n for every n, there exists a subfamily $\mathcal{U}_{n,U} \subseteq \mathcal{U}_n$ such that $U \cap F_n = \bigcup \mathcal{U}_{n,U}$. If $x^* \notin U$, then $r^{-1}(U) = U = \bigcup\limits_{n=1}^{\infty} \bigcup \mathcal{U}_{n,U}$. If $x^* \in U$, then $r^{-1}(U) = U \cup (X \setminus E) = \bigcup\limits_{n=1}^{\infty} \bigcup \mathcal{U}_{n,U} \cup (X \setminus E)$. Therefore, $\mathcal{B}$ is a strongly σ-discrete base for r. Hence, $r \in \Sigma^*(X, E)$.

By Theorem 3.1, $r \in B_1(X, E)$. □

Theorem 3.4. *Let X be a complete metric space and $E \subseteq X$ be an arcwise connected and locally arcwise connected G_δ-set. Then E is a B_1-retract of X.*

PROOF. Theorem 1.1 implies that there exists an H_1-retraction $r : X \to E$ of X onto E. Since X is complete, Hansell's Theorem [6, Theorem 3] implies that r is strongly σ-discrete. According to Theorem 3.1, $r \in B_1(X, E)$ and, therefore, r is a B_1-retraction of X onto E. □

4 Examples.

It is well-known that any retract of a locally connected space is also a locally connected space, and any retract of an arcwise connected space is an arcwise connected space too. We give an example which shows that for B_1-retracts it is not true.

We first prove the following auxiliary fact.

Lemma 4.1. *Let $A = [a, b] \times [c, d]$, $B_1 = A \setminus ((a, b) \times (c, d])$ and $B_2 = A \setminus ((a, b) \times [c, d))$. Then B_i is a retract of A for every $i = 1, 2$.*

PROOF. Let $p_1 = \left(\frac{a+b}{2}, d + 1\right)$ and $p_2 = \left(\frac{a+b}{2}, c - 1\right)$. For $i = 1, 2$ and $(x, y) \in A$ denote by $\ell_i(x, y)$ the line which connects (x, y) and p_i. Consider a function $\varphi_i : A \to B_i$ such that $\varphi_i(x, y)$ is the point of the intersection of $\ell_i(x, y)$ with B_i, $i = 1, 2$. It is easy to see that for every $i = 1, 2$ the function φ_i is a retraction of A onto B_i. □

Example 4.2. *There exists a connected closed B_1-retract of $[0,1]^2$ which is neither arcwise connected nor locally connected.*

Proof. Let

$$E=(\{0\}\times[0,1])\cup(\bigcup_{n=1}^{\infty}(\{\frac{1}{n}\}\times[0,1])\cup([\frac{1}{2n},\frac{1}{2n-1}]\times\{1\})\cup([\frac{1}{2n+1},\frac{1}{2n}]\times\{0\})).$$

It is not difficult to check that E is a connected closed subset of $[0,1]^2$, which is neither arcwise connected nor locally connected. We show that E is a B_1-retract of $[0,1]^2$.

Let

$$A_n=\left[\tfrac{1}{n+1},\tfrac{1}{n}\right]\times[0,1],\qquad B_n=\left[0,\tfrac{1}{n+1}\right]\times[0,1],\quad n\geq 1,$$

$$E_n=\left(\left[\tfrac{1}{n+1},\tfrac{1}{n}\right]\times[0,1]\right)\setminus\left(\left(\tfrac{1}{n+1},\tfrac{1}{n}\right)\times[0,1)\right) \text{ if } n \text{ is an odd number,}$$

$$E_n=\left(\left[\tfrac{1}{n+1},\tfrac{1}{n}\right]\times[0,1]\right)\setminus\left(\left(\tfrac{1}{n+1},\tfrac{1}{n}\right)\times(0,1]\right) \text{ if } n \text{ is an even number.}$$

By Lemma 4.1, E_n is a retract of A_n for every n. Denote by φ_n a retraction of A_n onto E_n, $n\in\mathbb{N}$. Let ψ_n be a continuous function $\psi_n: B_n\to\{\frac{1}{n+1}\}\times[0,1]$, $\psi_n(x,y)=(\frac{1}{n+1},y)$.

For every $n\geq 1$ and $x,y\in[0,1]$ define

$$r_n(x,y)=\begin{cases}\varphi_k(x,y), & (x,y)\in A_k, 1\leq k\leq n,\\ \psi_n(x,y), & (x,y)\in B_n.\end{cases}$$

The function $r_n:[0,1]^2\to\bigcup\limits_{k\leq n}E_k$ is correctly defined and continuous for every n, since $\varphi_k|_{A_k\cap A_{k+1}}=\varphi_{k+1}|_{A_k\cap A_{k+1}}$, $1\leq k<n$, and $\varphi_n|_{A_n\cap B_n}=\psi_n|_{A_n\cap B_n}$.

We show that $(r_n)_{n=1}^{\infty}$ is a pointwise convergent sequence on $[0,1]^2$. Fix an arbitrary $(x,y)\in[0,1]^2$. If $x\neq 0$, then there exists n_0 such that $(x,y)\in A_{n_0}$. Then $r_n(x,y)=\varphi_{n_0}(x,y)$ for all $n\geq n_0$, that is $r_n(x,y)\underset{n\to\infty}{\to}\varphi_{n_0}(x,y)$. Note that if $(x,y)\in E$, then $\varphi_{n_0}(x,y)=(x,y)$. If $x=0$, then $r_n(x,y)=\psi_n(x,y)=(\frac{1}{n+1},y)\underset{n\to\infty}{\to}(0,y)=(x,y)$. Hence, there exists $\lim\limits_{n\to\infty}r_n(x,y)$ for each $(x,y)\in[0,1]^2$. We remark that $\lim\limits_{n\to\infty}r_n(x,y)=(x,y)$ on E. Moreover, since E is closed, $\lim\limits_{n\to\infty}r_n(x,y)\in E$ for all $(x,y)\in[0,1]^2$.

Set $r(x,y)=\lim\limits_{n\to\infty}r_n(x,y)$ for every $(x,y)\in[0,1]^2$. Then $r:[0,1]^2\to E$ is a B_1-retraction of $[0,1]^2$ onto E. □

Note that for the set E from the previous example there exists an H_1-retraction $r:[0,1]^2\to E$. Though E is a complete metric separable connected space, we cannot apply Theorem 3.2 for r since E is not locally connected.

Therefore, it is natural to ask: is every connected H_1-retract of a complete metric separable connected and locally connected space its B_1-retract? The following example shows that the answer to this question is negative.

Example 4.3. *There exists a connected H_1-retract of a complete metric separable connected and locally connected space which is not its B_1-retract.*

PROOF. Let $\mathbb{Q}_0 = \mathbb{Q} \cap [0,1] = \{q_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ consider a function $f_n : [0,1] \to [-1,1]$, $f_n(x) = \sin \frac{1}{x-q_n}$ if $x \neq q_n$, and $f_n(q_n) = 0$. For every $x \in [0,1]$ define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x),$$

$$X = [0,1] \times [-1,1] \quad \text{and} \quad E = \mathrm{Gr}(f) = \{(x,y) \in X : y = f(x)\}.$$

For every n the function $f_n : [0,1] \to [-1,1]$ belongs to the first Baire class, since it is discontinuous only in one point $x = q_n$ (see for instance [10, p. 384]). Moreover, f_n is a Darboux function (i.e. $f_n(A)$ is connected for every connected set $A \subseteq [0,1]$) [2, p. 13]. Then f is a Darboux Baire-one function, as the sum of the uniform convergent series of Darboux Baire-one functions [2, p. 13]. Therefore, the set E, as the graph of f, is connected [2, p. 9] and G_δ [9, p. 393] in X. Hence, according to Theorem 1.1 the set E is an H_1-retract of X.

We prove that E is not a B_1-retract of X. Assume that there exists a function $r \in B_1(X,E)$ such that $r(p) = p$ for all $p \in E$. Let $(r_n)_{n=1}^{\infty}$ be a sequence of continuous functions $r_n : X \to E$, which is pointwise convergent to r on X. Since X is compact and connected, $E_n = r_n(X)$ is also compact and connected for every n. Note that at least one of E_n contains more than one point. Indeed, assume that all the sets E_n consist of one point, i.e. $E_n = \{p_n\}$, where $p_n \in E$, $n \in \mathbb{N}$. Choose two different points p' and p'' from E. Then $p_n = r_n(p') \underset{n\to\infty}{\to} p'$ and $p_n = r_n(p'') \underset{n\to\infty}{\to} p''$, a contradiction. Hence, there exists a number n_0 such that E_{n_0} contains at least two different points (to be more precise, the cardinality of E_{n_0} is equal to $\mathfrak{c}$ since E_{n_0} is a connected set).

Now fix $p, q \in E_{n_0}$. Since $E_{n_0} \subseteq \mathrm{Gr}(f)$, the points p and q are represented as $p = (a, f(a))$ and $q = (b, f(b))$. Without loss of generality we can assume that $a < b$. Note that $(x, f(x)) \in E_{n_0}$ for any $x \in (a,b)$. Indeed, if there exists a point $x_0 \in (a,b)$ such that $(x_0, f(x_0)) \notin E_{n_0}$, then the line $x = x_0$ does not intersect E_{n_0}. Then, since E_{n_0} is connected, it should be completely contained either in the left hand half-plane, or in the right hand half-plane with respect to the line $x = x_0$. But this contradicts the fact that both p and q belong to E_{n_0}.

Since $\mathbb{Q}_0$ is dense in $[0,1]$, there exists a number k such that $q_k \in (a,b)$. Note that f is discontinuous in every point of $\mathbb{Q}_0$, in particular, it is discontinuous in q_k. Then there exists a sequence $(x_n)_{n=1}^{\infty}$, $x_n \in (a,b)$, such that $\lim\limits_{n\to\infty} x_n = q_k$, but $\lim\limits_{n\to\infty} f(x_n) \neq f(q_k)$. Since $(x_n, f(x_n)) \in E_{n_0}$ for every n and E_{n_0} is closed, the point $(\lim\limits_{n\to\infty} x_n, \lim\limits_{n\to\infty} f(x_n)) = (q_k, \lim\limits_{n\to\infty} f(x_n))$ also belongs to E_{n_0}. But then it must be equal to $(q_k, f(q_k))$, a contradiction.

Hence, E is not a B_1-retract of X. □

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