RESEARCH

Real Analysis Exchange
Vol. 36(1), 2010/2011, pp. 107–148

E. M. Bonotto,* Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil. email: ebonotto@icmc.usp.br

M. Federson, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil. email: federson@icmc.usp.br

P. Muldowney, Magee College, University of Ulster, Derry, BT48 7JL, Northern Ireland. email: p.muldowney@ulster.ac.uk

A FEYNMAN-KAC SOLUTION TO A RANDOM IMPULSIVE EQUATION OF SCHRÖDINGER TYPE

Abstract

If a force is applied to a particle undergoing Brownian motion, the resulting motion has a state function which satisfies a diffusion or Schrödinger-type equation. We consider a process in which Brownian motion is replaced by a process which has Brownian transitions at all times other than random times at which the transitions have an additional "impulsive" displacement. Using a Feynman-Kac formulation based on generalized Riemann integration, we examine the resulting equation of motion.

1 Introduction.

As an introduction, we give a broad outline of the underlying ideas and methodology of the paper.

Mathematical Reviews subject classification: Primary: 28C20, 35R12; Secondary: 46G12, 46T12

Key words: Henstock integral, Feynman-Kac formula, partial differential equations, impulse, Brownian motion

Received by the editors October 10, 2010

Communicated by: Luisa Di Piazza

*Supported by CNPq (140326/2005-7) and CAPES (BEX 4681/06-1).

1.1 Some Underlying Ideas.

When some system parameter has a discontinuity, the term “impulse” or “jump” can be a vivid way of describing this characteristic of the system.

Sometimes the state of a system can be described by a differential equation. For instance, a diffusion can be described by a parabolic partial differential equation satisfied by some function of displacement and time.

The purpose of this paper is to examine the relationship between discontinuities in the state function which characterizes the diffusion, and impulsive changes in the underlying diffusion itself. We use a Feynman-Kac formulation to show the connection between these two classes of discontinuities.

Our method of analysis is based on the generalized Riemann approach of Henstock. In effect, our Feynman-Kac formulation of the problem is a generalized Riemann (or Henstock) integral.

The generalized Riemann integral is an adaptation of the standard Riemann integral such that Riemann sums can be used to give results for which Lebesgue methods are usually required. The general idea of this is as follows. We have some domain which is partitioned by means of a finite collection $\{I\}$ of disjoint sets, which we can think of as “intervals”, with $|I|$ denoting the measure of an interval I. By “shrinking” the partitions, we can estimate the Riemann integral of a function $f(x)$ of values x in the domain by forming the Riemann sums $\sum f(x)|I|$.

In the standard Riemann integral, in any term $f(x)|I|$ of the Riemann sum, the only restriction on the choice of the evaluation point x is that it should belong to the corresponding partitioning interval I. The generalized Riemann adaptation is to make the selection of each interval I in the partition depend on the choice of each corresponding evaluation point x in $\sum f(x)|I|$. What difference does this make? It means that we can form the Riemann sums in a way which is sensitive to, or responsive to, the local behavior of the integrand. For instance, if f is highly oscillatory in a particular neighborhood, taking very large positive and negative values there, we can force the local terms of the Riemann sum to correspond to the local behavior of f. So in a scenario where f has a positive value at x and a negative value at the nearby point x', the partitioning intervals I, I' can be chosen so that the Riemann sum $\ldots+f(x)|I|+f(x')|I'|+\ldots$ captures the variation of f; so that, in this scenario, we produce in the Riemann sum a cancellation effect from the neighboring x, x'.

In this way it is found that we can define an integral of f which is equal to the Lebesgue integral of f whenever the latter exists. We call this the generalized Riemann integral (also known as Henstock integral or Henstock-Kurzweil integral).

Instead of using the Lebesgue measure $|I|$, we can use arbitrary interval functions $\mu(I)$, and the resulting definition of an integral $\int f(x)\mu(I)$ by Riemann sums remains valid. More generally, instead of integrating a product $f(x)\mu(I)$, we can integrate functions $h(x, I)$ by taking Riemann sum estimates $\sum h(x, I)$ over x-dependent partitions $\{I\}$ of the domain of integration.

The discussion above can be read in a way which assumes that the domain of integration is a bounded real interval $[a, b]$, so that each of the partitioning intervals I is itself a bounded real interval. But the points made in the discussion remain valid if the domain of integration is a more general, multi-dimensional space, such as $\mathbb{R}^n$, in which some of the partitioning intervals are not bounded or compact.

The scenario we tackle in this paper requires us to consider displacements x_t at various times t in some time interval (τ', τ), and also to consider the possibility that, at arbitrary times $\tau' < t_1 < \cdots < t_{n-1} < \tau$, the displacements x_{t_j} satisfy $u_j \leq x_{t_j} \leq v_j$ for $1 \leq j \leq n-1$; or $x_j \in \mathrm{Cl}(I_j)$ (closure of I_j), where we write $I_j = [u_j, v_j)$ and $x_j = x_{t_j}$ for each j.

Writing

$$x = (x_t)_{t \in (\tau', \tau)} \quad \text{and} \quad I = \{x : x_j \in I_j,\ 1 \leq j \leq n-1\}$$

we are led to consider Riemann sums such as $\sum f(x)\mu(I)$. The corresponding integrals are $\int f(x)\mu(I)$. The domain of integration is the set $\{x\}$, where each x is a mapping of the form

$$x :\ (\tau', \tau)\ \mapsto \mathbb{R}, \quad \text{with} \ \ x_t = x(t) \in \mathbb{R} \quad \text{for} \ \ \tau' < t < \tau.$$

We denote this domain by $\mathbb{R}^{(\tau', \tau)}$, which can be viewed as a Cartesian product of $\mathbb{R}$ by itself uncountably many times. The partitioning intervals I are cylindrical subsets of $\mathbb{R}^{(\tau', \tau)}$.

The framework of generalized Riemann integration outlined above can be adapted to this scenario, and this is explained in more detail in [8].

Treating the elements x as sample paths in some version of the Brownian motion, we develop a Feynman-Kac representation

$$u(\xi, \tau) = \int_{\mathbb{R}^{(\tau', \tau)}} f(x)\mu(I),$$

with $\xi := x(\tau)$, of the solutions $u(\xi, \tau)$ of a partial differential equation

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + V(\xi)u = 0,$$

where V is a potential function.

With the aid of this theoretical framework, we can relate discontinuities in $u(\xi, \tau)$ to "impulses" in the sample paths x.

1.2 Outline of the Theory.

To begin, we define the different kinds of cylindrical intervals

$$I = P_N^{-1}(I_1 \times \cdots \times I_{n-1})$$

which are used to partition the domain of integration $\mathbb{R}^{(\tau', \tau)}$. Given

$$N = \{t_1, \ldots, t_n\} \subset (\tau', \tau), \quad \text{with } \tau' = t_0 < t_1 < \cdots < t_{n-1} < t_n = \tau,$$

P_N is the projection function which maps $\mathbb{R}^{(\tau', \tau)}$ to the n-dimensional product space $\mathbb{R}^N$ or $\mathbb{R}^n$.

In general, if we want to define an integral $\int_{\mathbb{R}^{(\tau',\tau)}} f(x)\mu(I)$, we must show how the approximating Riemann sums $\sum f(x)\mu(I)$ are constructed, and that is done in [8].

We are especially interested in volume functions (or measures) μ on the sets I which are related to the Brownian motion function in which each difference $x(t_j) - x(t_{j-1})$ is normally distributed with mean zero and variance $t_j - t_{j-1}$, giving

$$\mu(I) = \int_{I_1} \cdots \int_{I_{n-1}} \prod_{j=1}^{n} \left(\frac{e^{-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}}}{\sqrt{2\pi(t_j - t_{j-1})}} \right) dy_1 \ldots dy_{n-1} \tag{1}$$

as the joint probability that the motion takes a value x_j in I_j at time t_j, for $1 \leq j \leq n-1$. (There is a technical, notational reason for having $1, \ldots, n-1$, rather than $1, \ldots, n$, as the range of j). The finite-dimensional expression on the right hand side of (1) is used to give meaning to the function on the left which has an infinite-dimensional domain. The essential simplicity of the expression on the right helps to simplify the analysis, in comparison with other formulations of the theory which introduce measurable sets at this point, instead of the simpler cylindrical intervals I. Because if the integrand $f(x)$ takes the value 1 for all x, then $\int_{\mathbb{R}^{(\tau',\tau)}} f(x)\mu(I)$ is approximated by Riemann sums $\sum \mu(I)$ whose value, in every case, turns out to be

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=1}^{m} \left(\frac{e^{-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}}}{\sqrt{2\pi(t_j - t_{j-1})}} \right) dy_1 \ldots dy_{m-1},$$

where $\{t_1, \ldots, t_{m-1}\}$ is the maximal of the sets of times $\{\cdots, t_j, \cdots\}$ which appear, as variable sets, in each of the terms $\mu(I)$ of the Riemann sum.

The latter integral can be evaluated by iterated integration, and this is demonstrated in [8]. Alternatively, with $x_0 = x(t_0) = x(\tau')$ and $x_n = x(t_n) = x(\tau)$, basic probability theory tells us that this integral gives the probability density function of a normal random variable $x(\tau) - x(\tau')$ with mean zero and variance $\tau - \tau'$, so the value of the integral is

$$\frac{e^{-\frac{(x(\tau)-x(\tau'))^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau - \tau')}}.$$

Some of these ideas occur in the analysis of Brownian motion, but are often expressed somewhat differently.

1.3 Impulsive Processes with Drift.

When the underlying Brownian process undergoes impulsive changes of amounts J_k at specified times τ_k, then we get a process

$$z = \{z_t\} = \{z(t) : t \in (\tau', \tau)\}, \quad \text{where } z(t) = x(t) + \sum_{\tau_k \leq t} J_k.$$

From this we are led to consider a measure

$$\mu(I) = \int_{I_1} \cdots \int_{I_{n-1}} \prod_{j=1}^{n} \left(\frac{e^{-\frac{1}{2}\frac{(z_j - y_{j-1})^2}{t_j - t_{j-1}}}}{\sqrt{2\pi(t_j - t_{j-1})}} \right) dy_1 \ldots dy_{n-1},$$

where $z_j = y_j + J_j$, if t_j is one of the instants τ_k, and $z_j = y_j$ otherwise.

Our purpose is to investigate systems in which the impulsive process is subject to some external force which produces a further manifestation of drifting motion in the process, represented by a potential function V which, at any time t, depends on the displacement $x(t)$. The study of systems of this kind leads to examination of the function

$$\begin{aligned} f(x) &= \exp\left(-\sum_{j=1}^{n} V(z(t_j))(t_j - t_{j-1}) \right) \\ &= \exp\left(-\sum_{j=1}^{n} V\left(x(t_j) + \sum_{\tau_i \leq t_j} J_i \right) (t_j - t_{j-1}) \right) \end{aligned}$$

defined for $x \in \mathbb{R}^{(\tau',\tau)}$. (Indeed, f depends also on the choice of the set $\{t_1, \ldots, t_n\}$ which, like x, is a variable in successive terms of a Riemann sum. Suitable notation for this dependence is presented in a later section.)

The state function $u(\xi, \tau)$ describing the evolution of this system, with $\xi = x(\tau)$, is often obtained as a solution to an appropriate parabolic diffusion equation, and sometimes this has a Feynman-Kac representation

$$u(\xi, \tau) = \int_{\mathbb{R}^{(\tau',\tau)}} f(x)\mu(I).$$

We investigate each of these methods of determining $u(\xi, \tau)$ and, by examining Riemann sum estimates of u, we show how discontinuities in u are related to impulsive phenomena in the underlying process z. The latter can be regarded as initial conditions or boundary conditions for the constitutive partial differential equation.

Our investigation is restricted to impulses J which are functions of the displacement $x(t)$, at random times. In the concluding section of the paper, we illustrate the theory by explicit evaluation of u when each of the impulse functions is a constant.

2 The Henstock Integral in Function Space.

In this section we present the basic definitions and notation of the theory of Henstock Integral in Function Spaces. We also include some fundamental results which are necessary for understanding the basis of the theory.

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_+$ denote the set of positive real numbers. Let I be a real interval of the form:

$$(-\infty,\, v), \qquad [u,\, v) \quad \text{or} \quad [u,\, +\infty). \tag{2}$$

A partition of $\mathbb{R}$ is a finite collection of disjoint intervals I whose union is $\mathbb{R}$. We say that I is attached to x (or associated with x) if

$$x = -\infty, \qquad x = u \text{ or } v, \qquad x = +\infty,$$

respectively.

Let $\overline{\mathbb{R}}$ denote the union of the domain of integration $\mathbb{R}$ with the set of associated points x of the intervals I of $\mathbb{R}$, so that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

In generalized Riemann integration, the convention is that the domain of integration is the space which is partitioned by intervals. A point x is not always an element of the associated interval I to which it is attached. Thus the set of associated points x may constitute a set which differs from the domain of integration. In our case, the domain of integration is $\mathbb{R}$ and the set of associated points is $\overline{\mathbb{R}}$.

Let $\delta : \overline{\mathbb{R}} \to \mathbb{R}_+$ be a positive function defined for $x \in \overline{\mathbb{R}}$. If I is attached to x, we say that (x, I) is δ-fine if

$$v < -\frac{1}{\delta(x)}, \qquad v - u < \delta(x), \qquad \text{or} \qquad u > \frac{1}{\delta(x)}, \tag{3}$$

respectively.

In this version of the integral, the attached or associated points x of an interval I are its vertices. In another version (see [9]), they are chosen from the union of I with the vertices: that is, the closure of I in the open-interval topology. These two versions are equivalent whenever the integrator (measure or interval function) is finitely additive, because if x is an interior point of $[u, v)$, then $f(x)m([u, v)) = f(x)m([u, x)) + f(x)m([x, v))$, see [9]. In yet another version (see [7]), an equivalent of the Lebesgue integral is produced if the associated or attached intervals of a point x are the intervals I satisfying $I \subseteq (x - \delta(x), x + \delta(x))$. In this case, the attached points of an interval may be outside the closure of I in the open-interval topology. In all cases, the domain of integration is the space which is partitioned by the intervals.

If $N = \{t_1, ..., t_n\}$ is a finite set, with $\mathbb{R}_{t_j} = \mathbb{R}$ and $\overline{\mathbb{R}}_{t_j} = \overline{\mathbb{R}}$, let $x = (x(t_1), ..., x(t_n))$ denote any element of

$$\prod \{\overline{\mathbb{R}}_{t_j} : t_j \in N\} = \overline{\mathbb{R}}^N.$$

Denote $x(t_j)$ by x_j, $1 \le j \le n$. For each $t_j \in N$, let $I_j = I(t_j)$ denote an interval of form (2). Then $I = I_1 \times ... \times I_n$ is an interval of $\prod\{\mathbb{R}_{t_j} : t_j \in N\} = \mathbb{R}^N$. A pair (x, I) is associated, or attached, in $\mathbb{R}^N$ if each (x_j, I_j) is associated in $\mathbb{R}$, $1 \le j \le n$, that is, x is a vertex of I in $\overline{\mathbb{R}}^N$. Given a function $\delta : \overline{\mathbb{R}}^N \to \mathbb{R}_+$, an associated pair (x, I) of the domain $\mathbb{R}^N$ is δ-fine if each (x_j, I_j) satisfies one of the inequalities in (3), depending on the kind of interval I_j (see (2)). A finite collection $\mathcal{E} = \{(x_j, I_j)\}$ of associated pairs (x_j, I_j), where each (x_j, I_j) is associated in $\mathbb{R}^N$, is a *division* of $\mathbb{R}^N$ if the intervals I_j are disjoint with union $\mathbb{R}^N$, and the division is δ*-fine* if each of the pairs (x_j, I_j), $1 \le j \le n$, is δ-fine. A proof of the existence of such a δ-fine division is given in [9], Theorem 4.1.

Let B denote any infinite set, and let $\mathcal{F}(B)$ denote the family of finite subsets of B. In what follows, we consider the product space $\prod_{t \in B} \mathbb{R}_t$ with $\mathbb{R}_t = \mathbb{R}$ for each $t \in B$, that is, the set of all functions on B to $\mathbb{R}$. We prefer to use, for this product, the notation $\mathbb{R}^B$ which is usual in the theory of stochastic processes.

Let $x = x_B$ denote any element of $\overline{\mathbb{R}}^B$. With

$$N = N_B = \{t_1, ..., t_n\} \in \mathcal{F}(B),$$

let $x(N) = x(N_B)$ denote a point $(x_1, ..., x_n) = (x(t_1), ..., x(t_n))$ of $\overline{\mathbb{R}}^N$. Consider the projection

$$P_N : \mathbb{R}^B \to \mathbb{R}^N, \qquad P_N(x) = (x(t_1), ..., x(t_n)),$$

and similarly we define the projection $\overline{P}_N : \overline{\mathbb{R}}^B \to \overline{\mathbb{R}}^N$. Then, to each interval $I_1 \times ... \times I_n$ of $\mathbb{R}^N$ there corresponds a cylindrical interval $I[N] := P_N^{-1}(I_1 \times ... \times I_n)$, which is a subset of $\mathbb{R}^B$. It is often convenient to denote $I_1 \times ... \times I_n$ by $I(t_1) \times ... \times I(t_n)$ or $I(N)$, so $I[N] = I(N) \times \mathbb{R}^{B \setminus N}$. Similarly,

$$\overline{P}_N(x_B) = x(N) \in \overline{\mathbb{R}}^N, \quad \text{for} \quad x = x_B \in \overline{\mathbb{R}}^B.$$

Given $x \in \overline{\mathbb{R}}^B$ and $I[N] \subset \mathbb{R}^B$, we say that $(x, I[N])$ is associated in $\mathbb{R}^B$, if $(x(N), I(N))$ is an associated pair in $\mathbb{R}^N$. Our domain of integration is $\mathbb{R}^B$ and the set of associated points is $\overline{\mathbb{R}}^B$.

Definition 2.1. *A finite collection $\mathcal{E} = \{(x, I[N]) : x \in \overline{\mathbb{R}}^B$ and $N \in \mathcal{F}(\mathcal{B})\}$ of associated pairs is said to be a division of $\mathbb{R}^B$, if the intervals $I[N]$ are disjoint and have union $\mathbb{R}^B$.*

Divisions of cylindrical intervals in $\mathbb{R}^B$ are defined similarly.

We now address the question of a gauge for $\mathbb{R}^B$, that is, a rule which determines which associated point-interval pairs $(x, I[N])$ are admissible, as elements of a division, in forming a Riemann sum approximation of an integral in the infinite dimensional space $\mathbb{R}^B$. To do this, we define mappings L_B on the sets of associated points $\overline{\mathbb{R}}^B$ of the domain of integration $\mathbb{R}^B$, and mappings δ_B on $\overline{\mathbb{R}}^B \times \mathcal{F}(B)$, which give us an effective class of gauges. Let

$$L_B : \overline{\mathbb{R}}^B \to \mathcal{F}(B), \qquad L_B(x) \in \mathcal{F}(B);$$

$$\delta_B : \overline{\mathbb{R}}^B \times \mathcal{F}(B) \to \mathbb{R}_+, \qquad 0 < \delta_B(x, N) < +\infty.$$

A choice of L_B and δ_B gives us a representative member of this class of gauges:

$$\gamma_B := (L_B, \delta_B). \tag{4}$$

We say that an associated point-interval pair $(x, I[N])$ is γ_B-fine if

$$N \supseteq L_B(x), \quad \text{and} \quad (x(N), I(N)) \text{ is } \delta_B\text{-fine in } \mathbb{R}^N.$$

Definition 2.2. *A division $\mathcal{E} = \{(x, I[N]) : x \in \overline{\mathbb{R}}^B$ and $N \in \mathcal{F}(\mathcal{B})\}$ of the domain of integration is γ_B-fine, or is a γ_B-division, if each of the pairs $(x, I[N])$ is γ_B-fine. In this case, we denote $\mathcal{E}$ by $\mathcal{E}_{\gamma_B}$.*

The space $\mathbb{R}^B$ admits a γ_B-division. This result is stated next and a proof of it can be found in [8], Theorem 1.

Theorem 2.1. *For any infinite set B and for any given gauge γ_B, there exists a γ_B-fine division of $\mathbb{R}^B$.*

The generalized Riemann integral of a function h of an associated pair $(x, I[N])$ is defined as follows (see [14]).

Definition 2.3. *The function h is generalized Riemann integrable over $\mathbb{R}^B$, with integral $\alpha = \int_{\mathbb{R}^B} h$, if, given $\epsilon > 0$, there exists a gauge γ_B such that*

$$\left| \sum_{(x,\, I[N]) \in \mathcal{E}_{\gamma_B}} h(x,\, I[N]) - \alpha \right| < \epsilon$$

for every γ_B-division $\mathcal{E}_{\gamma_B}$ of $\mathbb{R}^B$.

Sometimes we integrate functions $h(I[N])$ which do not depend on the associated point x of the variable $I[N]$. In generalized Riemann integration, this must be handled carefully. We should think of the integrand as $h(x, I[N]) = h(I[N])$ for every x associated with $I[N]$. Thus, even though the variable x does not appear explicitly in the integrand, the terms $\sum h(I[N])$ of the Riemann sum still depend on the x's of the division $\{(x, I[N])\}$ which determines the Riemann sum.

Definition 2.4. *Two functions $h_1(x, I[N])$ and $h_2(x, I[N])$ are variationally equivalent in $\mathbb{R}^B$ if, given $\epsilon > 0$, there exists γ_B such that, for all divisions $\mathcal{E}_{\gamma_B}$,*

$$\sum_{(x,\, I[N]) \in \mathcal{E}_{\gamma_B}} |h_1(x,\, I[N]) - h_2(x,\, I[N])| < \epsilon.$$

If h_1 is integrable in $X \subseteq \mathbb{R}^B$ and if h_2 is variationally equivalent to h_1, then h_2 is integrable in X and $\int_X h_1 = \int_X h_2$ (see [14], Proposition 18, page 32 for a proof). This result is important because sometimes, when we want to establish a property of $\int_X h_1$, it is easier to demonstrate it first for an integral $\int_X h_2$, where h_2 is "equivalent", in the variational sense, to h_1.

3 Additional definitions.

The following result is a version of the Tonelli theorem for generalized Riemann integrals and it will be useful in the main results. See [22], Theorem 6.6.5, for a proof of it.

Theorem 3.1. *Let J be an interval in $\overline{\mathbb{R}}^n$, with $J = H \times K$, where H and K belong to $\overline{\mathbb{R}}^l$ and $\overline{\mathbb{R}}^m$, $n = l + m$. Let f be a function defined on $\overline{\mathbb{R}}^n$. If*

i) f is measurable on J;

ii) there is a function g such that $|f| \leq g$ on J and either

$$A_1 = \int_H \left(\int_K g(x, y)dy \right) dx < \infty$$

or

$$A_2 = \int_K \left(\int_H g(x, y)dx \right) dy < \infty.$$

Then, f is generalized Riemann integrable (or Henstock-Kurzweil integrable) on J and

$$\int \int_J f = \int_H \left(\int_K f(x, y)dy \right) dx.$$

The next result can be found in [22], Corollary 6.6.7.

Corollary 3.1. *If f is measurable and non-negative, then*

$$\int \int_J f = \int_H \left(\int_K f(x, y)dy \right) dx = \int_K \left(\int_H f(x, y)dx \right) dy,$$

provided that at least one of the three integrals exists and is finite.

The following result corresponds to Lebesgue's dominated convergence theorem. A proof of it can be found in [14], Proposition 33.

Theorem 3.2. *Suppose $h_j(x, N, I)$ is integrable in $\mathbb{R}^B$, $j = 1, 2, 3, ...,$ and for each associated pair $(x, I[N])$, the sequence $\{h_j(x, I, N)\}_{j \geq 1}$ converges to a value $h(x, N, I)$. Suppose, if $\epsilon > 0$ is given, there exists a gauge γ_1 such that, whenever $(x, I[N])$ is γ_1-fine, there exists $j_0 = j_0(x, I[N]) > 0$ such that $j > j_0$ implies*

$$|h(x, N, I) - h_j(x, N, I)| < \epsilon g_0(x, N, I),$$

where g_0 is a positive function, integrable in $\mathbb{R}^B$. Suppose there exist functions $g_1(x, N, I)$ and $g_2(x, N, I)$, integrable in $\mathbb{R}^B$, and a gauge γ_2 satisfying

$$g_1(x, N, I) \leq h_j(x, N, I) \leq g_2(x, N, I)$$

for each j and each γ_2-fine $(x, I[N])$. Then h is generalized Riemann integrable in $\mathbb{R}^B$ and

$$\lim_{j\to+\infty} \int_{\mathbb{R}^B} h_j(x, N, I) = \int_{\mathbb{R}^B} h(x, N, I).$$

4 The Main Results.

We divide this section into two parts. The first part is concerned with the properties of the volume function of a process with random impulses and with the integrability of this volume function. In the second part, we present an impulsive partial differential equation of Schrödinger type and we show that its solution can be represented by an integral with respect to the volume function of an underlying impulsive process. This is the Feynman-Kac representation.

4.1 The Volume Function of a Random Impulsive Process.

Let $\{x_t\}_{t\geq 0}$ be a Brownian motion. Suppose at time $t_{j-1} > 0$, the displacement is $x_{j-1} = x(t_{j-1})$. For the later time t_j, the increment $x_j - x_{j-1}$ is normally distributed, with mean zero and variance $t_j - t_{j-1}$. Therefore, the probability that $x_j = x(t_j) \in [u_j, v_j[$ $(u_j < v_j)$ is

$$\frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \int_{u_j}^{v_j} \exp\left(-\frac{1}{2}\frac{(y_j - x_{j-1})^2}{t_j - t_{j-1}}\right) dy_j.$$

Thus, given $x(t_0) = \xi'$ ($t_0 \geq 0$ and $\xi' \in \mathbb{R}$), the joint probability that $x_1 \in I_1$,..., $x_n \in I_n$, where $I_j = [u_j, v_j[$, $1 \leq j \leq n$, is

$$\int_{u_1}^{v_1} ... \int_{u_n}^{v_n} \prod_{j=1}^{n} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}} dy_1 ... dy_n. \tag{5}$$

Therefore, in Brownian motion, we are lead to consider expressions of the form

$$\prod_{j=1}^{n} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}}. \tag{6}$$

In the sequel, we shall present a version for the expressions (5) and (6), when the underlying Brownian process undergoes impulsive changes at some moments of time.

We start by giving some notations in order to define the impulsive process. Let $\{\omega_i : i = 1, 2, \ldots\}$ be a series of random variables with $\omega_i \in (0, T)$, $0 < T \leq +\infty$, where ω_i is independent of ω_j when $i \neq j$ for all $i, j = 1, 2, \ldots$. Let $J_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots$, be a collection of continuous functions. Let τ', τ be real numbers such that $0 < \tau' < \tau$. Now, let

$$\tau_j = \tau' + \sum_{i=1}^{j} \omega_i,$$

$j = 1, 2, \ldots$. We also assume that for any bounded interval $[a, b] \subseteq \mathbb{R}$ the set $\{\tau_i\}_{i \geq 1} \cap [a, b]$ is finite.

Given $x \in \mathbb{R}^{(\tau', \tau)}$, suppose $\tau' < \tau_1 < \ldots < \tau_p < \tau < \tau_{p+1}$. Consider the function $z \in \mathbb{R}^{(\tau', \tau)}$ such that

$$z(t) = x(t), \quad \text{for} \quad \tau' < t < \tau_1,$$

$$z(t) = x(t) + \sum_{\tau_j \leq t} J_j(x(\tau_j)), \quad \tau_j \leq t < \tau_{j+1}, \quad j = 1, 2, ..., p-1,$$

$$z(t) = x(t) + \sum_{\tau_j \leq t} J_j(x(\tau_j)), \quad \tau_p \leq t < \tau.$$

Figure 1 illustrates the behavior of the impulsive process $z(t)$, $\tau' < t < \tau$, when $x \in C((\tau', \tau))$, where $C((\tau', \tau))$ denotes the set of those x which are continuous at each t in (τ', τ).

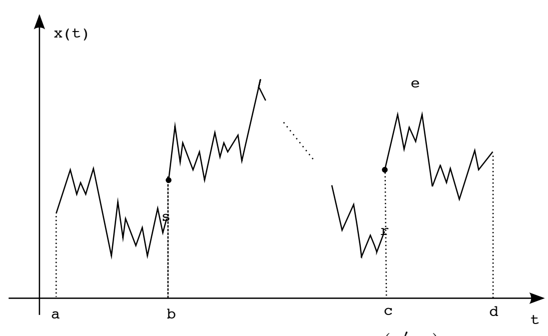

Figure 1: Process $z \in \mathbb{R}^{(\tau', \tau)}$.

Given the interval $(\tau',\,\tau)\subset\mathbb{R}$ let $N=\{t_1,...,t_{r-1}\}\subset(\tau',\,\tau)$, where $\tau'=t_0$ and $\tau=t_r$. We define N_ω by

$$N_\omega = N\cup\{\tau_1,\ldots,\tau_p\} \quad \text{whenever} \quad \tau_1,\ldots,\tau_p\in(\tau',\tau) \quad \text{with} \quad \tau_p<\tau<\tau_{p+1}.$$

Note that $N_\omega=N$ if $\tau_1>\tau$. If $\{\tau_1,\tau_2,...,\tau_p\}\subset(\tau',\,\tau)$, $p\geq 1$, we enumerate $N_\omega=\{t_1,t_2,\ldots,t_{n-1}\}$, where $\tau'=t_0$, $\tau=t_n$ and $\{\tau_1,\tau_2,...,\tau_p\}=\{t_{i_1},t_{i_2},...,t_{i_p}\}$ with $i_j\in\{1,2,...,n-1\}$ for $1\leq j\leq p$. Let $\mathcal{N}=\mathcal{N}(N_\omega)=\{1,2,...,n\}$ and $\mathcal{J}=\mathcal{J}(N_\omega)=\{i_1,i_2,...,i_p\}$.

We now define a volume function for the impulsive process. First, corresponding to $w(y,\,N)$ in the Brownian motion, that is

$$w(y,\,N)=\prod_{j=1}^{n}\frac{\exp\left(-\frac{1}{2}\frac{(y_j-y_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}},$$

(see [14], chapter 3, for more details), we define $g_{\mathcal{I}}(y,\,N_\omega)$ for the impulsive process by

$$\prod_{j\in\mathcal{N}\setminus\mathcal{J}}\frac{\exp\left(-\frac{1}{2}\frac{(y_j-y_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}}\prod_{j\in\mathcal{J}}\frac{\exp\left(-\frac{1}{2}\frac{(y_j-(y_{j-1}-J_j(y_j)))^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}}$$

which is equal to

$$\prod_{j\in\mathcal{N}\setminus\mathcal{J}}\frac{\exp\left(-\frac{1}{2}\frac{(y_j-y_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}}\prod_{j\in\mathcal{J}}\frac{\exp\left(-\frac{1}{2}\frac{(J_j(y_j)+y_j-y_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}}. \tag{7}$$

Note that if $\tau_1>\tau$ then $g_{\mathcal{I}}(y,\,N_\omega)=w(y,\,N)$.

Let $I(t_j)=I_j=[u_j,\,v_j)\subset\mathbb{R}$ and $\Delta I_j=v_j-u_j$, $1\leq j\leq n-1$. Recall that $I(N)=I_1\times...\times I_{n-1}$.

A volume function for the impulsive process is then given by

$$Q_{\mathcal{I}}(I[N_\omega])=Q_{\mathcal{I}}(I[N_\omega];\,\tau',\,\tau,\,\xi',\,\xi)=\int_{I(N_\omega)}g_{\mathcal{I}}(y,\,N_\omega)dy(N_\omega),$$

where $\xi',\,\xi\in\mathbb{R}$ and $0<\tau'<\tau$.

Let

$$\mathcal{Z}=\left\{J_j\in C(\mathbb{R},\mathbb{R}):\int_{-\infty}^{+\infty}\frac{\exp\left(-\frac{1}{2}\frac{(J_j(y_j)+y_j-y_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j-t_{j-1})}}dy_j=1,\text{for } j=1,2,\ldots\right\}$$

where $C(\mathbb{R},\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} : g \text{ is continuous}\}$. In particular, if J_j are constant functions for all $j = 1, 2, \ldots$, then clearly $\mathcal{Z} \neq \emptyset$.

If $J_j \in \mathcal{Z}$, $j = 1, 2, \ldots$, then $Q_{\mathcal{I}}(I[N_\omega])$ is the probability distribution function for the impulsive process which gives the probability that $x_j \in I_j$ for $1 \leq j \leq n-1$.

If (x, I) is an associated pair, where $I = I[N_\omega]$, let

$$G_{\mathcal{I}}(x,\, I[N_\omega]) := G_{\mathcal{I}}(x,\, I[N_\omega];\, \tau',\, \tau,\, \xi',\, \xi) = Q_{\mathcal{I}}(I[N_\omega]). \tag{8}$$

Now we shall prove that $G_{\mathcal{I}}(x,\, I[N_\omega])$ is generalized Riemann integrable in $\mathbb{R}^{(\tau',\tau)}$. In order to do that, we need to prove an auxiliary result. So, let us start by introducing some auxiliary functions.

Suppose that $\tau' < \tau_1 < \ldots < \tau_p < \tau < \tau_{p+1}$, where $\tau_j = \tau' + \sum_{i=1}^{j} \omega_i$, $j = 1, 2, \ldots$ as defined before. Then, we define $\phi_1,\, \phi_2 : \mathbb{R} \times (\tau',\, \tau) \longrightarrow \mathbb{R}$ and $\Phi_j : \mathbb{R} \times \mathbb{R} \times (\tau',\, \tau) \times (\tau',\, \tau) \longrightarrow \mathbb{R}$, $j = 1, 2, ..., p-1$, by

$$\phi_1(y_k,\, t_k) = \frac{1}{\sqrt{2\pi(t_k - \tau')}} \exp\left(-\frac{1}{2}\frac{(y_k - \xi')^2}{t_k - \tau'}\right),$$

for $k \in \{1, 2, ..., i_1 - 1\}$,

$$\phi_2(y_{i_p}, t_{i_p}) = \frac{1}{\sqrt{2\pi(\tau - t_{i_p})}} \exp\left(-\frac{1}{2}\frac{(\xi - y_{i_p})^2}{\tau - t_{i_p}}\right)$$

and

$$\Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1}) = \frac{1}{\sqrt{2\pi(t_{i_{j+1}-1} - t_{i_j})}} \exp\left(-\frac{1}{2}\frac{(y_{i_{j+1}-1} - y_{i_j})^2}{t_{i_{j+1}-1} - t_{i_j}}\right),$$

$j = 1, 2, ..., p-1$.

Analogously, define $\phi_1(J_k(y_k),\, t_k)$, for $k \in \mathcal{J}$, replacing y_k by $J_k(y_k) + y_k$ in the expression of $\phi_1(y_k,\, t_k)$, and define $\Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}})$ replacing $y_{i_{j+1}-1}$ by $J_{i_{j+1}}(y_{i_{j+1}}) + y_{i_{j+1}}$ and $t_{i_{j+1}-1}$ by $t_{i_{j+1}}$ in the expression of $\Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1})$, for $j \in \{1, 2, ..., p-1\}$.

We can prove the next lemma by completing the square in the exponential.

Lemma 4.1. *If a, b, u, $v \in \mathbb{R}$, with $a > 0$ and $b > 0$, then the function*

$$h(\alpha) = \sqrt{\frac{a}{\pi}} e^{-a(u-\alpha)^2} \sqrt{\frac{b}{\pi}} e^{-b(\alpha - v)^2}$$

is Riemann integrable and

$$\int_{-\infty}^{+\infty} \sqrt{\frac{a}{\pi}} e^{-a(u-\alpha)^2} \sqrt{\frac{b}{\pi}} e^{-b(\alpha-v)^2} d\alpha = \sqrt{\frac{ab}{\pi(a+b)}} \exp\left(-\frac{ab}{a+b}(u-v)^2\right).$$

Proposition 4.1 in the sequel says that, for $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$, the function $g_{\mathcal{I}}(y,\, N_\omega)$ defined by equation (7) is generalized Riemann integrable with respect to y in $\mathbb{R}^{n-1}$.

Proposition 4.1. *Let $N_\omega = \{t_1, t_2, ..., t_{n-1}\} \subset (\tau',\, \tau)$ be given with $\tau' = t_0$ and $\tau = t_n$. Let $g_{\mathcal{I}}$ be the function defined in (7), where $y(\tau') = y(t_0) = \xi'$ and $y(\tau) = y(t_n) = \xi$ $(\xi', \xi \in \mathbb{R})$. Then, $g_{\mathcal{I}}$ is generalized Riemann integrable with respect to y in $\mathbb{R}^{n-1}$ and*

$$\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N_\omega) dy_1 dy_2 ... dy_{n-1} = \frac{1}{\sqrt{2\pi(\tau-\tau')}} \exp\left(-\frac{1}{2}\frac{(\xi-\xi')^2}{\tau-\tau'}\right)$$

whenever $\tau_1 > \tau$. Also

$$\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N_\omega) dy_1 dy_2 ... dy_{n-1} = \int_{-\infty}^{+\infty} \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \phi_2(y_{i_1}, t_{i_1}) dy_{i_1}$$

whenever $\tau' < \tau_1 < \tau < \tau_2$, and

$$\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N_\omega) dy_1 dy_2 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \right) \times$$

$$\times \phi_2(y_{i_p}, t_{i_p}) \prod_{j=1}^{p} dy_{i_j}$$

whenever $\tau' < \tau_1 < \ldots < \tau_p < \tau < \tau_{p+1}$, $p \geq 2$.

PROOF. If $\tau_1 > \tau$ the result is proved in [14] (see Proposition 36). Let us prove the case when $\mathcal{J} = \{i_1, \ldots, i_p\}$, $p \geq 2$. Indeed, let $\mathcal{I} = \{\tau_1, \tau_2, ..., \tau_p\} = \{t_{i_1}, t_{i_2}, ..., t_{i_p}\} \subset (\tau',\, \tau)$ with $i_j \in \{1, 2, ..., n-1\}$ for $1 \leq j \leq p$, $p \geq 2$. Let

$$\mathcal{N} = \{1, 2, ..., i_1 - 1, i_1, i_1 + 1, ..., i_p - 1, i_p, i_p + 1, ..., n-1, n\}.$$

Define

$$\psi_j(y_j,\, y_{j-1}) = \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right), \quad j \in \mathcal{N} \setminus \mathcal{J},$$

and

$$\varphi_j(y_j,\, y_{j-1}) = \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left(-\frac{1}{2}\frac{(J_j(y_j) + y_j - y_{j-1})^2}{t_j - t_{j-1}}\right), \quad j \in \mathcal{J}.$$

By Lemma 4.1, we obtain

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \psi_1(y_1,\, y_0)...\psi_{i_1-1}(y_{i_1-1},\, y_{i_1-2})dy_1...dy_{i_1-2} =$$

$$= \frac{1}{\sqrt{2\pi(t_{i_1-1} - \tau')}} \exp\left(-\frac{1}{2}\frac{(y_{i_1-1} - \xi')^2}{t_{i_1-1} - \tau'}\right) = \phi_1(y_{i_1-1},\, t_{i_1-1}), \tag{9}$$

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \psi_{i_j+1}(y_{i_j+1},\, y_{i_j})...\psi_{i_{j+1}-1}(y_{i_{j+1}-1},\, y_{i_{j+1}-2})dy_{i_j+1}...dy_{i_{j+1}-2} =$$

$$= \frac{1}{\sqrt{2\pi(t_{i_{j+1}-1} - t_{i_j})}} \exp\left(-\frac{1}{2}\frac{(y_{i_{j+1}-1} - y_{i_j})^2}{t_{i_{j+1}-1} - t_{i_j}}\right) = \Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1}), \tag{10}$$

$j = 1, 2, ..., p-1$, and also

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \psi_{i_p+1}(y_{i_p+1},\, y_{i_p})...\psi_n(y_n,\, y_{n-1})dy_{i_p+1}...dy_{n-1} =$$

$$= \frac{1}{\sqrt{2\pi(\tau - t_{i_p})}} \exp\left(-\frac{1}{2}\frac{(\xi - y_{i_p})^2}{\tau - t_{i_p}}\right) = \phi_2(y_{i_p}, t_{i_p}). \tag{11}$$

Thus, taking $t_{i_p+\ell} := t_{n-1}$, $\ell \in \mathbb{N}$, from equations (9), (10) and (11), we have

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y,\, N_\omega) \left(\prod_{j=1}^{i_1-2} dy_j\right) \left(\prod_{j=1}^{p-1} dy_{i_j+1}...dy_{i_{j+1}-2}\right) \prod_{j=1}^{\ell} dy_{i_p+j} = \tag{12}$$

$$= \phi_1(y_{i_1-1}, t_{i_1-1}) \left(\prod_{j=1}^{p-1} \varphi_{i_j}(y_{i_j},\, y_{i_j-1})\Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1})\right) \times$$

$$\times \varphi_{i_p}(y_{i_p},\, y_{i_p-1})\phi_2(y_{i_p}, t_{i_p}).$$

Using Lemma 4.1, we have

$$\int_{-\infty}^{+\infty} \phi_1(y_{i_1-1}, t_{i_1-1})\varphi_{i_1}(y_{i_1},\, y_{i_1-1})dy_{i_1-1} = \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \tag{13}$$

and

$$\int_{-\infty}^{+\infty} \Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1})\varphi_{i_{j+1}}(y_{i_{j+1}}, y_{i_{j+1}-1})dy_{i_{j+1}-1} = \tag{14}$$

$$= \Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}), \quad j = 1, ..., p-1.$$

Then, from (12), (13) and (14), it follows that

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y, N_\omega) \prod_{j \in \mathcal{N} \setminus \mathcal{J}} dy_j =$$

$$= \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \right) \phi_2(y_{i_p}, t_{i_p}). \tag{15}$$

Define the functions f, $F : \mathbb{R}^p \to \mathbb{R}$ by

$$f(y_{i_1}, ..., y_{i_p}) = \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \right) \phi_2(y_{i_p}, t_{i_p})$$

and

$$F(y_{i_1}, ..., y_{i_p}) = \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) f(y_{i_1}, ..., y_{i_p}).$$

Thus F is a continuous function, $|F(y_{i_1}, ..., y_{i_p})| \leq \dfrac{f(y_{i_1}, ..., y_{i_p})}{\sqrt{2\pi(t_{i_1} - \tau')}}$ and

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \frac{f(y_{i_1}, ..., y_{i_p})}{\sqrt{2\pi(t_{i_1} - \tau')}} dy_{i_1} ... dy_{i_p} = \frac{1}{\sqrt{2\pi(t_{i_1} - \tau')}}.$$

By the Tonelli theorem (Theorem 3.1), the function F is generalized Riemann integrable in $\mathbb{R}^p$ and

$$\int_{\mathbb{R}^p} F(y_{i_1}, ..., y_{i_p}) dy_{i_1} ... dy_{i_p} = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} F(y_{i_1}, ..., y_{i_p}) dy_{i_1} ... dy_{i_p}$$

is finite. Hence, from (15),

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y, N_\omega) dy_1 dy_2 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} F(y_{i_1}, ..., y_{i_p}) dy_{i_1} ... dy_{i_p} < \infty.$$

Using Corollary 3.1, $g_{\mathcal{I}}(y,\, N_\omega)$ is generalized Riemann integrable with respect to y in $\mathbb{R}^{n-1}$ and

$$\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N_\omega) dy_1 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y,\, N_\omega) dy_1 dy_2 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \right) \times$$

$$\times \phi_2(y_{i_p}, t_{i_p}) \prod_{j=1}^{p} dy_{i_j},$$

which completes the proof. □

The next result says that $G_{\mathcal{I}}(x,\, I[N_\omega])$ given by (8) is generalized Riemann integrable in the function space $\mathbb{R}^{(\tau',\, \tau)}$.

Theorem 4.1. *The generalized Riemann integral*

$$\int_{\mathbb{R}^{(\tau',\, \tau)}} G_{\mathcal{I}}(x,\, I[N_\omega])$$

exists.

Proof. If $\tau_1 > \tau$ the result is proved in [14] (see Proposition 36). Suppose $\mathcal{J} = \{i_1, \ldots, i_p\}$, $p \geq 1$. Then, consider a division $\mathcal{E} = \{(x,\, I[N_\omega])\}$ of $\mathbb{R}^{(\tau',\, \tau)}$, with each N_ω chosen so that $\mathcal{I} = \{\tau_1, \ldots, \tau_p\} \subseteq N_\omega \in \mathcal{F}((\tau',\, \tau))$, $p \geq 1$. Then the Riemann sum of $G_{\mathcal{I}}$ is given by

$$\sum_{(x,\, I[N_\omega]) \in \mathcal{E}} G_{\mathcal{I}}(x,\, I[N_\omega]) = \sum_{(x,\, I[N_\omega]) \in \mathcal{E}} Q_{\mathcal{I}}(I[N_\omega]).$$

Let $M_\omega = \cup\{N_\omega : (x,\, I[N_\omega]) \in \mathcal{E}\}$ and enumerate M_ω as $\{t_1, ..., t_{m-1}\}$, where $\tau' = t_0$, $\tau = t_m$ and $t_0 < t_1 < ... < t_{m-1} < t_m$. Each term $Q_{\mathcal{I}}(I[N_\omega])$ of the Riemann sum can be rewritten as $Q_{\mathcal{I}}(I[M_\omega])$ by inserting additional y_j's in the expression of $g_{\mathcal{I}}$, $j \in \mathcal{N} \setminus \mathcal{J}$, and integrating from $-\infty$ to $+\infty$ on the extra y_j's. Then the Riemann sum becomes

$$\sum_{(x,\, I[M_\omega]) \in \mathcal{E}} Q_{\mathcal{I}}(I[M_\omega]).$$

with M_ω a fixed set of dimensions. So we are now dealing, in effect, with some Riemann sum estimate of an integral in $m-1$ dimensions. Then each term of the Riemann sum is an integral over $I[M_\omega] \subset \mathbb{R}^{m-1}$, and, by the finite additivity of these integrals in $\mathbb{R}^{m-1}$,

$$\sum_{(x,\, I[M_\omega]) \in \mathcal{E}} Q_{\mathcal{I}}(I([M_\omega]) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y,\, M_\omega) dy_1 ... dy_{m-1}. \tag{16}$$

By Proposition 4.1, the integral (16) exists and we can rewrite

$$\sum_{(x,\, I[M_\omega]) \in \mathcal{E}} Q_{\mathcal{I}}(I[M_\omega])$$

as

$$\int_{-\infty}^{+\infty} \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \phi_2(y_{i_1}, t_{i_1}) dy_{i_1} \tag{17}$$

if $p = 1$, and

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \phi_1(J_{i_1}(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J_{i_{j+1}}(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \right) \times$$

$$\times \phi_2(y_{i_p}, t_{i_p}) \prod_{j=1}^{p} dy_{i_j} \tag{18}$$

if $p \geq 2$. Let β_1 be the integral in (17) and β_2 be the integral in (18). Thus, given $\epsilon > 0$, for any gauge γ chosen so that $L(x) \supseteq \mathcal{I}$, we have that for every $(x,\, I[N_\omega]) \in \mathcal{E}_\gamma$, $\mathcal{I} \subseteq L(x) \subseteq N_\omega$ implies

$$\left| \sum_{(x,\, I[N_\omega]) \in \mathcal{E}_\gamma} G_{\mathcal{I}}(x,\, I[N_\omega]) - \beta_1 \right| < \epsilon \quad \text{if} \quad p = 1$$

and

$$\left| \sum_{(x,\, I[N_\omega]) \in \mathcal{E}_\gamma} G_{\mathcal{I}}(x,\, I[N_\omega]) - \beta_2 \right| < \epsilon \quad \text{if} \quad p \geq 2.$$

Therefore, $\int_{\mathbb{R}^{(\tau',\, \tau)}} G_{\mathcal{I}}(x,\, I[N_\omega]) = \beta_1$ if $p = 1$ or $\int_{\mathbb{R}^{(\tau',\, \tau)}} G_{\mathcal{I}}(x,\, I[N_\omega]) = \beta_2$ if $p \geq 2$ and the proof is complete. □

Let us show that the expressions $g_{\mathcal{I}}(x,\, N_\omega)\displaystyle\prod_{j=1}^{n-1}\Delta I_j$ and $G_{\mathcal{I}}(x,\, I[N_\omega])$ are variationally equivalent in $\mathbb{R}^{(\tau',\,\tau)}$. This result is a consequence of the next proposition.

If (x, I) is an associated pair, where $I = I[N_\omega]$, define the auxiliary function $q_{\mathcal{I}}(x,\, I[N_\omega])$ by

$$q_{\mathcal{I}}(x,\, I[N_\omega]) = g_{\mathcal{I}}(x,\, N_\omega)\prod_{j=1}^{n-1}\Delta I_j.$$

Proposition 4.2. *Let $k(x(N_\omega)) = k(x(t_1), ..., x(t_{n-1}))$ be a real-valued function which depends on $(x(t_1), ..., x(t_{n-1}))$. If k is jointly continuous in x_j, $1 \leq j \leq n-1$, then the expressions $k(x(N_\omega))q_{\mathcal{I}}(x,\, I[N_\omega])$ and $\displaystyle\int_{I(N_\omega)} k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega)$ are variationally equivalent in $\mathbb{R}^{(\tau',\,\tau)}$, provided at least one the two integrals exists.*

Proof. Let $\epsilon > 0$ be given. Let us consider the case when $\tau_1 \in (\tau',\, \tau)$. A proof for the case when $\tau_1 > \tau$ can be found in [14], Proposition 37. Since J_j, $j = 1, 2, \ldots$, are continuous functions, given $x \in \overline{\mathbb{R}}^{(\tau',\tau)}$, we can choose $L(x)$ and $\delta(x,\, N_\omega)$ such that, if $N_\omega \supseteq L(x) \supseteq \mathcal{I} = \{\tau_1, \ldots, \tau_p\}$, then $(I(N_\omega),\, x(N_\omega))$ is $\delta-$fine, and if $y \in I(N_\omega)$, then

$$|k(x(N_\omega))g_{\mathcal{I}}(x,\, N_\omega) - k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)| < \frac{\epsilon}{4}\sqrt{2\pi(t_{i_1} - \tau')}g_{\mathcal{I}}(x,\, N_\omega)$$

and

$$g_{\mathcal{I}}(y,\, N_\omega) > \frac{1}{2}g_{\mathcal{I}}(x,\, N_\omega).$$

Thus,

$$\left|k(x(N_\omega))q_{\mathcal{I}}(x,\, I[N_\omega]) - \int_{I(N_\omega)} k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega)\right| =$$

$$= \left|k(x(N_\omega))g_{\mathcal{I}}(x,\, N_\omega)\prod_{j=1}^{n-1}\Delta I_j - \int_{I(N_\omega)} k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega)\right| =$$

$$= \left|\int_{I(N_\omega)} \left[k(x(N_\omega))g_{\mathcal{I}}(x,\, N_\omega) - k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)\right] dy(N_\omega)\right| \leq$$

$$\leq \frac{\epsilon}{2}\sqrt{2\pi(t_{i_1} - \tau')}\int_{I(N_\omega)} g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega).$$

Now, we can choose a gauge γ such that for each division $\mathcal{E}_\gamma$,

$$\sum_{(x,\, I[N_\omega])\in\mathcal{E}_\gamma} \left| k(x(N_\omega))q_{\mathcal{I}}(x,\, I[N_\omega]) - \int_{I(N_\omega)} k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega) \right| \leq$$

$$\leq \frac{\epsilon}{2}\sqrt{2\pi(t_{i_1}-\tau')} \sum_{(x,\, I[N_\omega])\in\mathcal{E}_\gamma} \int_{I(N_\omega)} g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega) =$$

$$= \frac{\epsilon}{2}\sqrt{2\pi(t_{i_1}-\tau')} \int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega) < \epsilon.$$

Therefore,

$$\int_{\mathbb{R}^{(\tau',\tau)}} k(x(N_\omega))q_{\mathcal{I}}(x,\, I[N_\omega]) = \int_{\mathbb{R}^{(\tau',\tau)}} \int_{I(N_\omega)} k(y(N_\omega))g_{\mathcal{I}}(y,\, N_\omega)dy(N_\omega),$$

which completes the proof. □

As a direct consequence of Proposition 4.2, we have the next corollary.

Corollary 4.1. *The expressions* $q_{\mathcal{I}}(x,\, I[N_\omega])$ *and* $G_{\mathcal{I}}(x,\, I[N_\omega])$ *are variationally equivalent in* $\mathbb{R}^{(\tau',\,\tau)}$.

From now up to the end of this section, we are going to consider that $(\tau',\,\tau)\subset\mathbb{R}$ contains at least one point of the sequence $\{\tau_j\}_{j\geq 1}$, $\tau_j=\tau'+\sum_{i=1}^{j}\omega_i$, $j=1,2,\ldots$. When $\tau_1>\tau$ the reader can find the next results in [14] chapter 3.

Let $\tau'<T_1<\tau$ and $D_1=\{x\in\mathbb{R}^{(\tau',\,\tau)}:\, x \text{ is discontinuous at } T_1\}$. We intend to prove that $\int_{D_1} G_{\mathcal{I}}(x,\, I[N_\omega])$ exists and equals zero, because this result will be useful in the next section. We need to show some auxiliary results in order to prove this.

Let $M=\{T_1,...,T_m\}\subset(\tau',\,\tau)$ be fixed and suppose a functional h satisfies $h(x)=h(x(M))$ for all $x\in\mathbb{R}^{(\tau',\,\tau)}$. Then h is called a cylinder functional. Note that h depends only on the values taken by x at $T_1,...,T_m$, and we can treat it as a function of $x(M)\in\mathbb{R}^m$ or as a function of $x\in\mathbb{R}^{(\tau',\tau)}$. Thus, consider the particular case when $M=\{T_1,T_2\}$ and $h(x)=h(x(M))$. Let $H_{\mathcal{I}}(I[N_\omega])$ be given by

$$H_{\mathcal{I}}(I[N_\omega]) = \int_{I(N_\omega)} h(x(M))g_{\mathcal{I}}(x,\, N_\omega)dx_1...dx_{n-1}.$$

If $\tau_k < T_1 < T_2 < \tau_{k+1}$ for some $k \in \{0, 1, 2, ..., p\}$ and $\tau_0 = \tau'$, define

$$H_1(x, M) = h(x(M)) \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_1} - x_{\tau_k})^2}{T_1 - \tau_k}\right)}{\sqrt{2\pi(T_1 - \tau_k)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right)}{\sqrt{2\pi(T_2 - T_1)}} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{k+1}} + J_{i_{k+1}}(x_{\tau_{k+1}}) - x_{T_2})^2}{\tau_{k+1} - T_2}\right)}{\sqrt{2\pi(\tau_{k+1} - T_2)}} \times$$

$$\times \left(\prod_{j=k+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau} - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}}$$

and, if $T_1 = \tau_k$ and $T_1 < T_2 < \tau_{k+1}$ for some $k \in \{1, 2, ..., p\}$, define

$$H_2(x, M) = h(x(M)) \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{\tau_k})^2}{T_2 - \tau_k}\right)}{\sqrt{2\pi(T_2 - \tau_k)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{k+1}} + J_{i_{k+1}}(x_{\tau_{k+1}}) - x_{T_2})^2}{\tau_{k+1} - T_2}\right)}{\sqrt{2\pi(\tau_{k+1} - T_2)}} \times$$

$$\times \left(\prod_{j=k+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau} - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}}.$$

The next theorem states conditions on $H_1(x, M)$ or $H_2(x, M)$ so that the function $H_{\mathcal{I}}(I[N_\omega])$ is generalized Riemann integrable in $\mathbb{R}^{(\tau', \tau)}$.

Theorem 4.2. *Suppose h, considered as a function of $x(M) = (x(T_1), x(T_2)) \in \mathbb{R}^2$, is almost everywhere continuous and positive.*

1. *If* $\tau_k < T_1 < T_2 < \tau_{k+1}$ *for some* $k \in \{0, 1, ..., p\}$, $p \geq 1$, *and* $H_1(x, M)$ *is generalized Riemann integrable in* $\mathbb{R}^{p+2}$ *with respect to the variables* $x_{\tau_1}, ..., x_{\tau_k},\ x_{T_1},\ x_{T_2},\ x_{\tau_{k+1}}, ..., x_{\tau_p}$, *then* $H_{\mathcal{I}}(I[N_\omega])$ *is generalized Riemann integrable in* $\mathbb{R}^{(\tau', \tau)}$, *and*

$$\int_{\mathbb{R}^{(\tau',\tau)}} H_{\mathcal{I}}(I[N_\omega]) = \int_{\mathbb{R}^{p+2}} H_1(x,\, M) dx_{\tau_1} ... dx_{\tau_k} dx_{T_1} dx_{T_2} dx_{\tau_{k+1}} ... dx_{\tau_p}.$$

2. *If* $T_1 = \tau_k$ *and* $T_1 < T_2 < \tau_{k+1}$ *for some* $k \in \{1, 2, ..., p\}$, $p \geq 1$, *and* $H_2(x, M)$ *is generalized Riemann integrable in* $\mathbb{R}^{p+1}$ *with respect to the variables* $x_{\tau_1}, ..., x_{\tau_k},\ x_{T_2},\ x_{\tau_{k+1}}, ..., x_{\tau_p}$, *then* $H_{\mathcal{I}}(I[N_\omega])$ *is generalized Riemann integrable in* $\mathbb{R}^{(\tau', \tau)}$, *and*

$$\int_{\mathbb{R}^{(\tau',\tau)}} H_{\mathcal{I}}(I[N_\omega]) = \int_{\mathbb{R}^{p+1}} H_2(x,\, M) dx_{\tau_1} ... dx_{\tau_k} dx_{T_2} dx_{\tau_{k+1}} ... dx_{\tau_p}.$$

PROOF. Let us prove item 1. Let $\mathcal{E} = \{(x,\, I[N_\omega])\}$ be a division of $\mathbb{R}^{(\tau',\, \tau)}$ with each N_ω satisfying $\mathcal{I} \subseteq N_\omega \in \mathcal{F}((\tau',\, \tau))\}$. We recall that

$$H_{\mathcal{I}}(I[N_\omega]) = \int_{I(N_\omega)} h(x(M)) g_{\mathcal{I}}(x,\, N_\omega) dx_1 ... dx_{n-1}.$$

Let $\mathcal{O} = \cup\{N_\omega : (x,\, I[N_\omega]) \in \mathcal{E}\}$ and enumerate $\mathcal{O}$ as $\{t_1, ..., t_{r-1}\}$, where $\tau' = t_0$, $\tau = t_r$ and $t_0 < t_1 < ... < t_{r-1} < t_r$. As in the proof of Theorem 4.1, each term $H_{\mathcal{I}}(I[N_\omega])$ of the Riemann sum can be rewritten as $H_{\mathcal{I}}(I[\mathcal{O}])$. Then, by the finite additivity of these integrals, the Riemann sum becomes

$$\sum_{(x,\, I[N_\omega]) \in \mathcal{E}} H_{\mathcal{I}}(I[N_\omega]) = \sum_{(x,\, I[\mathcal{O}]) \in \mathcal{E}} H_{\mathcal{I}}(I[\mathcal{O}]) =$$

$$= \sum_{(x,\, I[\mathcal{O}]) \in \mathcal{E}} \int_{I(\mathcal{O})} h(x(M)) g_{\mathcal{I}}(x,\, \mathcal{O}) dx_1 ... dx_{r-1}.$$

But,

$$\sum_{(x,\, I[\mathcal{O}]) \in \mathcal{E}} \int_{I(\mathcal{O})} h(x(M)) g_{\mathcal{I}}(x,\, \mathcal{O}) dx_1 ... dx_{r-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x(M)) g_{\mathcal{I}}(x,\, \mathcal{O}) dx_1 ... dx_{r-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} H_1(x, M) dx_{\tau_1} ... dx_{\tau_k} dx_{T_1} dx_{T_2} dx_{\tau_{k+1}} ... dx_{\tau_p},$$

where the last equality follows from Lemma 4.1.

Let β be the value of

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} H_1(x, M) dx_{\tau_1} ... dx_{\tau_k} dx_{T_1} dx_{T_2} dx_{\tau_{k+1}} ... dx_{\tau_p}.$$

Given $\epsilon > 0$, we can choose a gauge γ, with $\mathcal{I} \subset L(x) \subseteq N_\omega$, such that

$$\left| \sum_{(x,\, I[N_\omega]) \in \mathcal{E}_\gamma} H_{\mathcal{I}}(I[N_\omega]) - \beta \right| < \epsilon.$$

for every division $\mathcal{E}_\gamma$. Therefore $\int_{\mathbb{R}^{(\tau',\, \tau)}} H_{\mathcal{I}}(I[N_\omega]) = \beta$.

Analogously, we prove item 2. □

Let $\tau' < T_1 < \tau$ and let $D_1 = \{x \in \mathbb{R}^{(\tau',\, \tau)} : x \text{ is discontinuous at } T_1\}$. Let $\tau' < T_2 < \tau$, $T_2 \neq T_1$, and let

$$X^1 = \left\{ x \in \mathbb{R}^{(\tau',\, \tau)} : \lim \sup_{T_2 \to T_1} |x(T_2) - x(T_1)|^2 \geq 1 \right\},$$

$$X^j = \left\{ x \in \mathbb{R}^{(\tau',\, \tau)} : \frac{1}{j} \leq \lim \sup_{T_2 \to T_1} |x(T_2) - x(T_1)|^2 \leq \frac{1}{j-1} \right\},$$

$j = 2, 3, ...$ Let $D^r = \bigcup_{j=1}^{r} X^j$. Then, $D_1 = \bigcup_{r=1}^{+\infty} D^r$.

In the next lines, we prove that $G_{\mathcal{I}}(x,\, I[N_\omega])$ is generalized Riemann integrable in D^r with integral zero. Then we conclude that this function is generalized Riemann integrable in D_1 with integral zero.

Lemma 4.2. *For* $r = 1, 2, 3, ...$, $\int_{D^r} G_{\mathcal{I}}(x,\, I[N_\omega])$ *exists and equals zero.*

Proof. At first, suppose $T_1 \notin \{\tau_1, ..., \tau_p\}$, $p \geq 1$. We can suppose, without loss of generality, that $\tau_k < T_1 < T_2 < \tau_{k+1}$ for some $k \in \{0, 1, 2, ..., p-1\}$, $\tau_0 := \tau'$. If $\tau_p < T_1 < T_2 < \tau$ the case is handled analogously. Note that

$$\frac{1}{\sqrt{2\pi(T_2 - T_1)}} \int_{-\infty}^{+\infty} (x_{T_2} - x_{T_1})^2 \exp\left(-\frac{1}{2} \frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1} \right) dx_{T_1} =$$

$$= \frac{2(T_2 - T_1)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u^2 \exp(-u^2) du = \frac{2(T_2 - T_1)}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) =$$

$$= \frac{2(T_2 - T_1)}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = T_2 - T_1 = |T_2 - T_1|.$$

Then,

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \frac{|x_{T_2} - x_{T_1}|^2}{\sqrt{2\pi(\tau_1 - \tau')}} \prod_{j=2}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_1} - x_{\tau_k})^2}{T_1 - \tau_k}\right)}{\sqrt{2\pi(T_1 - \tau_k)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right)}{\sqrt{2\pi(T_2 - T_1)}} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{k+1}} + J_{i_{k+1}}(x_{\tau_{k+1}}) - x_{T_2})^2}{\tau_{k+1} - T_2}\right)}{\sqrt{2\pi(\tau_{k+1} - T_2)}} \times$$

$$\times \prod_{j=k+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(J_{i_j}(x_{\tau_j}) + x_{\tau_j} - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_\tau - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}} dx_{\tau_1}...dx_{\tau_k} dx_{T_1} dx_{T_2} dx_{\tau_{k+1}}...dx_{\tau_p} = \frac{|T_2 - T_1|}{\sqrt{2\pi(\tau_1 - \tau')}}.$$

Let ς be the last integral. Let $h(x(M)) = (x_{T_2} - x_{T_1})^2$ in the expression of $H_1(x, M)$. Then, by Theorem 4.2, item 1., we have

$$\int_{\mathbb{R}^{(\tau',\, \tau)}} H_{\mathcal{I}}(I[N_\omega]) =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} |x_{T_2} - x_{T_1}|^2 \prod_{j=1}^{k} \left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_1} - x_{\tau_k})^2}{T_1 - \tau_k}\right)}{\sqrt{2\pi(T_1 - \tau_k)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right)}{\sqrt{2\pi(T_2 - T_1)}} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{k+1}} + J_{i_{k+1}}(x_{\tau_{k+1}}) - x_{T_2})^2}{\tau_{k+1} - T_2}\right)}{\sqrt{2\pi(\tau_{k+1} - T_2)}} \times$$

$$\times \prod_{j=k+2}^{p} \left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J_{i_j}(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_\tau - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}} dx_{\tau_1}...dx_{\tau_k} dx_{T_1} dx_{T_2} dx_{\tau_{k+1}}...dx_{\tau_p}$$

$$\leq \varsigma = \frac{|T_2 - T_1|}{\sqrt{2\pi(\tau_1 - \tau')}},$$

where the last inequality follows from the Tonelli theorem (Theorem 3.1).

Given $\epsilon > 0$ and $j \in \mathbb{N}$, we can choose T_2 and a division $\mathcal{E}_\gamma$ such that

$$\frac{\epsilon}{j} > \sum_{(x,\, I[N_\omega]) \in \mathcal{E}_\gamma} H_{\mathcal{I}}(I[N_\omega]) =$$

$$\overset{(*)}{=} \sum_{(x,\, I(\mathcal{O})) \in \mathcal{E}_\gamma} \int_{I(\mathcal{O})} (x_{T_2} - x_{T_1})^2 g_{\mathcal{I}}(x,\, \mathcal{O}) dx_1...dx_{r-1} \geq$$

$$\geq \sum_{(x,\, I[\mathcal{O}]) \in \mathcal{E}_\gamma} \chi(X^j, x) \int_{I(\mathcal{O})} (x_{T_2} - x_{T_1})^2 g_{\mathcal{I}}(x,\, \mathcal{O}) dx_1...dx_{r-1} \geq$$

$$\geq \frac{1}{j} \sum_{(x,\, I[\mathcal{O}]) \in \mathcal{E}_\gamma} \chi(X^j, x) \int_{I(\mathcal{O})} g_{\mathcal{I}}(x,\, \mathcal{O}) dx_1...dx_{r-1} =$$

$$= \frac{1}{j} \sum_{(x,\, I[\mathcal{O}]) \in \mathcal{E}_\gamma} \chi(X^j, x) G_{\mathcal{I}}(x,\, I[\mathcal{O}]).$$

The symbol $\mathcal{O}$ after the first equality $(*)$ denotes $\mathcal{O} = \cup\{N_\omega : (x,\, I[N_\omega]) \in \mathcal{E}_\gamma\}$. Since $\epsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^{(\tau',\, \tau)}} \chi(X^j, x) G_{\mathcal{I}}(x,\, I[N_\omega]) = 0$$

for every $j = 1, 2, ...$ Then, by the finite additivity of the integral,

$$\int_{\mathbb{R}^{(\tau',\, \tau)}} \chi(D^r, x) G_{\mathcal{I}}(x,\, I[N_\omega]) = 0.$$

If $T_1 \in \{\tau_1, ..., \tau_p\}$, then $T_1 = \tau_k$ for some $k \in \{1, 2, ..., p\}$. Considering $T_1 < T_2 < \tau_{k+1}$ (denote in this case $\tau := \tau_{p+1}$), since

$$\frac{1}{\sqrt{2\pi(T_2-\tau_k)}}\int_{-\infty}^{+\infty}|x_{T_2}-x_{\tau_k}|^2\exp\left(-\frac{1}{2}\frac{(x_{T_2}-x_{\tau_k})^2}{T_2-\tau_k}\right)dx_{\tau_k}=|T_2-T_1|,$$

then

$$\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty}\frac{|x_{T_2}-x_{T_1}|^2}{\sqrt{2\pi(\tau_1-\tau')}}\prod_{j=2}^{k}\left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j}+J_{i_j}(x_{\tau_j})-x_{\tau_{j-1}})^2}{\tau_j-\tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}\right)\times$$

$$\times\frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2}-x_{\tau_k})^2}{T_2-\tau_k}\right)}{\sqrt{2\pi(T_2-\tau_k)}}\frac{\exp\left(-\frac{1}{2}\frac{(J_{i_{k+1}}(x_{\tau_{k+1}})+x_{\tau_{k+1}}-x_{T_2})^2}{\tau_{k+1}-T_2}\right)}{\sqrt{2\pi(\tau_{k+1}-T_2)}}\times$$

$$\times\prod_{j=k+2}^{p}\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j}+J_{i_j}(x_{\tau_j})-x_{\tau_{j-1}})^2}{\tau_j-\tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}\frac{\exp\left(-\frac{1}{2}\frac{(x_\tau-x_{\tau_p})^2}{\tau-\tau_p}\right)}{\sqrt{2\pi(\tau-\tau_p)}}\times$$

$$\times dx_{\tau_1}...dx_{\tau_k}dx_{T_2}dx_{\tau_{k+1}}...dx_{\tau_p}=\frac{|T_2-T_1|}{\sqrt{2\pi(\tau_1-\tau)}}.$$

Thus,

$$\int_{\mathbb{R}^{(\tau',\,\tau)}}H_{\mathcal{I}}(I[N_\omega])\leq\frac{|T_2-T_1|}{\sqrt{2\pi(\tau_1-\tau')}},$$

where

$$\int_{\mathbb{R}^{(\tau',\tau)}}H_{\mathcal{I}}(I[N_\omega])=\int_{\mathbb{R}^{p+1}}H_2(x,\,M)dx_{\tau_1}...dx_{\tau_k}dx_{T_2}dx_{\tau_{k+1}}...dx_{\tau_p}$$

with $h(x(M)) = (x_{T_2} - x_{T_1})^2$ in the expression of $H_2(x, M)$.

The rest of the proof follows by analogous argument. □

Theorem 4.3. *The integral* $\int_{D_1} G_{\mathcal{I}}(x,\, I[N_\omega])$ *exists and equals zero.*

PROOF. We have $D_1 = \bigcup_{r=1}^{+\infty} D^r$ and $D^r \subset D^{r+1}$. For each associated pair $(x,\, I[N_\omega])$, define

$$f_k(x,\,I[N_\omega])=\chi(D^k,x)G_{\mathcal{I}}(x,\,I[N_\omega]),\quad k=1,2,3....$$

Given $x \in D_1$, there exists a positive integer k_0 such that $x \in D^k$ for $k \geq k_0$. Thus, $\chi(D^k, x) = \chi(D_1, x)$ for $k \geq k_0$. Consequently, for each associated pair $(x, I[N_\omega])$, we have

$$f_k(x, I[N_\omega]) \overset{k \to +\infty}{\longrightarrow} f_0(x, I[N_\omega]),$$

where $f_0(x, I[N_\omega]) = \chi(D_1, x) G_{\mathcal{I}}(x, I[N_\omega])$. Note that

$$|f_0(x, I[N_\omega])| \leq G_{\mathcal{I}}(x, I[N_\omega]) \quad \text{and} \quad |f_k(x, I[N_\omega])| \leq G_{\mathcal{I}}(x, I[N_\omega]),$$

$k = 1, 2, 3, \ldots$.

Given $\epsilon > 0$, there exists $k_1 > 0$ such that

$$|f_0(x, I[N_\omega]) - f_k(x, I[N_\omega])| < \epsilon G_{\mathcal{I}}(x, I[N_\omega]),$$

for $k > k_1$ and all associated pairs $(x, I[N_\omega])$. By Theorem 4.1, $G_{\mathcal{I}}(x, I[N_\omega])$ is generalized Riemann integrable in $\mathbb{R}^{(\tau', \tau)}$. By Theorem 3.2, f_0 is generalized Riemann integrable in $\mathbb{R}^{(\tau', \tau)}$ and

$$\int_{\mathbb{R}^{(\tau', \tau)}} f_0(x, I[N_\omega]) = \lim_{k \to +\infty} \int_{\mathbb{R}^{(\tau', \tau)}} f_k(x, I[N_\omega]).$$

Using Lemma 4.2, we obtain

$$\int_{\mathbb{R}^{(\tau', \tau)}} \chi(D_1, x) G_{\mathcal{I}}(x, I[N_\omega]) = 0$$

and the proof is complete. $\square$

4.2 An equation of Schrödinger type random with impulses.

We start by defining the solution of an impulsive partial differential equation. The idea of this definition is inspired by [13].

Suppose $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_p < \tau$ are given numbers and $\tau \in]0, +\infty[$. Define

$$\Delta = \mathbb{R} \times [0, \tau],$$

$$\Gamma_k = \{(\psi, t) : \psi \in \mathbb{R}, t \in (\tau_k, \tau_{k+1})\}, \quad 0 \leq k \leq p-1,$$

$$\overline{\Gamma}_k = \{(\psi, t) : \psi \in \mathbb{R}, t \in [\tau_k, \tau_{k+1})\}, \quad 0 \leq k \leq p-1,$$

$$\Gamma_p = \{(\psi, t) : \psi \in \mathbb{R}, t \in (\tau_p, \tau)\},$$

$$\overline{\Gamma}_p = \{(\psi, t) : \psi \in \mathbb{R}, t \in [\tau_p, \tau)\},$$

$$\Gamma = \bigcup_{k=0}^{p} \Gamma_k \quad \text{and} \quad \overline{\Gamma} = \bigcup_{k=0}^{p} \overline{\Gamma}_k.$$

Let $\mathcal{K}(\Delta, \mathbb{R})$ be the class of all functions $u : \Delta \to \mathbb{R}$ such that

i) the functions $u|_{\Gamma_k}$, $k = 0, 1, ..., p$, are continuous.

ii) for each k, $k = 1, ..., p$, the limit $\lim_{(\nu,\, t)\to(\psi,\, \tau_k^-)} u(\nu, t) = u(\psi, \tau_k^-)$, $\psi \in \mathbb{R}$, exists.

iii) for each k, $k = 1, ..., p$, the limit $\lim_{(\nu,\, t)\to(\psi,\, \tau_k^+)} u(\nu, t) = u(\psi, \tau_k^+)$, $\psi \in \mathbb{R}$, exists.

iv) for each k, $k = 1, ..., p$, we have $u(\psi, \tau_k) = u(\psi, \tau_k^+)$, $\psi \in \mathbb{R}$.

We consider the equation of Schrödinger type in Γ

$$\frac{\partial}{\partial t}u(\psi, t) - \frac{1}{2}\frac{\partial^2}{\partial \psi^2}u(\psi, t) + V(\psi)u(\psi, t) = 0, \tag{19}$$

subject to the impulse condition

$$u(\psi, \tau_k) - u(\psi, \tau_k^-) = \mathbb{I}(\psi, \tau_k, u(\psi, \tau_k)), \tag{20}$$

where $k = 1, 2, ..., p$, and $V : \mathbb{R} \to \mathbb{R}$ and $\mathbb{I} : \mathbb{R}^3 \to \mathbb{R}$ are functions taking real values and $\mathbb{I}$ is not identically zero.

Definition 4.1. *The function $u : \Delta \to \mathbb{R}$ is called a solution of the problem* (19) – (20) *if:*

i) $u \in \mathcal{K}(\Delta, \mathbb{R})$;

ii) the derivatives $u_t(\psi, t)$ and $u_{\psi\psi}(\psi, t)$ exist, for $(\psi, t) \in \Gamma$;

iii) u satisfies (19) *in Γ and* (20) *at each τ_k, $k = 1, 2, ..., p$.*

From now up to the end of this section we are going to consider that given $0 < \tau' < \tau$, $\{\tau_p\}_{p\geq 1} \cap (\tau', \tau) \neq \emptyset$, where $\tau_j = \tau' + \sum_{i=1}^{j} \omega_i$, $j = 1, 2, \ldots$, and $\{\omega_i : i = 1, 2, \ldots\}$ is a sequence of random variables with $\omega_i \in]0, T[$, $0 < T \leq +\infty$, where ω_i is independent of ω_j when $i \neq j$ for all $i, j = 1, 2, \ldots$. When $\{\tau_p\}_{p\geq 1} \cap (\tau', \tau) = \emptyset$ the main result of this section, namely Theorem 4.7, is found in [14], Proposition 57.

Let $U_{\mathcal{I}} : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Given $s \in (\tau', \tau)$ and $\varsigma \in \mathbb{R}$, let $N_\omega^{(s)}$ be the set $\{t_1, ..., t_{r-1}\}$, where $t_0 = \tau'$ and $t_r = s$ ($r = r(s) \in \mathbb{N}$). From now, we denote $N_\omega^{(s)}$ simply by $N^{(s)}$. Then define

$$v_{\mathcal{I}}(N^{(s)}, I^{(s)}; \varsigma, s) = \int_{I(N^{(s)})} g_{\mathcal{I}}(y, N^{(s)}) e^{-U_{\mathcal{I}}(x_{r(s)-1})(s-\tau')} dy(N^{(s)})$$

and

$$q_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)}) = g_{\mathcal{I}}(x,\, N^{(s)}) \prod_{j=1}^{r(s)-1} \Delta I_j,$$

where $I^{(s)} = I[N^{(s)}]$.

Let $W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)};\, \varsigma,\, s) = q_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)}) e^{-U_{\mathcal{I}}(x_{r(s)-1})(s-\tau')}$. If $W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)}; \varsigma,\, s)$ is generalized Riemann integrable in $\mathbb{R}^{(\tau',\, s)}$, define

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \int_{\mathbb{R}^{(\tau',\, s)}} W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)}; \varsigma,\, s).$$

The proof given in [14], shows that the expressions $W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)}; \varsigma,\, s)$, $v_{\mathcal{I}}(N^{(s)},\, I^{(s)};\, \varsigma,\, s)$ and $e^{-U_{\mathcal{I}}(x_{r(s)-1})(s-\tau')} G_{\mathcal{I}}(x,\, I[N^{(s)}])$ are variationally equivalent in $\mathbb{R}^{(\tau',\, s)}$. Therefore we have the following result.

Proposition 4.3. *The following equalities hold*

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \int_{\mathbb{R}^{(\tau',\, s)}} v_{\mathcal{I}}(N^{(s)}\, I^{(s)};\, \varsigma,\, s) = \int_{\mathbb{R}^{(\tau',\, s)}} e^{-U_{\mathcal{I}}(x_{r(s)-1})(s-\tau')} G_{\mathcal{I}}(x,\, I[N^{(s)}]),$$

whenever any of the integrals exist.

Note that, here, the domain of integration is $\mathbb{R}^{(\tau',\, s)}$ rather than $\mathbb{R}^{(\tau',\, \tau)}$, and the elements x, $N^{(s)}$ and $I^{(s)}$ are taken in $\mathbb{R}^{(\tau',\, s)}$.

We shall show that $\phi_{\mathcal{I}}(\varsigma,\, s)$ satisfies the equation of Schrödinger type in Γ

$$\frac{\partial}{\partial s} u(\varsigma,\, s) - \frac{1}{2} \frac{\partial^2}{\partial \varsigma^2} u(\varsigma,\, s) + U_{\mathcal{I}}(\varsigma) u(\varsigma,\, s) = 0, \tag{21}$$

subject to the impulse condition

$$u(\xi_k,\, \tau_k) - u(\xi_k,\, \tau_k^-) = \mathbb{I}(\xi_k,\, \tau_k,\, u(\xi_k,\, \tau_k)), \tag{22}$$

where $\tau_j = \tau' + \sum_{i=1}^{j} \omega_i$, $j = 1, 2, \ldots$, $\{\omega_i : i = 1, 2, \ldots\}$ is a sequence of random variables with $\omega_i \in]0,\, T[$, $0 < T \leq +\infty$, ω_i is independent of ω_j when $i \neq j$ for all $i, j = 1, 2, \ldots$, $x(\tau_k) = \xi_k \in \mathbb{R}$, $k = 1, 2, ..., p$, $p \geq 1$, and $\mathbb{I}(\xi_k,\, \tau_k,\, u(\xi_k,\, \tau_k))$ is some function which is not identically zero taking values in $\mathbb{R}$.

In the next result, we establish the integrability of the function $W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)};\, \varsigma,\, s)$.

Proposition 4.4. *Let $\tau' < s < \tau$ and $x(s) = \varsigma$. Then $W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)};\, \varsigma,\, s)$ is generalized Riemann integrable in $\mathbb{R}^{(\tau',\, s)}$ and*

$$\int_{\mathbb{R}^{(\tau',\, s)}} W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)};\varsigma,\, s) = e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')}\times$$

$$\times \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\prod_{j=1}^{r(s)+1}\frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J_{i_j}(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right)dx_{i_1}\ldots dx_{i_{r(s)}},$$

where $t_{i_0} = t_0 = \tau'$, $r(s) = \max\big\{j : j \in \{1, 2, ..., p\}$ and $t_{i_j} < s\big\}$, $p \geq 1$, and $t_{i_{r(s)+1}} = s$.

PROOF. Since $U_{\mathcal{I}}$ is continuous, given $\epsilon > 0$, Theorem 4.3 says that for $x \in \mathbb{R}^{(\tau',\, s)}$ continuous at s, we can choose $L(x)$ such that

$$N^{(s)} = \{t_1, ..., t_{r-1}\} \supseteq L(x) \supseteq \mathcal{I}$$

implies

$$\left|e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')} - e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')}\right| < \frac{\epsilon}{\varphi(\varsigma,\, s)},$$

where

$$\varphi(\varsigma,\, s) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left(\prod_{j=1}^{r(s)+1}\frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J_{i_j}(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right)dx_{i_1}\ldots dx_{i_{r(s)}}.$$

By Proposition 4.1, $0 < \varphi(\varsigma,\, s) < +\infty$, for every $s \in (\tau',\, \tau)$ and $\varsigma \in \mathbb{R}$. By Theorem 4.1, we have

$$\int_{\mathbb{R}^{(\tau',\, s)}} G_{\mathcal{I}}(x,\, I[N^{(s)}]) = \varphi(\varsigma,\, s).$$

Then, we can choose a gauge γ such that, for every division $\mathcal{E}_\gamma$,

$$\left|\sum_{(x, I[N^{(s)}])\in\mathcal{E}_\gamma}\left[e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')}G_{\mathcal{I}}(x,\, I[N^{(s)}]) - e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')}G_{\mathcal{I}}(x,\, I[N^{(s)}])\right]\right| <$$

$$< \frac{\epsilon}{\varphi(\varsigma,\, s)}\sum_{(x, I[N^{(s)}])\in\mathcal{E}_\gamma} G_{\mathcal{I}}(x,\, I[N^{(s)}]) < \frac{\epsilon}{\varphi(\varsigma,\, s)}(\epsilon + \varphi(\varsigma,\, s)).$$

Hence, we have the result. □

Now, we give a result which establishes conditions for the continuity of the function $\phi_{\mathcal{I}}$ at intervals with no impulse action.

Proposition 4.5. *Let $s \in (\tau', \tau) \setminus \{\tau_1, ..., \tau_p\}$ $(p \geq 1)$ and $\varsigma \in \mathbb{R}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that, if $|s_1 - s| < \delta$ and $|\varsigma_1 - \varsigma| < \delta$, then $|\phi_{\mathcal{I}}(\varsigma_1, s_1) - \phi_{\mathcal{I}}(\varsigma, s)| < \epsilon$.*

Proof. Let $s \in (\tau', \tau) \setminus \{\tau_1, ..., \tau_p\}$, $p \geq 1$. We can suppose, without loss of generality, that $\tau_k < s < \tau_{k+1}$, for some $k \in \{0, 1, ..., p\}$, where $\tau_0 = t_0 = \tau'$ and $\tau_{p+1} := \tau$ (in this case $\tau_{p+1} := \tau$ is not an impulsive point). Then, there exits $\delta > 0$ such that $]s - \delta, s + \delta[\subset (\tau_k, \tau_{k+1})$. By Proposition 4.4, we have

$$\phi_{\mathcal{I}}(\psi, \beta) = e^{-U_{\mathcal{I}}(\psi)(\beta - \tau')} \times$$

$$\times \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{k+1} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j} - t_{i_{j-1}})}} \right) dx_{i_1} ... dx_{i_k},$$

for every $\beta \in]s - \delta, s + \delta[$ and every $\psi \in \mathbb{R}$, where $t_{i_0} = \tau'$, $t_{i_{k+1}} = \beta$, $x(\tau') = \xi'$, $x(\beta) = \psi$ and $J(x_{i_{k+1}}) = 0$. Given $\beta \in]s - \delta, s + \delta[$, $\beta \neq s$, consider the following expressions

$$\kappa_{\mathcal{I}}(\varsigma, s) = e^{-U_{\mathcal{I}}(\varsigma)(s - \tau')} \times$$

$$\times \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j} - t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma - x_{i_k})^2}{s - t_{i_k}}\right)}{\sqrt{2\pi(s - t_{i_k})}}$$

and

$$\kappa_{\mathcal{I}}(\psi, \beta) = e^{-U_{\mathcal{I}}(\psi)(\beta - \tau')} \times$$

$$\times \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j} - t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(\psi - x_{i_k})^2}{\beta - t_{i_k}}\right)}{\sqrt{2\pi(\beta - t_{i_k})}}.$$

Since $U_{\mathcal{I}}$ is continuous, we have $\kappa_{\mathcal{I}}(\psi, \beta) \to \kappa_{\mathcal{I}}(\varsigma, s)$ as $\psi \to \varsigma$ and $\beta \to s$. Note that $\kappa_{\mathcal{I}}(\varsigma, s) > 0$. Then, given $\epsilon > 0$, there exists $\delta_1 > 0$, with $\delta_1 < \delta$, such that

$$|\kappa_{\mathcal{I}}(\psi, \beta) - \kappa_{\mathcal{I}}(\varsigma, s)| < \epsilon \kappa_{\mathcal{I}}(\varsigma, s),$$

whenever $0 < |\beta - s| < \delta_1$ and $0 < |\psi - \varsigma| < \delta_1$. By the dominated convergence theorem (Theorem 3.2, see also [9], Theorem 9.2), we have

$$\phi_{\mathcal{I}}(\psi,\, \beta) \to \phi_{\mathcal{I}}(\varsigma,\, s)$$

as $\psi \to \varsigma$ and $\beta \to s$. Therefore the result is proved. □

Theorem 4.4 below says the lateral limits of the function $\phi_{\mathcal{I}}$ at the moments $\{\tau_1, ..., \tau_p\}, p \geq 1$, exist.

Theorem 4.4. *Let $x(\tau_k) = \xi_k \in \mathbb{R}$, $k = 1, 2, ..., p$, $p \geq 1$. Then, the limits*

$$\lim_{(\varsigma,\, s) \to (\xi_k,\, \tau_k^+)} \phi_{\mathcal{I}}(\varsigma,\, s) \quad \text{and} \quad \lim_{(\varsigma,\, s) \to (\xi_k,\, \tau_k^-)} \phi_{\mathcal{I}}(\varsigma,\, s)$$

exist for $k = 1, 2, ..., p$, $p \geq 1$.

PROOF. Let τ_k where $k \in \{1, 2, ...\}$ and $\tau' < \tau_k < \tau$. Let $\delta > 0$ be arbitrarily small. By Proposition 4.4, we have

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k + \delta) = e^{-U_{\mathcal{I}}(\varsigma)(\tau_k + \delta - \tau')} \times$$

$$\times \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2} \frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}} \right)}{\sqrt{2\pi (t_{i_j} - t_{i_{j-1}})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2} \frac{(\varsigma - \xi_k)^2}{\delta} \right)}{\sqrt{2\pi\delta}} dx_{i_1} ... dx_{i_k}.$$

Now, note that

$$\left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2} \frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}} \right)}{\sqrt{2\pi (t_{i_j} - t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2} \frac{(\varsigma - \xi_k)^2}{\delta} \right)}{\sqrt{2\pi\delta}} \leq$$

$$\leq \left(\frac{1}{\sqrt{2\pi (t_{i_1} - t_{i_0})}} \right) \left(\prod_{j=2}^{k} \frac{\exp\left(-\frac{1}{2} \frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}} \right)}{\sqrt{2\pi (t_{i_j} - t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2} \frac{(\varsigma - \xi_k)^2}{\delta} \right)}{\sqrt{2\pi\delta}}.$$

Define α as being the righthand side of the inequality above. Then, the integral $\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \alpha dx_{i_1}...dx_{i_k}$ exists and it is equal to

$$\frac{1}{\sqrt{2\pi(t_{i_1} - t_{i_0})}} = \frac{1}{\sqrt{2\pi(\tau_1 - \tau')}}.$$

Then,

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k + \delta) \leq \frac{e^{-U_{\mathcal{I}}(\varsigma)(\tau_k + \delta - \tau')}}{\sqrt{2\pi(\tau_1 - \tau')}}.$$

Hence the limit $\lim\limits_{(\varsigma,\, s) \to (\xi_k,\, \tau_k^+)} \phi_{\mathcal{I}}(\varsigma,\, s)$ exists.

Now, we note that

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k - \delta) = e^{-U_{\mathcal{I}}(\varsigma)(\tau_k - \delta - \tau')} \times$$

$$\times \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{k-1} \frac{\exp\left(-\frac{1}{2} \frac{(x_{i_j} + J_{i_j}(x_{i_j}) - x_{i_{j-1}})^2}{t_{i_j} - t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j} - t_{i_{j-1}})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2} \frac{(\varsigma - \xi_{k-1})^2}{(\tau_k - \delta - \tau_{k-1})}\right)}{\sqrt{2\pi(\tau_k - \delta - \tau_{k-1})}} dx_{i_1}...dx_{i_{k-1}}.$$

Analogously, we have

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k - \delta) \leq \frac{e^{-U_{\mathcal{I}}(\varsigma)(\tau_k - \delta - \tau')}}{\sqrt{2\pi(\tau_1 - \delta - \tau')}} \quad \text{if} \quad k = 1,$$

and

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k - \delta) \leq \frac{e^{-U_{\mathcal{I}}(\varsigma)(\tau_k - \delta - \tau')}}{\sqrt{2\pi(\tau_1 - \tau')}} \quad \text{if} \quad k = 2, 3, ..., p,\ p \geq 2.$$

Thus, the limit $\lim\limits_{(\varsigma,\, s) \to (\xi_k,\, \tau_k^-)} \phi_{\mathcal{I}}(\varsigma,\, s)$ also exists. □

As a consequence of Proposition 4.4 and Theorem 4.4, we have the following result.

Theorem 4.5. *Let τ_k, $k \in \{1, 2, ...\}$, such that $\tau' < \tau_k < \tau$ and $x(\tau_k) = \xi_k$. The function $\phi_{\mathcal{I}}$ satisfies the condition given by (22) where $\mathbb{I}(\xi_k,\, \tau_k,\, \phi_{\mathcal{I}}(\xi_k,\, \tau_k))$ $= \phi_{\mathcal{I}}(\xi_k,\, \tau_k) - \phi_{\mathcal{I}}(\xi_k,\, \tau_k^-)$ and $\phi_{\mathcal{I}}$ is given by Proposition 4.4.*

Now, let us denote $W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)};\, \varsigma,\, s)$ by $\omega_{\mathcal{I}}(\varsigma,\, s)$, where

$$W_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)};\, \varsigma,\, s) = q_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)})e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')}$$

and

$$q_{\mathcal{I}}(x,\, N^{(s)},\, I^{(s)}) = g_{\mathcal{I}}(x,\, N^{(s)})\prod_{j=1}^{r-1}\Delta I_j.$$

By differentiation, for $s \neq \tau_j$, $j = 1, 2, \ldots$, we get

$$\frac{\partial\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} = -U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s) - \frac{1}{2(s-t_{i_r})}\omega_{\mathcal{I}}(\varsigma,\, s) + \frac{1}{2}\left(\frac{\varsigma - x_{i_r}}{s-t_{i_r}}\right)^2\omega_{\mathcal{I}}(\varsigma,\, s),$$

$$\frac{\partial\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial\varsigma} = -\frac{(\varsigma - x_{i_r})}{s-t_{i_r}}\omega_{\mathcal{I}}(\varsigma,\, s)$$

and

$$\frac{\partial^2\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial\varsigma^2} = -\frac{1}{s-t_{i_r}}\omega_{\mathcal{I}}(\varsigma,\, s) + \left(\frac{\varsigma - x_{i_r}}{s-t_{i_r}}\right)^2\omega_{\mathcal{I}}(\varsigma,\, s).$$

Thus,

$$\frac{\partial\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial\varsigma^2} + U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s) = 0. \tag{23}$$

The next result says that $U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s)$ is generalized Riemann integrable. Since $U_{\mathcal{I}}$ is continuous the proof of Proposition 4.6 is analogously to the proof of Proposition 4.4.

Proposition 4.6. *Let $\tau' < s < \tau$, $s \neq \tau_j$ for every $j = 1, 2, ..., p$ and $x(s) = \varsigma$. Then the function $U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s)$ is generalized Riemann integrable and*

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r} = U_{\mathcal{I}}(\varsigma)\phi_{\mathcal{I}}(\varsigma,\, s).$$

By Proposition 4.6, the expression $\frac{\partial\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial\varsigma^2}$ in equation (23) is generalized Riemann integrable and

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \left(\frac{\partial\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial\varsigma^2}\right)dx_{i_1}...dx_{i_r} = -U_{\mathcal{I}}(\varsigma)\phi_{\mathcal{I}}(\varsigma,\, s).$$

The problem now is to prove the following equalities

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \frac{\partial\omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s}dx_{i_1}...dx_{i_r} = \frac{\partial}{\partial s}\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r}$$

and

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{\partial^2 \omega_{\mathcal{I}}(\varsigma, s)}{\partial \varsigma^2} dx_{i_1} \ldots dx_{i_r} = \frac{\partial^2}{\partial \varsigma^2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma, s) dx_{i_1} \ldots dx_{i_r}$$

and to conclude that

$$\frac{\partial \phi_{\mathcal{I}}}{\partial s}(\varsigma, s) - \frac{1}{2} \frac{\partial^2 \phi_{\mathcal{I}}}{\partial \varsigma^2}(\varsigma, s) + U_{\mathcal{I}}(\varsigma) \phi_{\mathcal{I}}(\varsigma, s) = 0$$

for $(\varsigma, s) \in \Gamma$. Thus, in order to prove the equalities, we introduce some additional notations below.

Given $f(\varsigma, s)$, let

$$D_{abc} f(\varsigma, s) = \frac{1}{a} f_a(\varsigma, s) - \frac{1}{2bc} f_{bc}(\varsigma, s),$$

where

$$f_a(\varsigma, s) = f(\varsigma, s + a) - f(\varsigma, s)$$

and

$$f_{bc}(\varsigma, s) = f(\varsigma + b + c, s) - f(\varsigma + b, s) - f(\varsigma + c, s) + f(\varsigma, s)$$

for non-zero real numbers a, b, c. Then the limit

$$\lim_{a,b,c \to 0} D_{abc} f(\varsigma, s)$$

exists and equals

$$\frac{\partial f}{\partial s}(\varsigma, s) - \frac{1}{2} \frac{\partial^2 f}{\partial \varsigma^2}(\varsigma, s),$$

if and only if the partial derivatives

$$\frac{\partial f}{\partial s}, \ \frac{\partial^2 f}{\partial \varsigma^2}$$

exist.

In our case, since the derivatives $\frac{\partial \omega_{\mathcal{I}}}{\partial s}(\varsigma, s)$ and $\frac{\partial^2 \omega_{\mathcal{I}}}{\partial \varsigma^2}(\varsigma, s)$ exist, we have

$$\lim_{a,b,c \to 0} D_{abc} \omega_{\mathcal{I}}(\varsigma, s) = \frac{\partial \omega_{\mathcal{I}}}{\partial s}(\varsigma, s) - \frac{1}{2} \frac{\partial^2 \omega_{\mathcal{I}}}{\partial \varsigma^2}(\varsigma, s) = -U_{\mathcal{I}}(x_{r-1}) \omega_{\mathcal{I}}(\varsigma, s). \quad (24)$$

By Proposition 4.6, the limit $\lim_{a,b,c \to 0} D_{abc} \omega_{\mathcal{I}}(\varsigma, s)$ is generalized Riemann integrable and

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \lim_{a,b,c \to 0} D_{abc} \omega_{\mathcal{I}}(\varsigma, s) dx_{i_1} \ldots dx_{i_r} = -U_{\mathcal{I}}(\varsigma) \phi_{\mathcal{I}}(\varsigma, s). \quad (25)$$

Thus, if we prove that

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \lim_{a,b,c\to 0} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r} =$$

$$= \lim_{a,b,c\to 0} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r},$$

then we can conclude that $\dfrac{\partial\phi_{\mathcal{I}}(\varsigma,\, s)}{\partial s}$ and $\dfrac{\partial^2\phi_{\mathcal{I}}(\varsigma,\, s)}{\partial\varsigma^2}$ exist.

The next theorem shows the existence of the derivatives $\dfrac{\partial\phi_{\mathcal{I}}}{\partial s}$ and $\dfrac{\partial^2\phi_{\mathcal{I}}}{\partial\varsigma^2}$.

Theorem 4.6. *Let $\tau' < s < \tau$, $s \neq \tau_j$ for every $j = 1, 2, ..., p$, $p \geq 1$, and $x(s) = \varsigma \in \mathbb{R}$. Then the partial derivatives $\dfrac{\partial\phi_{\mathcal{I}}}{\partial s}(\varsigma,\, s)$ and $\dfrac{\partial^2\phi_{\mathcal{I}}}{\partial\varsigma^2}(\varsigma,\, s)$ exist for $(\varsigma,\, s) \in \Gamma$.*

Proof. Let $\epsilon > 0$ be given. By equation (24), we can choose $\mu > 0$ such that $0 < |\alpha| < \mu$, $0 < |\beta| < \mu$ and $0 < |\gamma| < \mu$ imply

$$|D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s) + U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s)| < \omega_{\mathcal{I}}(\varsigma,\, s)\epsilon.$$

Given x, N, I, choose α_0, β_0 and γ_0 satisfying $0 < \alpha_0 < \mu$, $0 < \beta_0 < \mu$ and $0 < \gamma_0 < \mu$ such that

$$\sup_{\substack{0<|\alpha|<\alpha_0 \\ 0<|\beta|<\beta_0 \\ 0<|\gamma|<\gamma_0}} |D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s) + U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s)| < \omega_{\mathcal{I}}(\varsigma,\, s).$$

Since $0 < |\alpha| < \alpha_0$, $0 < |\beta| < \beta_0$, $0 < |\gamma| < \gamma_0$, we have

$$-\omega_{\mathcal{I}}(\varsigma,\, s) \leq D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s) + U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s) \leq \omega_{\mathcal{I}}(\varsigma,\, s).$$

By the dominated convergence test (Theorem 3.2, see also [9], Theorem 9.2), we have

$$\lim_{\alpha,\, \beta,\, \gamma\to 0} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r} =$$

$$= -\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} U_{\mathcal{I}}(x_{r-1})\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r} = -U_{\mathcal{I}}(\varsigma)\phi_{\mathcal{I}}(\varsigma,\, s) =$$

$$\overset{(25)}{=} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \lim_{\alpha,\, \beta,\, \gamma\to 0} D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_r}.$$

Then, since

$$\lim_{\alpha,\,\beta,\,\gamma\to 0}\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\,s)dx_{i_1}...dx_{i_r} = \frac{\partial\phi_{\mathcal{I}}(\varsigma,\,s)}{\partial s} - \frac{1}{2}\frac{\partial^2\phi_{\mathcal{I}}(\varsigma,\,s)}{\partial\varsigma^2},$$

the result is proved. □

Thus we conclude the following result.

Theorem 4.7. *Let $\tau' < s < \tau$ and $\varsigma \in \mathbb{R}$. The function*

$$\phi_{\mathcal{I}}(\varsigma,\,s) = \int_{\mathbb{R}^{(\tau',\,s)}} W_{\mathcal{I}}(x,\,N^{(s)},\,I^{(s)};\varsigma,\,s)$$

satisfies the partial differential equation of Schrödinger type in Γ

$$\frac{\partial}{\partial s}u(\varsigma,\,s) - \frac{1}{2}\frac{\partial^2}{\partial\varsigma^2}u(\varsigma,\,s) + U_{\mathcal{I}}(\varsigma)u(\varsigma,\,s) = 0,$$

subject to the impulse condition

$$u(\xi_k,\,\tau_k) - u(\xi_k,\,\tau_k^-) = \mathbb{I}(\xi_k,\,\tau_k,\,u(\xi_k,\,\tau_k)),$$

where $\tau_j = \tau' + \sum_{i=1}^{j}\omega_i$, $j = 1,2,\ldots$, $\{\omega_i : i = 1,2,\ldots\}$ *is a sequence of random variables with* $\omega_i \in]0,T[$, $0 < T \leq +\infty$, ω_i *is independent of* ω_j *when* $i \neq j$ *for all* $i,j = 1,2,\ldots$, $x(\tau_k) = \xi_k \in \mathbb{R}$ *and* $\mathbb{I}(\xi_k,\,\tau_k,\,u(\xi_k,\,\tau_k)) = \phi_{\mathcal{I}}(\xi_k,\,\tau_k) - \phi_{\mathcal{I}}(\xi_k,\,\tau_k^-)$, $k = 1,2,...,p$, $p \geq 1$.

4.3 Example.

Now, we illustrate the theory by explicit evaluation of $\phi_{\mathcal{I}}$, when each of the impulses is a constant and the function $U_{\mathcal{I}}(t) = \beta$ for every $t \in \mathbb{R}$, with $\beta \in \mathbb{R}$. Consider the continuous functions $J_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\ldots$, given by $J_{i_j}(x(\tau_j)) = \alpha_j$, $j = 1,2,\ldots$, and $\tau_j = \tau' + \sum_{i=1}^{j}\omega_i$, $j = 1,2,\ldots$, $\{\omega_i : i = 1,2,\ldots\}$ is a sequence of random variables with $\omega_i \in]0,T[$, $0 < T \leq +\infty$, and ω_i is independent of ω_j when $i \neq j$ for all $i,j = 1,2,\ldots$. Let $s \in (\tau',\,\tau)$, $\varsigma \in \mathbb{R}$ and $N^{(s)} = \{t_1,....,t_{r-1}\}$, with $t_0 = \tau'$ and $t_r = s$.

If $\tau_1 > \tau$, then

$$\phi_{\mathcal{I}}(\varsigma,\,s) = \frac{e^{-\beta(s-\tau')}}{\sqrt{2\pi(s-\tau')}}\exp\left(-\frac{1}{2}\frac{(\varsigma-\xi')^2}{s-\tau'}\right)$$

is a solution of the partial differential equation of Schrödinger type

$$\frac{\partial}{\partial s}u(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2}{\partial \varsigma^2}u(\varsigma,\, s) + \beta u(\varsigma,\, s) = 0$$

in $\mathbb{R} \times (\tau',\, \tau)$.

But if $\{\tau_p\}_{p\geq 1} \cap (\tau',\, \tau) \neq \emptyset$ the solution is given as follows.

Consider the following auxiliary function $\varrho : \mathbb{R} \to \mathbb{R}$ given by

$$\varrho(x(t)) = \begin{cases} x(t), & \text{if} \quad \tau' < t < \tau \text{ and } t \neq \tau_j \text{ for all } j = 1, 2, ..., \\ x(t) + J_{i_j}(x(t)), & \text{if} \quad \tau' < t < \tau \text{ and } t = \tau_j \text{ for any } j = 1, 2, ... \end{cases}$$

Then

$$\kappa_{\mathcal{I}}(\varsigma,\, s) = e^{-\beta(s-\tau')}\left(\prod_{j=1}^{r}\frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+\alpha_j-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right)\frac{\exp\left(-\frac{1}{2}\frac{(\varrho(\varsigma)-x_{i_r})^2}{s-t_{i_r}}\right)}{\sqrt{2\pi(s-t_{i_r})}}.$$

Then, by using Lemma 4.1, we obtain

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} ... dx_{i_r} =$$

$$= \frac{e^{-\beta(s-\tau')}}{\sqrt{2\pi(s-\tau')}}\exp\left(-\frac{1}{2}\frac{(\varrho(\varsigma)-\xi'+\alpha_1+...+\alpha_r)^2}{s-\tau'}\right)$$

and by Theorem 4.7, we have

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \frac{e^{-\beta(s-\tau')}}{\sqrt{2\pi(s-\tau')}}\exp\left(-\frac{1}{2}\frac{\left(\varrho(\varsigma)-\xi'+\sum_{t_j\leq s}\alpha_j\right)^2}{s-\tau'}\right)$$

is a solution of the partial differential equation of Schrödinger type in Γ

$$\frac{\partial}{\partial s}u(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2}{\partial \varsigma^2}u(\varsigma,\, s) + \beta u(\varsigma,\, s) = 0,$$

subject to the impulse condition

$$u(\xi_1,\, \tau_1) - u(\xi_1,\, \tau_1^-) =$$

$$= \frac{1}{\sqrt{2\pi(\tau_1 - \tau')}} \left[\exp\left(-\frac{1}{2} \frac{(\xi_1 - \xi' + \alpha_1)^2}{\tau_1 - \tau'} \right) - \exp\left(-\frac{1}{2} \frac{(\xi_1 - \xi')^2}{\tau_1 - \tau'} \right) \right],$$

and

$$u(\xi_k, \tau_k) - u(\xi_k, \tau_k^-) = \frac{1}{\sqrt{2\pi(\tau_k - \tau')}} \times$$

$$\times \left[\exp\left(-\frac{1}{2} \frac{\left(\xi_k - \xi' + \sum_{i=1}^{k} \alpha_i \right)^2}{\tau_k - \tau'} \right) - \exp\left(-\frac{1}{2} \frac{\left(\xi_k - \xi' + \sum_{i=1}^{k-1} \alpha_i \right)^2}{\tau_k - \tau'} \right) \right],$$

for $k = 2, 3, \ldots$ such that $\tau' < \tau_k < \tau$.

5 Some Final Remarks.

The classical Black-Scholes equation, for pricing European call options, is obtained from a stochastic differential equation using the Itô calculus. The present paper presents a theory that will be very useful to obtain the Black-Scholes equation with random jumps by using the Feynman-Kac formulation based on generalized Riemann integration, [4].

Acknowledgment. The authors wish to thank the referees for their comments.

References

[1] L. Bachelier, *Théorie de la spéculation*, Ann. Sci. École Norm. Sup., **17** (1900), 21-86.

[2] M. Baxter and A. Rennie, *Financial Calculus*, Cambridge University Press, 1996.

[3] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, **81** (1973), 637-659.

[4] E. M. Bonotto, M. Federson and P. Muldowney, *The Henstock integral and the Black-Scholes equation with impulse action*, preprint.

[5] A. Einstein, *Investigations on the theory of the Brownian movement*, Dover, New York, 1959.

[6] A. Etheridge, *A course in financial calculus*, Cambridge University Press, 2002.

[7] R. Gordon, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Amer. Math. Soc., Providence, RI, 1994.

[8] R. Henstock, P. Muldowney and V. A. Skvortsov, *Partitioning infinite-dimensional spaces for generalized Riemann integration*, Bull. London Math. Soc., **38** (2006), 795-803.

[9] R. Henstock, *Lectures on the theory of integration*, World Scientific, Singapore, 1988.

[10] M. Kac, *Probability and related topics in the Physical Sciences*, Interscience, New York, 1957.

[11] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Graduate texts in Mathematics, Springer-Verlag, **113**, 1991.

[12] D. Lamberton and B. Lapeyre, *Introduction to stochastic calculus applied to finance*, Chapman & Hall/CRC, 2000.

[13] J. Luo, *Oscillation of hyperbolic partial differential equations with impulses*, Appl. Math. Comput, **133(2-3)** (2002), 309-318.

[14] P. Muldowney, *A general theory of integration in function spaces*, Pitman Research Notes in Mathematics, no. 153, Longman, 1987.

[15] P. Muldowney, *Financial valuation and the Henstock integral*, Seminário Brasileiro de Análise, **60** (2000), 79-108.

[16] P. Muldowney, *Introduction to Feynman integration*, J. Math. Study, **27(1)** (1994), 127-132.

[17] P. Muldowney, *Topics in probability using generalised Riemann integration*, Math. Proc. R. Ir. Acad., **99A(1)** (1999), 39-50.

[18] P. Muldowney, *Feynman's path integrals and Henstock's non-absolute integration*, J. Appl. Anal., **6(1)** (2000), 1-24.

[19] P. Muldowney, *The Henstock integral and the Black-Scholes theory of derivative asset pricing*, Real Anal. Exchange, **26(1)** (2001), 117-132.

[20] P. Muldowney, *Lebesgue integrability implies generalized Riemann integrability in* $\mathbb{R}^{]0,1]}$, Real Anal. Exchange, **27(1)** (2001/2002), 223-234.

[21] N. Wiener, *Generalised harmonic analysis*, Acta Math., **55** (1930), 117-258.

[22] L. P. Yee and R. Výborný, *The integral: An easy approach after Kurzweil and Henstock*, Austral. Math. Soc. Lect. Ser., Cambridge University Press, **14**, 2000.