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A SIMPLE PROOF THAT $(s)/(s^0)$ IS A COMPLETE BOOLEAN ALGEBRA

Abstract

Let X be a complete separable metric space, let (s) be the set of all Marczewski [5] measurable subsets of X, and let (s^0) be the the set of all Marczewski null subsets of X. It is already known that $(s)/(s^0)$ is a complete Boolean algebra, but the known proofs of this involve complicated preliminaries. We present a simple proof that $(s)/(s^0)$ is a complete Boolean algebra.

Let X and Y be complete, separable, metric spaces. In [4], Kenneth Schilling constructed a (s)-set in $X \times Y$ which was not in the σ-algebra generated by sets of the form $A \times B$, where A and B are (s)-sets in X and Y respectively. The proof used the fact that $(s)/(s^0)$ is a complete Boolean Algebra, but the two proofs [1],[3] of this fact which were described in [4] both involve complicated preliminaries (see [4] for more details). The present paper gives a simple, direct proof that $(s)/(s^0)$ is a complete Boolean Algebra. First, we give the basic definitions.

Definition 1. Let X be a complete separable metric space. A set $A \subseteq X$ is said to be *Marczewski measurable*, or an (s)-set (written $A \in (s)$) iff for every perfect set $P \subseteq X$, there is a perfect set $Q \subseteq P$ such that either $Q \subseteq A$ or $Q \cap A = \emptyset$. $A \subseteq X$ is said to be *Marczewski null*, or an (s^0)-set (written $A \in (s^0)$) iff for every perfect set $P \subseteq X$, there is a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$ (or, equivalently, $A \in (s^0)$ iff $A \in (s)$ and A does not contain a perfect subset).

Rather than introduce the basic definitions from Boolean Algebra, we shall translate the statement that $(s)/(s^0)$ is a complete Boolean Algebra

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into a statement which is more convenient for our purposes. Thus, define an equivalence relation $\sim$ on (s) by $Y \sim Z$ iff $Y \Delta Z \in (s^0)$, where $Y \Delta Z = (Y \cup Z) \setminus (Y \cap Z)$ is the symmetric difference of Y and Z. Define the relation $\leq$ on (s) (which we may think of as "almost" a subset of, mod (s^0)), by $Y \leq Z$ iff $Y \setminus Z \in (s^0)$, and note that $Y \sim Z$ iff $Y \leq Z$ and $Z \leq Y$. We could form the Boolean Algebra $(s)/(s^0)$ by going to equivalence classes, but it will be more convenient to avoid the additional layer of notation which that would entail. The terms "upper bound", "lower bound", "least upper bound", and "greatest lower bound" (with respect to the relation $\leq$) are then defined in the obvious way, with l.u.b.'s (and g.l.b.'s) not necessarily being unique. However, it is easy to see that l.u.b.'s and g.l.b.'s are unique modulo the equivalence relation $\sim$ (if they exist). The statement that $(s)/(s^0)$ is a complete Boolean Algebra is then easily seen to be equivalent to the statement that "Every subset of (s) has a least upper bound in (s) with respect to the relation $\leq$." (This also implies the existence of g.l.b.'s.)

The proof given here was motivated by a similar result due to John Walsh [2]. If $\mathcal{A}$ is a σ-algebra of subsets of some set X, and $\mathcal{I} \subseteq \mathcal{A}$ is a σ-ideal, then we say that the pair $(\mathcal{A}, \mathcal{I})$ has the *hull property* iff whenever $U \subseteq X$ there is a $V \in \mathcal{A}$ such that $U \subseteq V$, and if $W \in \mathcal{A}$ such that $U \subseteq W$, then $V \setminus W \in \mathcal{I}$. Walsh's result was that $((s), (s^0))$ has the hull property. In fact, the argument given below turns out to be somewhat simpler than Walsh's corresponding argument for the hull property. The reason for this is that, in the proof below, every element of $\mathcal{B}$ is an (s)-set. In the hull property proof given in [2], U is not necessarily an (s)-set, and the set corresponding to $\mathcal{C}$ in that proof is necessarily more complicated.

Theorem 1. *Let X be a complete separable metric space, and let (s) and (s^0) be as above. Then $(s)/(s^0)$ is a complete Boolean algebra.*

Proof. Let $\mathcal{B} \subseteq (s)$, and let $\mathcal{P}$ be the set of all perfect subsets of X. Let $\mathcal{C} = \{C \in \mathcal{P}$: For some $B \in \mathcal{B}, C \subseteq B\}$, and $\mathcal{D} = \{D \in \mathcal{P}$: For every $B \in \mathcal{B}, D \cap B \in (s^0)\}$. Observe that, if $C \in \mathcal{C}$ and $D \in \mathcal{D}$, then $C \cap D$ is a closed set which cannot contain a perfect subset, and is therefore countable. Without loss of generality, we may assume that $\mathcal{C}$ and $\mathcal{D}$ are both nonempty, since otherwise either X or $\emptyset$ would be a least upper bound of $\mathcal{B}$. Thus $\mathcal{C}$ and $\mathcal{D}$ both have cardinality $\mathfrak{c}$ (the cardinality of the continuum), and we may enumerate $\mathcal{C} = \{C_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{D} = \{D_\alpha : \alpha < \mathfrak{c}\}$. Let $L = \bigcup_{\alpha < \mathfrak{c}} (C_\alpha \setminus \bigcup_{\beta < \alpha} D_\beta)$, and we claim that L is the desired least upper bound. First, to show that $L \in (s)$, let $P \in \mathcal{P}$. There are then two cases.

Case 1: $P \in \mathcal{D}$. Then $P = D_\alpha$ for some $\alpha < \mathfrak{c}$. Thus $P \cap L = D_\alpha \cap L \subseteq D_\alpha \cap \bigcup_{\beta \leq \alpha} C_\beta$, which has cardinality less than $\mathfrak{c}$. Thus, there is a perfect set $Q \subseteq P$ such that $Q \cap L = \emptyset$. Thus $L \in (s)$.

Case 2: $P \notin \mathcal{D}$. Then there is a $B \in \mathcal{B}$ such that $P \cap B \notin (s^0)$, and if we pick a perfect subset P' of $P \cap B$, then $P' \in \mathcal{C}$. Then $P' = C_\alpha$ for some $\alpha < \mathfrak{c}$. Then $C_\alpha \setminus \bigcup_{\beta<\alpha} D_\beta \subseteq L$, and since $C_\alpha \cap \bigcup_{\beta<\alpha} D_\beta$ has cardinality less than $\mathfrak{c}$, $L \cap P'$ contains a perfect subset, and $L \in (s)$.

To see that L is an upper bound of $\mathcal{B}$, let $B \in \mathcal{B}$. Let P be a perfect subset of B. Then $P = C_\alpha$ for some $\alpha < \mathfrak{c}$. Since $C_\alpha \cap \bigcup_{\beta<\alpha} D_\beta$ has cardinality less than $\mathfrak{c}$, we see that $P \cap L \neq \emptyset$, and thus, since P was arbitrary, $B \setminus L$ does not contain a perfect subset. Thus, since $B \setminus L \in (s)$, $B \setminus L \in (s^0)$, and $B \leq L$. Thus, L is an upper bound of $\mathcal{B}$.

Now, let $U \in (s)$ be any upper bound of $\mathcal{B}$. We need to show that $L \leq U$. Let P be any perfect subset of L. Then P cannot be equal to D_α for any $\alpha < \mathfrak{c}$, for the construction of L guarantees that L intersects D_α in fewer than $\mathfrak{c}$ points. Thus, $P \notin \mathcal{D}$, so there is a $B \in \mathcal{B}$ such that $P \cap B \notin (s^0)$. Thus, since $P \cap B \in (s)$, $P \cap B$ must contain a perfect subset Q. Since U is an upper bound of $\mathcal{B}$, $B \leq U$, and therefore $Q \cap U \neq \emptyset$. Thus $P \cap U \neq \emptyset$, and since P was an arbitrary perfect subset of L, $L \setminus U \notin (s^0)$. Thus, since $L \setminus U \in (s)$, $L \leq U$, and we are done. $\square$

The similarity between the above proof and Walsh's proof for the hull property motivates the following obvious question, for which there appears to be no obvious answer:

Question: If $\mathcal{A}$ is a σ-algebra of subsets of some set X, and $\mathcal{I} \subseteq \mathcal{A}$ is a σ-ideal, then does either of the statements "$\mathcal{A}/\mathcal{I}$ is a complete Boolean Algebra" and "$(\mathcal{A}, \mathcal{I})$ has the hull property" imply the other?

References

[1] John Morgan II, *Point Set Theory*, Marcel Dekker, New York, 1990.

[2] J. T. Walsh, "Marczewski Sets, Measure, and the Baire Property, II," *Proc. of the Amer. Math. Soc.* **106** (1989), 1027–1030.

[3] K. Schilling, "Some category bases which are equivalent to topologies", *Real Analysis Exchange* **14** (1988-89), 210–214.

[4] ________________, "A Tale of Two (s)-ities," to appear in *Real Analysis Exchange*.

[5] E. Szprilrajn (Marczewski), " Sur une classe de fonction de M. Sierpinski et la classe correspondante d'ensembles", *Fund. Math.* **24** (1935), 17–34.