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ON A PROBLEM OF FAURE AND GUÉNARD

Abstract

In [3], Faure and Guénard put the following problem: Characterize the Denjoy*-integrable functions $f : [a,b] \to \overline{\mathbb{R}}$ that can be approximated by two Baire 1 functions g_ϵ and h_ϵ, $\epsilon > 0$, that are $\mathcal{D}^*$-integrable. In the present article we show that this class of functions coincides with the class of all $\mathcal{D}^*$-integrable functions $f : [a,b] \to \overline{\mathbb{R}}$.

In [3], Faure and Guénard put the following problem, that can be written as follows:

Problem 1. *Characterize the Denjoy*-integrable (short $\mathcal{D}^*$-integrable) functions $f : [a,b] \to \overline{\mathbb{R}}$ that can be approximated by two Baire 1 functions g_ϵ and h_ϵ that are $\mathcal{D}^*$-integrable, i.e., $\epsilon > 0$, $g_\epsilon \le f \le h_\epsilon$, $g_\epsilon < \infty$, $h_\epsilon > -\infty$ and*

$$(\mathcal{D}^*) \int_a^b (h_\epsilon - g_\epsilon)(t)\, dt = (\mathcal{L}) \int_a^b (h_\epsilon - g_\epsilon)(t)\, dt < \epsilon\,.$$

In what follows we show that the class of functions from Problem 1 coincides with the class of all $\mathcal{D}^*$-integrable functions $f : [a,b] \to \overline{\mathbb{R}}$.

We shall use the following well known classes of functions: $\mathcal{C}$ (the continuous functions), $\mathcal{B}_1$ (the Baire 1 functions), AC^*G and $\underline{AC}^*G$ (see for example [6] or [2]).

Definition 1. ([6, p. 241] or [2, p. 175]). A function $f : [a,b] \to \mathbb{R}$ is said to be $\mathcal{D}^*$-integrable on $[a,b]$ if there exists $F : [a,b] \to \mathbb{R}$ such that $F \in AC^*G \cap \mathcal{C}$ on $[a,b]$ and $F'(x) = f(x)$ *a.e.* on $[a,b]$. Then $(\mathcal{D}^*) \int_a^b f(t)\, dt = F(b) - F(a)$.

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Remark 1. There exist a lot of integrals of Perron type that are equivalent to the $\mathcal{D}^*$-integral. In [2] we gathered many of these definitions and classified them: for example the Perron type integral defined in [4, p. 158] is called $\mathcal{P}_{7,7}$ in [2] (note that this is the same integral as the one used in [3]), and the $\mathcal{P}_0$ integral of [6] is called $\mathcal{P}_{3,3}$ in [2]. In what follows we shall need the $\mathcal{P}_{1,1}$-integral and the $\mathcal{P}_{8,8}$-integral of [2].

Definition 2. [2, p. 31] Let $P \subseteq [a,b]$, $x_o \in P$ and $F : P \to \mathbb{R}$. F is said to be $\mathcal{C}_i$ at x_o if $\limsup_{x \nearrow x_o, x \in P} F(x) \leq F(x_o)$, whenever x_o is a left accumulation point for P, and $F(x_o) \leq \liminf_{x \searrow x_o, x \in P} F(x)$, whenever x_o is a right accumulation point for P. F is said to be $\mathcal{C}_i$ on P if it is so at each point $x \in P$.

Definition 3. [2, pp. 174-175] Let $f : [a,b] \to \overline{\mathbb{R}}$. We define the following classes of majorants:

- $\overline{\mathcal{M}}_1(f) = \{M : [a,b] \to \mathbb{R} \,:\, M(a) = 0,\ M \in AC^*G \cap \mathcal{C};\ M'(x)$ exists (finite or infinite); $f(x) \leq M'(x) \neq -\infty\}$;
- $\overline{\mathcal{M}}_8(f) = \{M : [a,b] \to \mathbb{R} \,:\, M(a) = 0,\ M \in \underline{AC}^*G \cap \mathcal{C}_i$ on $[a,b]$; $f(x) \leq M'(x)$ *a.e.* on $[a,b]\}$;

We define the following classes of minorants:

- $\underline{\mathcal{M}}_1(f) = \{m : [a,b] \to \mathbb{R} \,:\, -m \in \overline{M}_1(f)\}$;
- $\underline{\mathcal{M}}_8(f) = \{m : [a,b] \to \mathbb{R} \,:\, -m \in \overline{M}_8(f)\}$;

If $\overline{\mathcal{M}}_1(f) \neq \emptyset$ (respectively $\overline{\mathcal{M}}_8(f) \neq \emptyset$), then we denote by $\overline{I}_1(b)$ (respectively $\overline{I}_8(b)$) the lower bound of all $M(b)$, M in $\overline{\mathcal{M}}_1(f)$ (respectively $\overline{\mathcal{M}}_8(f)$).
If $\underline{\mathcal{M}}_1(f) \neq \emptyset$ (respectively $\underline{\mathcal{M}}_8(f) \neq \emptyset$) then we denote by $\underline{I}_1(b)$ (respectively $\underline{I}_8(b)$) the upper bound of all $m(b)$, m in $\underline{M}_1(f)$ (respectively $\underline{M}_8(f)$).

- f is said to have a $\mathcal{P}_{1,1}$-integral on $[a,b]$ if $\overline{M}_1(f) \times \underline{M}_1(f) \neq \emptyset$ and $\overline{I}_1(b) = \underline{I}_1(b) = (\mathcal{P}_{1,1}) \int_a^b f(t)\,dt$.
- f is said to have a $\mathcal{P}_{8,8}$-integral on $[a,b]$ if $\overline{M}_8(f) \times \underline{M}_8(f) \neq \emptyset$ and $\overline{I}_8(b) = \underline{I}_8(b) = (\mathcal{P}_{8,8}) \int_a^b f(t)\,dt$.

Remark 2. Following Bruckner [1], let $\Delta = \{f : [a,b] \to \overline{\mathbb{R}} \,:\, f$ is a derivative (finite or infinite), i.e., there exists $F : [a,b] \to \mathbb{R}$ such that $F' = f\}$. Then $\Delta \subset \mathcal{B}_1$ (see Theorem 2.2.4 of [2, p. 30], or see Corollary 2.4 and Remark 90 of [1, p. 90] and the classification of the Zahorski classes, or [7], or [5]).

Definition 4. We denote by $\underline{\mathcal{B}}_1$ the class of all functions $f : [a,b] \to \overline{\mathbb{R}}$ that are lower semi Baire 1, i.e., the set $\{x \in [a,b] : f(x) > \alpha\}$ is of F_σ-type for $\alpha \in [-\infty, +\infty)$. Similarly $\overline{\mathcal{B}}_1$ is the class of all functions $f : [a,b] \to \overline{\mathbb{R}}$ that are upper semi Baire 1, i.e., the set $\{x \in [a,b] : f(x) < \alpha\}$ is of F_σ-type for $\alpha \in (-\infty, +\infty]$. Clearly $\mathcal{B}_1 = \underline{\mathcal{B}}_1 \cap \overline{\mathcal{B}}_1$.

Note that in [3], these two classes are denoted by $\mathcal{C}^{-+}$ respectively $\mathcal{C}^{+-}$.

Theorem 1. *Let $f : [a,b] \to \overline{\mathbb{R}}$. The following assertions are equivalent:*

(i) f is $\mathcal{D}^*$-integrable on $[a,b]$;

(ii) For any $\epsilon > 0$ there exist two $\mathcal{D}^*$-integrable functions $g_\epsilon : [a,b] \to [-\infty, +\infty)$ and $h_\epsilon : [a,b] \to (-\infty, +\infty]$ such that:

- *$g_\epsilon, h_\epsilon \in \Delta$ on $[a,b]$;*
- *$g_\epsilon \le f \le h_\epsilon$ and $(\mathcal{D}^*) \int_a^b (h_\epsilon - g_\epsilon)(t)\, dt < \epsilon$;*

(iii) Replace in (ii) "$g_\epsilon, h_\epsilon \in \Delta$" by "$g_\epsilon, h_\epsilon \in \mathcal{B}_1$";

(iv) Replace in (ii) "$g_\epsilon, h_\epsilon \in \Delta$" by "$g_\epsilon \in \overline{\mathcal{B}}_1,\ h_\epsilon \in \underline{\mathcal{B}}_1$".

Proof. (i) $\Rightarrow$ (ii) From [2, Corollary 5.9.1], it follows that f is $\mathcal{D}^*$ integrable on $[a,b]$ if and only if f is $\mathcal{P}_{1,1}$ integrable on $[a,b]$, and then

$$(\mathcal{D}^*) \int_a^b f(t)\, dt = (\mathcal{P}_{1,1}) \int_a^b f(t)\, dt\,.$$

By [2, Lemma 5.7.5], we have that f is $\mathcal{D}^*$-integrable on $[a,b]$ if and only if for $\epsilon > 0$ there is a pair $(M_\epsilon, m_\epsilon) \in \overline{\mathcal{M}}_1(f) \times \underline{\mathcal{M}}_1(f) \neq \emptyset$ such that $M_\epsilon(b) - m_\epsilon(b) < \epsilon$. Putting $g_\epsilon = m'_\epsilon$ and $h_\epsilon = M'_\epsilon$ we obtain (ii).

(ii) $\Rightarrow$ (iii) See Remark 2.

(iii) $\Rightarrow$ (iv) This is evident.

(iv) $\Rightarrow$ (i) Let

$$M_\epsilon(x) = (\mathcal{D}^*) \int_a^x h_\epsilon(t)\, dt \quad \text{and} \quad m_\epsilon(x) = (\mathcal{D}^*) \int_a^x g_\epsilon(t)\, dt\,.$$

Then $(M_\epsilon, m_\epsilon) \in \overline{\mathcal{M}}_8(f) \times \underline{\mathcal{M}}_8(f) \neq \emptyset$. By [2, Lemma 5.7.5] it follows that f is $\mathcal{P}_{8,8}$-integrable on $[a,b]$, and by [2, Corollary 5.9.1], we obtain that f is $\mathcal{D}^*$-integrable on $[a,b]$. $\square$

Remark 3. Theorem 1, (i), (iv) is in fact Theorem C, 1), 2) of Faure and Guénard [3]. Theorem 1, (i), (iii) solves the Problem 1.

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