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SUMS OF DARBOUX-LIKE FUNCTIONS FROM $\mathbb{R}^n$ TO $\mathbb{R}^m$

Abstract

The additivity $\mathrm{A}(\mathcal{F})$ of a family $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ is the minimum cardinality of a $G \subseteq \mathbb{R}^{\mathbb{R}}$ with the property that $f+G \subseteq \mathcal{F}$ for no $f \in \mathbb{R}^{\mathbb{R}}$. The values of A have been calculated for many families of Darboux-like functions in $\mathbb{R}^{\mathbb{R}}$. We extend these results to include some families of Darboux-like functions in $\mathbb{R}^{\mathbb{R}^n}$. To do this we must define (n,k)-additivity which is much more flexible than additivity.

1 Preliminaries and (n,k)-additivity

We use standard notation as in [3]. In particular, $\mathbb{R}$ will stand for the real numbers. For any $r \in \mathbb{R}$ let $\lfloor r \rfloor$ denote the greatest integer less than or equal to r. For sets X and Y we denote the set of all functions from X into Y by Y^X. If f and g are functions with domain X we let $[f = g] = \{x \in X \colon f(x) = g(x)\}$. For a family of functions $\mathcal{F} \subseteq Y^X$, where Y has an appropriate algebraic structure, we let $-\mathcal{F} = \{-f \colon f \in \mathcal{F}\}$ and $n\mathcal{F} = \{f_1 + \cdots + f_n \colon f_i \in \mathcal{F}\}$. The cardinality of a set X will be denoted by $|X|$. We let $\mathfrak{c}$ stand for the cardinality of $\mathbb{R}$ and ω will denote the cardinality of the natural numbers. We say a subset A of $\mathbb{R}$ is $\mathfrak{c}$-dense provided that $|A \cap (a,b)| = \mathfrak{c}$ for all $a, b \in \mathbb{R}$ such that $a < b$. For a cardinal κ we let κ^+ denote its cardinal successor. The symbol $\oplus$ will stand for cardinal addition. Recall that $\kappa \oplus \lambda = \kappa + \lambda$ if κ and λ are finite cardinals, and that $\kappa \oplus \lambda = \max(\kappa, \lambda)$ if either one of κ or λ is an infinite cardinal. Given a topological space X and a natural number n we let X^n stand for the product of the space X with itself n-times with the usual topology.

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We now discuss the notion of (n,k)-additivity. In what follows we will assume that X is a non-empty set and Y is an Abelian group (i.e., $(\forall x, y \in Y)(x+y=y+x)$) with identity 0. We denote by $\theta \in Y^X$ the function with range $\{0\}$. We will assume throughout our discussion that Y contains more than one member; so $|Y^X|>1$. The cardinal function A (the additivity function) has been defined in [14] for families $\mathcal{F} \subseteq Y^X$:

$$\mathrm{A}(\mathcal{F}) = \min(\{|F| \colon F \subseteq Y^X \& (\forall g \in Y^X)(\exists f \in F)(f+g \notin \mathcal{F})\} \cup \{(|Y^X|)^+\}).$$

We refer to the cardinal $\mathrm{A}(\mathcal{F})$ as the additivity of $\mathcal{F}$.

We define the repeatability of $\mathcal{F}$ to be

$$\mathrm{Rep}(\mathcal{F}) = \min(\{n \in \omega \colon n\mathcal{F} = Y^X\} \cup \{\omega\}).$$

The reader interested in these and other cardinal functions in real analysis is referred to the survey article [4]. Below we list some basic facts about the additivity function which can be found in [13].

Proposition 1.1. *Let $\mathcal{G}, \mathcal{F} \subseteq Y^X$. Then,*

(i) $\mathrm{A}(\mathcal{F}) = 1$ *if and only if* $\mathcal{F} = \emptyset$*;*

(ii) $\mathrm{A}(\mathcal{F}) \leq |Y^X|$ *if and only if* $\mathcal{F} \neq Y^X$*;*

(iii) *if* $\mathcal{F} \subset \mathcal{G}$ *then* $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$*;*

(iv) $2 < \mathrm{A}(\mathcal{F})$ *if and only if* $\mathcal{F} - \mathcal{F} = Y^X$*.*

When $\mathcal{F} \subseteq Y^X$ has the property that $\mathcal{F} = -\mathcal{F}$ the statements (ii) and (iv) of Proposition 1.1 imply, using the notation of repeatability,

$$(*) \qquad 2 < \mathrm{A}(\mathcal{F}) \leq |Y^X| \text{ if and only if } \mathrm{Rep}(\mathcal{F}) = 2.$$

One implication of $(*)$ is that for reasonable families $\mathcal{F}$ of functions the additivity is a generalization of repeatability when $\mathrm{Rep}(\mathcal{F}) = 2$. One might also notice that under certain conditions we have a nice restatement of the the definition of A in terms of coding functions as sums of functions in $\mathcal{F}$.

Proposition 1.2. *Let $\mathcal{F} = -\mathcal{F}$ and $\mathrm{A}(\mathcal{F}) \geq \omega$. Then $\mathrm{A}(\mathcal{F})$ is the largest cardinal κ such that for any $F \subseteq Y^X$ of cardinality less than κ there exists a $g \in \mathcal{F}$ such that*

$$(\forall f \in F)(\exists g_f \in \mathcal{F})(g + g_f = f). \tag{1}$$

Proof. Let $F \subseteq Y^X$ and $|F| < \mathrm{A}(\mathcal{F})$. We show that there is a $g \in \mathcal{F}$ which satisfies (1). Since $\mathrm{A}(\mathcal{F}) \geq \omega$, we may assume the zero function θ is an element of F. There is a $g' \in Y^X$ such that $g' + F \subseteq \mathcal{F}$, notice that since $\theta \in F$ we actually have $g' \in \mathcal{F}$. For each $f \in F$ let $g_f = g' + f \in \mathcal{F}$. Since $g' \in \mathcal{F}$ and $-\mathcal{F} = \mathcal{F}$ we have $-g' \in \mathcal{F}$. Now $g = -g'$ satisfies (1).

Now assume κ is a cardinal that is larger than $\mathrm{A}(\mathcal{F})$. We will find a family $F \subseteq Y^X$ such that $|F| < \kappa$ and there is no $g \in \mathcal{F}$ which satisfies (1). Let $F \subseteq Y^X$ witness the definition of $\mathrm{A}(\mathcal{F})$, i.e. $|F| = \mathrm{A}(\mathcal{F}) < \kappa$ and

$$(\forall g \in Y^X)(\exists f \in F)(f + g \notin \mathcal{F}).$$

By way of contradiction assume that there is some $g \in \mathcal{F}$ satisfying (1). Then $f + (-g) = g_f \in \mathcal{F}$ for each $f \in F$ which contradicts our choice of F. □

When the repeatability of the family of functions is greater than 2 the additivity function tells us at most that the repeatability is not 2. Since we will be considering families of functions with repeatabilities that may be larger than 2 we want to define a cardinal function which will be of use for these families. We would like for this cardinal function (actually it will be a family of cardinal functions) to satisfy some statements similar to $(*)$ and Proposition 1.2. It would also be appealing if we could have a statement like $(*)$ that held for all $\mathcal{F} \subseteq Y^X$ not just those where $\mathcal{F} = -\mathcal{F}$. We might also want to remove some of the hypothesizes of Proposition 1.2 as well. Removal of the need for the hypotheses $\mathcal{F} = -\mathcal{F}$ and $\mathrm{A}(\mathcal{F}) \geq \omega$ is fairly simple. We define

$$\mathrm{A}^*(\mathcal{F}) = \min(\{|F| \colon F \subseteq Y^X \& (\forall g \in \mathcal{F})(\exists f \in F)(f - g \notin \mathcal{F})\} \cup \{(|Y^X|)^+\}).$$

We must now ask how closely A^* corresponds to A as a function. The following proposition shows that the correspondence is very close for reasonable families of functions.

Proposition 1.3. *Let $\mathcal{F} \subseteq Y^X$ be such that $\mathcal{F} = -\mathcal{F}$. Then $\mathrm{A}(\mathcal{F}) = 1 \oplus \mathrm{A}^*(\mathcal{F})$. In particular, if $\mathrm{A}(\mathcal{F}) \geq \omega$, then $\mathrm{A}^*(\mathcal{F}) = \mathrm{A}(\mathcal{F})$.*

Proof. We first note that the inequality

$$\mathrm{A}^*(\mathcal{F}) \leq \mathrm{A}(\mathcal{F}) \tag{2}$$

is immediate from the definitions. We show that $\mathrm{A}(\mathcal{F}) \leq 1 \oplus \mathrm{A}^*(\mathcal{F})$. Let $F \subseteq Y^X$ be such that $|F| = \mathrm{A}^*(\mathcal{F})$ and

$$(\forall g \in \mathcal{F})(\exists f \in F)(f + g \notin \mathcal{F}).$$

Put $F_1 = F \cup \{\theta\}$ and notice that

$$(\forall g \in Y^X)(\exists f \in F_1)(f + g \notin \mathcal{F}).$$

Thus,

$$\mathrm{A}(\mathcal{F}) \leq 1 \oplus \mathrm{A}^*(\mathcal{F}). \tag{3}$$

For the other inequality we consider two cases. First assume $\mathrm{A}(\mathcal{F}) \geq \omega$. Then, by (3), $\mathrm{A}^*(\mathcal{F})$ is infinite so $1 \oplus \mathrm{A}^*(\mathcal{F}) = \mathrm{A}^*(\mathcal{F})$. Now by (2) we have $1 \oplus \mathrm{A}^*(\mathcal{F}) = \mathrm{A}^*(\mathcal{F}) \leq \mathrm{A}(\mathcal{F})$.

Next assume $\mathrm{A}(\mathcal{F}) < \omega$. Note that (2) implies that $\mathrm{A}^*(\mathcal{F})$ is finite. We will show that $1 \oplus \mathrm{A}^*(\mathcal{F}) \leq \mathrm{A}(\mathcal{F})$. First assume that $\mathrm{A}^*(\mathcal{F}) = 0$. Since the empty set $\emptyset$ is the only subset of Y^X with cardinality zero we have, by the definition of $\mathrm{A}^*(\mathcal{F})$,

$$(\forall g \in \mathcal{F})(\exists f \in \emptyset)(f - g \notin \mathcal{F}).$$

The only way the above statement may be true is if $\mathcal{F} = \emptyset$. By Proposition 1.1, $\mathrm{A}(\emptyset) = 1$. So in this case $1 \oplus \mathrm{A}^*(\mathcal{F}) = \mathrm{A}(\mathcal{F})$. So we may assume that $\mathrm{A}^*(\mathcal{F}) > 0$. Take $F \subseteq Y^X$ with $|F| < 1 \oplus \mathrm{A}^*(\mathcal{F})$. Fix $f \in F$ and put $F_1 = (-f + F) \setminus \{\theta\}$. We have $|F_1| < \mathrm{A}^*(\mathcal{F})$. So there is a $g \in \mathcal{F}$ such that $-g + F_1 \subseteq \mathcal{F}$. Since $\mathcal{F} = -\mathcal{F}$, we have $-g + (F_1 \cup \{\theta\}) \subseteq \mathcal{F}$. Now $(-g - f) + F = -g + (F_1 \cup \{\theta\}) \subseteq \mathcal{F}$. Thus, $1 \oplus \mathrm{A}^*(\mathcal{F}) \leq \mathrm{A}(\mathcal{F})$. $\square$

Now the A^*-analog of $(*)$ is that for any $\mathcal{F} \subseteq Y^X$

$$(*_1) \qquad 1 < \mathrm{A}^*(\mathcal{F}) \leq |Y^X| \text{ if and only if } \mathrm{Rep}(\mathcal{F}) = 2.$$

The version of Proposition 1.2 for A^* is more simple to state and prove.

Proposition 1.4. $\mathrm{A}^*(\mathcal{F})$ *is the largest cardinal* κ *such that for any* $F \subseteq Y^X$ *of cardinality less than* κ *there exists a* $g \in \mathcal{F}$ *such that*

$$(\forall f \in F)(\exists g_f \in \mathcal{F})(g + g_f = f). \tag{4}$$

PROOF. Let $F \subseteq Y^X$ and $|F| < \mathrm{A}^*(\mathcal{F})$. We show that there is a $g \in \mathcal{F}$ which satisfies (4). Clearly there is a $g \in \mathcal{F}$ such that $-g + F \subseteq \mathcal{F}$. For each $f \in \mathcal{F}$ let $g_f = -g + f \in \mathcal{F}$. Then g satisfies (4).

Now assume κ is a cardinal that is larger than $\mathrm{A}^*(\mathcal{F})$. We find a family $F \subseteq Y^X$ such that $|F| < \kappa$ and there is no $g \in \mathcal{F}$ which satisfies (4). Let $F \subseteq Y^X$ witness the definition of $\mathrm{A}^*(\mathcal{F})$, i.e., $|F| = \mathrm{A}^*(\mathcal{F}) < \kappa$ and

$$(\forall g \in \mathcal{F})(\exists f \in F)(f - g \notin \mathcal{F}).$$

By way of contradiction assume that there is some $g \in \mathcal{F}$ satisfying

$$(\forall f \in F)(\exists g_f \in \mathcal{F})(g + g_f = f)$$

Then $f - g = g_f \in \mathcal{F}$ for each $f \in F$ which contradicts our choice of F. $\square$

Now our goal will be be to construct a family of cardinal functions that will allow us to generalize $(*_1)$ and Proposition 1.4 to include families with repeatabilities greater than 2. Let $n, k \in \omega$ be such that $k < n$ and let $\mathcal{F} \subseteq Y^X$. We define the (n,k)-additivity of $\mathcal{F}$ to be

$$\mathrm{A}_{n,k}(\mathcal{F}) = \min(\{|F| \colon F \subseteq Y^X \,\&\, \Psi_{n,k}(F, \mathcal{F}) \text{ holds}\} \cup \{|Y^X|^+\})$$

where $\Psi_{n,k}(F, \mathcal{F})$ denotes the statement

$$(\forall g \in (n-k)\mathcal{F})(\exists f \in F)(f - g \notin (k+1)\mathcal{F}).$$

Notice that $\mathrm{A}^* = \mathrm{A}_{1,0}$. We can now restate Proposition 1.3 in this language.

Proposition 1.5. *Let $\mathcal{F} \subseteq Y^X$ be such that $\mathcal{F} = \{-f \colon f \in \mathcal{F}\}$. Then $\mathrm{A}(\mathcal{F}) = 1 \oplus \mathrm{A}_{1,0}(\mathcal{F})$. In particular, if $\mathrm{A}(\mathcal{F}) \geq \omega$, then $\mathrm{A}_{1,0}(\mathcal{F}) = \mathrm{A}(\mathcal{F})$.*

We see that Proposition 1.4 is generalized as follows. We leave the proof to the reader since it is of the same form as the proof of Proposition 1.4.

Proposition 1.6. *$\mathrm{A}_{n,k}(\mathcal{F})$ is the largest cardinal κ such that for any $F \subseteq Y^X$ of cardinality less than κ there exist $g^1, \ldots, g^{n-k} \in \mathcal{F}$ such that*

$$(\forall f \in F)(\exists g_f^1, \ldots, g_f^{k+1} \in \mathcal{F})(g^1 + \ldots + g^{n-k} + g_f^1 + \ldots + g_f^{k+1} = f).$$

We state an expanded version of Proposition 1.1 for (n,k)-additivity and include some other facts.

Proposition 1.7. *Let $\mathcal{F}, \mathcal{G} \subseteq Y^X$ and $n \in \omega$. Then*

(i) *if $k < n$, then $\mathrm{A}_{n,k}(\mathcal{F}) = 0$ if and only if $\mathcal{F} = \emptyset$;*

(ii) *if $k < n$, and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathrm{A}_{n,k}(\mathcal{F}) \leq \mathrm{A}_{n,k}(\mathcal{G})$;*

(iii) *if $i < k < n$, then $\mathrm{A}_{n,i}(\mathcal{F}) \leq \mathrm{A}_{n,k}(\mathcal{F})$;*

(iv) *if $k < n$, then $\mathrm{A}_{n,k}(\mathcal{F}) \leq |Y^X|$ if and only if $\mathrm{Rep}(\mathcal{F}) > k + 1$;*

(v) *if $k < n$, then $1 < \mathrm{A}_{n,k}(\mathcal{F})$ if and only if $\mathrm{Rep}(\mathcal{F}) \leq n + 1$;*

(vi) *$1 < \mathrm{A}_{n,k}(\mathcal{F}) \leq |Y^X|$ for all $k < n$ if and only if $\mathrm{Rep}(\mathcal{F}) = n + 1$;*

(vii) *if* $\mathrm{Rep}(\mathcal{F}) \geq n+1$ *and* $\mathcal{F} = -\mathcal{F}$*, then* $\mathrm{A}_{n,k}(\mathcal{F}) \leq 2$ *for all* $k < \lfloor n/2 \rfloor$*;*

(viii) *if* $k < n$*, then* $\mathrm{A}_{n,k}(\mathcal{F}) \leq \mathrm{A}((k+1)\mathcal{F})$*.*

PROOF. Items (i) and (ii) are straight forward and will be left without proof.

We prove (iii). Let $F \subseteq Y^X$ and $|F| < \mathrm{A}_{n,i}(\mathcal{F})$. By Proposition 1.6 there exist $g^1, \dots, g^{n-i} \in \mathcal{F}$ such that

$$(\forall f \in F)(\exists g_f^1, \cdots, g_f^{i+1} \in \mathcal{F})(g^1 + \cdots + g^{n-i} + g_f^1 + \cdots + g_f^{i+1} = f). \tag{5}$$

Let $m = k - i$. By (5) we have for each $f \in F$,

$$g_f^1 + \cdots + g_f^{i+1} + g^1 + \cdots + g^m = f - (g^{m+1} + \cdots + g^{n-i}). \tag{6}$$

Note that $n - i - (m+1) + 1 = n - k$ so $g = g^{m+1} + \cdots + g^{n-i} \in (n-k)\mathcal{F}$. Notice also that $i+1+m = k+1$, so $g_f^1 + \cdots + g_f^{i+1} + g^1 + \cdots + g^m \in (k+1)\mathcal{F}$. So (6) implies that there is a $g \in (n-k)\mathcal{F}$ such that $-g + F \subseteq (k+1)\mathcal{F}$. Thus $\mathrm{A}_{n,i}(\mathcal{F}) \leq \mathrm{A}_{n,k}(\mathcal{F})$.

We prove (iv). Suppose $\mathrm{A}_{n,k}(\mathcal{F}) = |Y^X|^+$. Then there is a $g \in (n-k)\mathcal{F}$ such that $(-g) + (Y^X) \subseteq (k+1)\mathcal{F} \subseteq Y^X$. Thus $(k+1)\mathcal{F} = Y^X$. Hence $\mathrm{Rep}(\mathcal{F}) \leq k+1$. To see the other implication assume that $\mathrm{Rep}(\mathcal{F}) \leq k+1$ which is to say $(k+1)\mathcal{F} = Y^X$. Then $g + (k+1)\mathcal{F} = Y^X$ for any $g \in Y^X$. In particular, we may pick g to be in $(n-k)\mathcal{F}$. Thus, $(-g) + (Y^X) \subseteq (k+1)\mathcal{F}$ for some $g \in (n-k)\mathcal{F}$. So $\mathrm{A}_{n,k}(\mathcal{F}) = |Y^X|^+$.

We prove (v). Suppose $\mathrm{A}_{n,k}(\mathcal{F}) \geq 2$. Then for any $f \in Y^X$ there is a $g \in (n-k)\mathcal{F}$ such that $f - g \in (k+1)\mathcal{F}$, which is to say $f \in (n+1)\mathcal{F}$. Thus $(n+1)\mathcal{F} = Y^X$ and so $\mathrm{Rep}(\mathcal{F}) \leq n+1$. Now, assume $\mathrm{A}_{n,k}(\mathcal{F}) \leq 1$ and let $G = \{g\}$ witness the definition of $\mathrm{A}_{n,k}(\mathcal{F})$ (i.e., $f - g \notin (k+1)\mathcal{F}$ for every $g \in (n-k)\mathcal{F}$). Clearly, $f \notin (n+1)\mathcal{F}$ so $(n+1)\mathcal{F} \neq Y^X$ which is to say $\mathrm{Rep}(\mathcal{F}) > n+1$.

Item (vi) is a direct consequence of (iv) and (v).

We now prove (vii). Let $k < \lfloor n/2 \rfloor$. Since $\mathrm{Rep}(\mathcal{F}) \geq n+1$ there is a function $h\colon X \to Y$ that is not an element of $(2k+2)\mathcal{F}$. By way of contradiction assume $\mathrm{A}_{n,k}(\mathcal{F}) > 2$. Pick $f_1, f_2 \in Y^X$ such that $f_1 - f_2 = h$. Since $\mathrm{A}_{n,k}(\mathcal{F}) > 2$, there is a $g \in (n-k)\mathcal{F}$ such that $\{f_1 - g, f_2 - g\} \subseteq (k+1)\mathcal{F}$. Note that $-(f_2 - g) \in (k+1)\mathcal{F}$ since $\mathcal{F} = -\mathcal{F}$. Thus,

$$h = f_1 - f_2 = (f_1 - g) - (f_2 - g) \in (2k+2)\mathcal{F}$$

contradicting our choice of h.

We prove (viii). Let $F \subseteq Y^X$ and $|F| < \mathrm{A}_{n,k}(\mathcal{F})$. There is a $g \in (n-k)\mathcal{F}$ such that $-g + F \subseteq (k+1)\mathcal{F}$. Thus, $\mathrm{A}_{n,k}(\mathcal{F}) \leq \mathrm{A}((k+1)\mathcal{F})$. $\square$

Notice that (vi) of Proposition 1.7 is a good generalization of $(*_1)$.

2 Results

We calculate the generalized additivities of the following families of functions. Descriptions of these functions for general topological spaces are given.

Dar: $f \in Y^X$ is a Darboux function if and only if $f[C]$ is connected in Y for any connected set C of X.

Con: $f \in Y^X$ is a connectivity function if and only if the graph of f restricted to C is connected in $X \times Y$ for every connected set C of X.

AC: $f \in Y^X$ is an almost continuous function if and only if every open set in $X \times Y$ containing f also contains some continuous function $g \in Y^X$.

Ext: $f \in Y^X$ is an extendable function if and only if there is a connectivity function $g : X \times [0,1] \to Y$ such that $f(x) = g(0,x)$ for every $x \in X$.

PC: $f \in Y^X$ is a peripherally continuous function if and only if for every $x \in X$ and pair of open sets $U \subset X$ and $V \subset Y$ such that $x \in U$ and $f(x) \in V$ there is an open neighborhood W of x with $\mathrm{cl}(W) \subset U$ and $f[\mathrm{bd}(W)] \subset V$, where $\mathrm{cl}(W)$ and $\mathrm{bd}(W)$ denote the boundary and closure of W, respectively.

SZ: $f \in Y^X$ is a Sierpiński-Zygmund function if and only if f is continuous on no set of cardinality $|X|$.

We note that in [12] it is shown that $\mathrm{Con}(\mathbb{R}^n, \mathbb{R}^m) = \mathrm{PC}(\mathbb{R}^n, \mathbb{R}^m)$ when $n > 1$. The equality $\mathrm{Ext}(\mathbb{R}^n, \mathbb{R}) = \mathrm{Con}(\mathbb{R}^n, \mathbb{R})$ is shown in [8] for $n > 1$. The relationships of containment between the above families for the cases $n = 1$ and $n > 1$ are shown in the diagrams below. Each arrow denotes proper containment (see [2] for $n = 1$ and for $n > 1$ see [16] and [14, Examples 1.6 and 1.7]).

$n = 1$: $\mathrm{Ext} \longrightarrow \mathrm{AC} \longrightarrow \mathrm{Con} \longrightarrow \mathrm{Dar} \longrightarrow \mathrm{PC}$

$$\begin{tikzcd} & & \mathrm{Dar} \\ n > 1: & \mathrm{Ext} = \mathrm{PC} = \mathrm{Con} \arrow[ur] \arrow[dr] & \\ & & \mathrm{AC} \end{tikzcd}$$

To state some of the theorems we will need to define the following cardinal number.

$$e_{\mathfrak{c}} = \min\{|F| \colon F \subseteq \mathfrak{c}^{\mathfrak{c}} \;\&\; (\forall g \in \mathfrak{c}^{\mathfrak{c}})(\exists f \in F)(|[f = g]| < \mathfrak{c})\}.$$

It is easy to prove that $\mathfrak{c} < e_{\mathfrak{c}} \leq 2^{\mathfrak{c}}$ and in [5] it is shown that this is about all that can be said about its value in ZFC. We intend to prove the following six theorems. The first three deal with the generalized additivities of some the families of functions listed above. The last three theorems are concerned with some containments between between the above families of functions.

Theorem 2.1. *Let $n > 1$. Then,*

$$\mathrm{A}_{n,n-1}(\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})) = \mathrm{A}(n\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})) = \mathfrak{c}^+.$$

Theorem 2.2. *Let $n, m \geq 1$. Then,*

$$\mathrm{A}_{n,n-1}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) = \mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) = e_{\mathfrak{c}}.$$

Theorem 2.3. *Let $n, m \geq 1$. Then,* $\mathrm{A}(\mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)) = e_{\mathfrak{c}}$.

Theorem 2.4. *Let $n, m \geq 1$. Then*

$$\mathrm{A}_{n,j}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) = \mathrm{A}((j+1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) = \mathfrak{c}^+$$

for $n - 1 > j \geq \lfloor n/2 \rfloor$.

Before stating the other three main Theorems we consider some implications of the two Theorems above. Using Propositions 1.3 and 1.7(vi) we see that Theorems 2.1, 2.2 and 2.3 generalize the following results.

Proposition 2.5.

(i) [6] $\mathrm{A}(\mathrm{Ext}(\mathbb{R}, \mathbb{R})) = \mathfrak{c}^+$.

(ii) [7] $\mathrm{Rep}(\mathcal{F}(\mathbb{R}^n, \mathbb{R})) = n + 1$ for $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{Con}\}$.

(iii) [5] $\mathrm{A}(\mathrm{Dar}(\mathbb{R},\mathbb{R})) = \mathrm{A}(\mathrm{AC}(\mathbb{R},\mathbb{R})) = e_{\mathfrak{c}}$.

Theorem 2.2 has as a corollary, using Proposition 1.7(v), the answer to a question of Ciesielski and Wojciechowski [7].

Corollary 2.6. *Let $n, m \geq 1$. Then* $\mathrm{Rep}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) = n + 1$.

Theorems 2.1 and 2.2 also have the following two corollaries:

Corollary 2.7. *If $k < \lfloor n/2 \rfloor$ then $\mathrm{A}_{n,k}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) = 2$, and if $m > k \geq n$ then $\mathrm{A}_{m,k}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) = (2^{\mathfrak{c}})^+$.*

Proof. By Theorem 2.1 and Proposition 1.7(v) $\mathrm{Rep}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) = n+1$. Since $\mathrm{Ext}(\mathbb{R}^n,\mathbb{R}) = -\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$, Proposition 1.7(vii) implies that if $k < \lfloor n/2 \rfloor$, then $\mathrm{A}_{n,k}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) = 2$. Since $\mathrm{Rep}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R}) = n+1$, it follows from Proposition 1.7(iv) that if $m > k \geq n$ then $\mathrm{A}_{m,k}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) = (2^{\mathfrak{c}})^+$. □

Corollary 2.8. *If $k < \lfloor n/2 \rfloor$ then $\mathrm{A}_{n,k}(\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)) = 2$, and if $m > k \geq n$, then $\mathrm{A}_{m,k}(\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)) = (2^{\mathfrak{c}})^+$.*

Proof. Repeat the proof of Corollary 2.7 but with $\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$ in place of $\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$ and using Theorem 2.2 instead of Theorem 2.1. □

The above results for $n > 1$ are summarized in the tables below.

	$k < p < n$	$k < n = p$	$n \leq k < p$
$\mathcal{F} = \mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$	1	see table below	$(2^{\mathfrak{c}})^+$
$\mathcal{F} = \mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$	1	see table below	$(2^{\mathfrak{c}})^+$

Figure 3.1

	$k < \lfloor n/2 \rfloor$	$\lfloor n/2 \rfloor \leq k \leq n-2$	$k = n-1$
$\mathcal{F} = \mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$	2	?	$\mathfrak{c}^+$
$\mathcal{F} = \mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$	2	$\mathfrak{c}^+$	$e_{\mathfrak{c}}$

Figure 3.2

The first table gives the values of $\mathrm{A}_{p,k}(\mathcal{F})$; the second table gives the values of $\mathrm{A}_{n,k}$. We now state the remaining theorems.

Theorem 2.9. $\mathrm{AC}(\mathbb{R}^n,\mathbb{R}^m) \subsetneq n\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$. *Moreover, there is an almost continuous function f such that $f \notin (n-1)\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$.*

Theorem 2.10. $n\mathrm{Dar}(\mathbb{R}^n,\mathbb{R})\cap \mathrm{SZ}(\mathbb{R}^n,\mathbb{R})=\emptyset$ *for* $n>1$.

An immediate Corollary of Theorems 2.9 and 2.10 is

Corollary 2.11. *If* $n>1$ *then* $\mathrm{AC}(\mathbb{R}^n,\mathbb{R})\cap \mathrm{SZ}(\mathbb{R}^n,\mathbb{R})=\emptyset$.

In [1] it is shown that the conclusions of Theorem 2.10 and Corollary 2.11 are independent of ZFC when $n=1$.

Corollary 2.6 and Theorem 2.9 might lead one to conjecture that every function from $\mathbb{R}^n$ into $\mathbb{R}$ is the sum of an almost continuous function and a Darboux function. The following example which is constructed in Section 4 shows that this is very much not the case when $n>1$.

Example 2.1. *There exist a Baire class 1 function* $f\colon \mathbb{R}^n\to\mathbb{R}$ *such that* $f\notin n\mathrm{Dar}(\mathbb{R}^n,\mathbb{R})$. *Moreover, for* $n>1$ *we have that* f *is not the sum of an almost continuous function and* $n-1$ *Darboux functions.*

3 Proofs of Theorems 2.2 and 2.10.

Lemma 3.1. *Let* $n,m\geq 1$, $f\colon\mathbb{R}^n\to\mathbb{R}^m$ *be a function and* $A,B\subseteq\mathbb{R}$ *be a partition of* $\mathbb{R}$ *into two disjoint* $\mathfrak{c}$*-dense sets. If* $\pi:\mathbb{R}^n\to\mathbb{R}$ *denotes the projection of* $\mathbb{R}^n$ *onto a fixed coordinate and*

(i) $f|_{\pi^{-1}(y)}\in\mathrm{Dar}(\pi^{-1}(y),\mathbb{R}^m)$ *for every* $y\in B$,

(ii) $f|_{\pi^{-1}(y)}$ *is constant for every* $y\in A$, *and*

(iii) $\{y\in A\colon f[\pi^{-1}(y)]=\{r\}\}$ *is dense in* $\mathbb{R}$ *for every* $r\in\mathbb{R}^m$,

then $f\in\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$.

PROOF. Suppose $C\subseteq\mathbb{R}^n$ is connected. We show that $f[C]$ is a connected subset of $\mathbb{R}^m$. There are two possible cases that may occur. The first case is when there is a $y\in\mathbb{R}$ such that $C\subseteq\pi^{-1}(y)$. In this case, one of (i) or (ii) applies to show that $f[C]$ is a connected set. The other case is when there exist distinct $y_1<y_2\in\mathbb{R}$ such that $C\cap\pi^{-1}(y_1)\neq\emptyset\neq C\cap\pi^{-1}(y_2)$. Since C is connected, we have $C\cap\pi^{-1}(y)\neq\emptyset$ for all $y\in[y_1,y_2]$. So, by (iii), $f[C]=\mathbb{R}^m$, which is connected. □

We now prove one of the main inequalities of Theorem 2.2.

Lemma 3.2. $\mathrm{A}_{n,n-1}(\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m))\geq e_{\mathfrak{c}}$.

PROOF. We proceed by induction on n. The inequality is proven in [5] for the case $\mathrm{Dar}(\mathbb{R},\mathbb{R})$, and the methods used in [5] can clearly be used to establish the inequality for $\mathrm{Dar}(\mathbb{R},\mathbb{R}^m)$ when $m>1$. So we may assume that $n>1$ and $\mathrm{A}_{n-1,n-2}(\mathrm{Dar}(\mathbb{R}^{n-1},\mathbb{R}^m))\geq e_{\mathfrak{c}}$. Let $F\subseteq(\mathbb{R}^m)^{\mathbb{R}^n}$ be an arbitrary collection of functions such that $|F|<e_{\mathfrak{c}}$. We must find a $g\in\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$ such that $-g+F\subseteq n\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$. Let $\{A_k\}_{k\in n+1}$ be a partition of $\mathbb{R}$ into $n+1$ $\mathfrak{c}$-dense sets. Define $h\colon\mathbb{R}\to\mathbb{R}^m$ so that for each $p\in\mathbb{R}^m$ and $k\in n+1$

$$\{y\in A_k\colon h(y)=p\}\text{ is dense in }\mathbb{R}.$$

Let $\pi\colon\mathbb{R}^n\to\mathbb{R}$ denote the projection of $\mathbb{R}^n$ onto a fixed coordinate. Note that $\pi^{-1}(y)$ is homeomorphic to $\mathbb{R}^{n-1}$ for every $y\in\mathbb{R}$. For each $y\in\mathbb{R}$ let $F^y=\{f|_{\pi^{-1}(y)}-h(y)\colon f\in F\}$. Since $|F^y|\leq|F|<e_{\mathfrak{c}}$, we may apply the inductive hypothesis to find for each $l\in n$ and $y\in A_l$ a Darboux function $g^y\colon\pi^{-1}(y)\to\mathbb{R}^m$ such that $-g^y+F^y\subseteq(n-1)\mathrm{Dar}(\pi^{-1}(y),\mathbb{R}^m)$. So, for every $f\in F$ and $y\in A_l$ there exist $\{g_k^{f,y}\in\mathrm{Dar}(\pi^{-1}(y),\mathbb{R}^m)\colon k\in n\setminus\{l\}\}$ such that

$$g^y+\sum_{\{k\in n\colon k\neq l\}}g_k^{f,y}=f|_{\pi^{-1}(y)}-h(y). \tag{7}$$

Define $g\colon\mathbb{R}^n\to\mathbb{R}^m$ by

$$g(p)=\begin{cases}h(y) & \text{if }\pi(p)=y\in A_n,\\ g^y(p) & \text{if }\pi(p)=y\notin A_n.\end{cases}$$

We claim g is as desired. To see that g is Darboux let $A=A_n$, $B=\mathbb{R}\setminus A_n$, and apply Lemma 3.1. By inductive hypothesis, $\mathrm{A}_{n-1,n-2}(\mathrm{Dar}(\pi^{-1}(y),\mathbb{R}^m)\geq e_{\mathfrak{c}}$ for each $y\in A_n$. By Proposition 1.7(v), $\mathrm{Rep}(\mathrm{Dar}(\pi^{-1}(y),\mathbb{R}^m))\leq n$. Thus, for each $y\in A_n$ and $f\in F$, we may find $\{g_k^{f,y}\in\mathrm{Dar}(\pi^{-1}(y),\mathbb{R}^m)\colon k\in n\}$ such that

$$\sum_{k=0}^{n-1}g_k^{f,y}=f|_{\pi^{-1}(y)}-h(y). \tag{8}$$

For each $f\in F$ and $k\in n$ define $g_k^f\colon\mathbb{R}^n\to\mathbb{R}^m$ so that

$$g_k^f(p)=\begin{cases}h(y) & \text{if }\pi(p)=y\in A_k,\\ g_k^{f,y} & \text{if }\pi(p)=y\notin A_k.\end{cases} \tag{9}$$

Note that $g_k^f\in\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$ for each $f\in F$ and $k\in n$. To see it, take $A=A_k$, $B=\mathbb{R}\setminus A_k$ and apply Lemma 3.1. We now show that $-g+F\subseteq n\mathrm{Dar}(\mathbb{R}^n,\mathbb{R}^m)$.

More precisely we show that for each $f \in F$

$$g + \sum_{k=0}^{n-1} g_k^f = f. \tag{10}$$

Let $p \in \mathbb{R}^n$. We must consider two cases. First, assume there is an $l \in n$ and a $y \in A_l$ such that $p \in \pi^{-1}(y)$. Then,

$$\begin{aligned} g(p) + \sum_{k=0}^{n-1} g_k^f(p) =& g^y(p) + \left(\sum_{\{k \in n:\ k \neq l\}} g_k^{f,y}(p) + g_l^f(p) \right) \\ =& \left(g^y(p) + \sum_{\{k \in n:\ k \neq l\}} g_k^{f,y}(p) \right) + g_l^f(p) \\ =& (f(p) - h(y)) + h(y) = f(p), \end{aligned} \tag{11}$$

where (11) follows from (7) and (9). We now consider the case in which there exists a $y \in A_n$ such that $p \in \pi^{-1}(y)$. Then,

$$\begin{aligned} g(p) + \sum_{k=0}^{n-1} g_k^f(p) =& h(y) + \sum_{k=0}^{n-1} g_k^{f,y}(p) \\ =& h(y) + (f(p) - h(y)) = f(p), \end{aligned} \tag{12}$$

where (12) follows from (8). So, (10) holds completing the inductive step. □

Our next goal is to show that $\mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq e_{\mathfrak{c}}$. Since Proposition 1.7(viii) implies that $\mathrm{A}_{n,n-1}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq \mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m))$, this will complete the proof. We will need some results and definitions from dimension theory, all of which may be found in [11].

The dimension of a topological space is defined as follows.

(i) $\dim X = -1$ if and only if $X = \emptyset$.

(ii) $\dim X \leq n$ if for any $p \in X$ and any open neighborhood W of p there exists an open neighborhood $U \subseteq W$ of p such that $\dim \mathrm{bd}_X(U) \leq n-1$.

(iii) $\dim X = n$ if $\dim X \leq n$ and it is not true that $\dim X \leq n-1$.

An n-dimensional Cantor manifold ($n \geq 1$) is a compact n-dimensional space that cannot be disconnected by a subset of dimension less than $n-1$. We will need three facts about n-dimensional Cantor manifolds. We say X is a *continuum* if X is a compact, connected, nonempty metric space. We say a continuum X is *degenerate* if $|X| = 1$; otherwise, we say X is *non-degenerate*.

Proposition 3.3. *Every n-dimensional Cantor manifold is a continuum.*

Proposition 3.4. *If X is a compact n-dimensional space, then X contains an n-dimensional Cantor manifold.*

Proposition 3.5. $[0,1]^n$ *is an n-dimensional Cantor manifold for all $n > 0$.*

To establish the inequality $\mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq e_{\mathfrak{c}}$, we must first prove some lemmas.

Lemma 3.6. *Suppose $n > 1$ and M is an n-dimensional Cantor manifold. If $B \subseteq M$ disconnects M, then there is an $(n-1)$-dimensional Cantor manifold contained in B .*

PROOF. It is widely known that if a subset S of a continuum X disconnects X, then there is a compact set $F \subseteq S$ that disconnects X. In particular, we may find a compact set $C \subseteq B$ such that C disconnects M. By the definition of Cantor manifold, the dimension of C is at least $n-1$. So, by Proposition 3.4, there is an $(n-1)$-dimensional Cantor manifold N such that $N \subseteq C \subseteq B$. $\square$

Lemma 3.7. *Assume that $n > 1$, M is an n-dimensional Cantor manifold, and $f\colon M \to \mathbb{R}$ is Darboux. There is a collection $\{N_\alpha\}_{\alpha \in \mathfrak{c}}$ of pairwise disjoint $(n-1)$-dimensional Cantor manifolds such that $f|_{N_\alpha}$ is constant for each $\alpha \in \mathfrak{c}$.*

PROOF. We first assume f is not constant. Let r be in the interior of $f[M]$. Since $f[M]\setminus\{r\}$ is not connected and f is Darboux, it follows that $M \setminus f^{-1}(\{r\})$ is not connected. By Lemma 3.6 there is a $(n-1)$-dimensional Cantor manifold N such that $N \subseteq f^{-1}(\{r\})$. Thus, $f|_N$ is constant. Since we may do this for each r in the interior of $f[M]$ and preimages of distinct points are disjoint, we can find the desired collection of continua.

If f is a constant function, the lemma reduces to the question of whether there exist $\mathfrak{c}$-many pairwise disjoint $(n-1)$-dimensional Cantor manifolds contained in M. By the first part of the proof of this lemma, it is enough to show there is a non-constant Darboux function $h\colon M \to \mathbb{R}$. Fix $x_0 \in M$ and let $h\colon M \to \mathbb{R}$ be the continuous function defined by $h(x) = \mathrm{dist}(\mathrm{x}, \mathrm{x}_0)$. Clearly, h is non-constant and Darboux. $\square$

Lemma 3.8. *Suppose that $n > 1$ and M is a n-dimensional Cantor manifold. If $n \geq k \geq 1$ and $f \in k\mathrm{Dar}(M, \mathbb{R})$, then there is a collection of pairwise disjoint non-degenerate continua $\{C_\alpha\}_{\alpha \in \mathfrak{c}}$ such that $f|_{C_\alpha}$ is Darboux for all $\alpha \in \mathfrak{c}$. Moreover, if $n > k$ then we may assume $f|_{C_\alpha}$ is constant for all $\alpha \in \mathfrak{c}$.*

PROOF. Let M be an n-dimensional Cantor manifold and $f \in k\mathrm{Dar}(M,\mathbb{R})$. We proceed by induction on n. First, we establish the lemma for $n = 2$. When $n = 2$ and $k = 1$, the result is immediate by Lemma 3.7. When $n = 2$ and $k = 2$, then $f = g_1 + g_2$ where $g_1, g_2 \in \mathrm{Dar}(M,\mathbb{R})$. By Lemma 3.7, there is a collection of pairwise disjoint 1-dimensional Cantor manifolds $\{C_\alpha\}_{\alpha\in\mathfrak{c}}$ such that $g_1|_{C_\alpha}$ is constant for every $\alpha \in \mathfrak{c}$. It follows that $f|_{C_\alpha}$ is Darboux for every $\alpha \in \mathfrak{c}$. This completes the proof of the lemma for $n = 2$ since each C_α is a continuum.

So, suppose the lemma holds for all m less than n. We show it also holds for n. If $k = 1$, the result is immediate by Lemma 3.7. So, we may assume $k > 1$. Let $f = g_1 + \cdots + g_k$ where $g_1, \ldots, g_k \in \mathrm{Dar}(M,\mathbb{R})$. By Lemma 3.7 there is an $(n-1)$-dimensional Cantor manifold N such that $g_1|_N$ is a constant function. Thus, $f|_N \in (k-1)\mathrm{Dar}(N,\mathbb{R})$. By inductive hypothesis there is a collection of pairwise disjoint continua, $\{C_\alpha\}_{\alpha\in\mathfrak{c}}$, such that $C_\alpha \subseteq N \subseteq M$ and $f|_{C_\alpha}$ is Darboux for every $\alpha \in \mathfrak{c}$. Finally, if $n > k$ then $n - 1 > k - 1$. So we may assume by inductive hypothesis that $f|_{C_\alpha}$ is constant for every $\alpha \in \mathfrak{c}$ since $f|_N$ is constant. So, the lemma holds for all $n \geq 2$. $\square$

Lemma 3.9. *Let $n > 1$, $n \geq k \geq 1$, and $f \in k\mathrm{Dar}(\mathbb{R}^n,\mathbb{R})$. Then, there is a collection of pairwise disjoint non-degenerate continua $\{C_\alpha\}_{\alpha\in\mathfrak{c}}$ such that $f|_{C_\alpha}$ is Darboux for each $\alpha \in \mathfrak{c}$. Moreover, if $n > k$, then we may assume $f|_{C_\alpha}$ is constant for every $\alpha \in \mathfrak{c}$.*

PROOF. If f is Darboux then its restriction to $[0,1]^n$ is also Darboux. By Proposition 3.5, $[0,1]^n$ is an n-dimensional Cantor manifold. The lemma now follows from Lemma 3.8. $\square$

Lemma 3.9 has an easy consequence.

Lemma 3.10. *Let $n > 1$, $n \geq k \geq 1$, and $f \in k\mathrm{Dar}(\mathbb{R}^n,\mathbb{R})$. Then either f is constant on some non-degenerate continuum or there is a rational number q such that $|f^{-1}(\{q\})| = \mathfrak{c}$.*

PROOF. Let $\{C_\alpha\}_{\alpha\in\mathfrak{c}}$ be as in Lemma 3.9 with $M = \mathbb{R}^n$. If f is constant on no non-degenerate continuum, $f[C_\alpha]$ must be a non-degenerate interval for each $\alpha \in \mathfrak{c}$ and must contain a rational number. It follows that $|f^{-1}(q)| = \mathfrak{c}$ for some $q \in \mathbb{Q}$. $\square$

We now prove Theorem 2.10.

PROOF OF THEOREM 2.10. By Lemma 3.10, if $f \in n\mathrm{Dar}(\mathbb{R}^n,\mathbb{R})$, then f is constant on a set of cardinality continuum. In particular, f is not a Sierpiński-Zygmund function. $\square$

To get the upper bound of Theorem 2.2, we will need the following combinatorial lemma which concerns the cardinal $e_{\mathfrak{c}}$.

Lemma 3.11. *$e_{\mathfrak{c}} = \kappa$ where*

$$\kappa = \min\{|F| \colon F \subseteq \mathfrak{c}^{\mathfrak{c}} \& \ (\forall G \in [\mathfrak{c}^{\mathfrak{c}}]^{\omega})(\exists f \in F)(\forall g \in G)(|[f = g]| < \mathfrak{c})\}.$$

PROOF. We show that $\kappa \leq e_{\mathfrak{c}}$. Let $V = \mathfrak{c} \times \omega$ and $W = \mathfrak{c}^{\omega}$. Take $F \subseteq \mathfrak{c}^{V}$, witnessing the definition of $e_{\mathfrak{c}}$, i.e., $|F| = e_{\mathfrak{c}}$ and

$$(\forall g \in \mathfrak{c}^{V})(\exists f \in F)(|[f = g]| < \mathfrak{c}). \tag{13}$$

It is enough to construct a family $F^* \subseteq W^{\mathfrak{c}}$ such that $|F^*| \leq |F|$ and

$$(\forall G \in [W^{\mathfrak{c}}]^{\omega})(\exists f \in F^*)(\forall g \in G)(|[f = g]| < \mathfrak{c}). \tag{14}$$

For every $f \in F$ let $f^* \in W^{\mathfrak{c}}$ be defined so that $f^*(\alpha)(n) = f(\alpha, n)$ for every $\alpha \in \mathfrak{c}$ and $n \in \omega$. Let $F^* = \{f^* \colon f \in F\}$ and note that $|F^*| \leq |F|$. We show that F^* satisfies (14). Let $G \in [W^{\mathfrak{c}}]^{\omega}$ and enumerate G by $G = \{g_n \colon n \in \omega\}$. Define $g' \in \mathfrak{c}^{V}$ so that $g'(\alpha, n) = g_n(\alpha)(n)$ for $\alpha \in \mathfrak{c}$ and $n \in \omega$. By (13), there is an $f \in F$ such that $|[f = g'| < \mathfrak{c}$. We show that $|[f^* = g_n]| < \mathfrak{c}$ for every $n \in \omega$. By way of contradiction, assume that $|[f^* = g_n]| = \mathfrak{c}$ for some $n \in \omega$. Then, $f^*(\alpha)(k) = g_n(\alpha)(k)$ for every $k \in \omega$ and $\alpha \in [f^* = g_n]$. In particular, we have $f^*(\alpha)(n) = g_n(\alpha)(n)$ for each $\alpha \in [f^* = g_n]$. So, $f(\alpha, n) = g'(\alpha, n)$ for every $\alpha \in [f^* = g_n]$. Since $|[f^* = g_n]| = \mathfrak{c}$ we have $|[f = g']| = \mathfrak{c}$, contradicting our choice of f. So F^* satisfies (14) and $\kappa \leq e_{\mathfrak{c}}$. The other inequality is trivial. □

We may now prove the remaining inequality for $m = 1$.

Lemma 3.12. *If $n > 1$, then* $\mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})) \leq e_{\mathfrak{c}}$.

PROOF. By Lemma 3.11 there is a family $F \subseteq (\mathbb{R})^{\mathbb{R}^n}$ such that $|F| = e_{\mathfrak{c}}$ and

$$\left(\forall H \in \left[(\mathbb{R})^{\mathbb{R}^n}\right]^{\omega}\right)(\exists f \in F)\,(\forall h \in H)\,(|[f = h]| < \mathfrak{c})\,. \tag{15}$$

It is enough to show that F satisfies

$$\left(\forall g \in (\mathbb{R})^{\mathbb{R}^n}\right)(\exists f \in F)\,(f + g \notin n\mathrm{Dar}\,(\mathbb{R}^n, \mathbb{R}))\,. \tag{16}$$

So let $g \in (\mathbb{R})^{\mathbb{R}^n}$ be arbitrary. To find the appropriate element of F we must first define some other functions.

Let $\{r_\alpha\}_{\alpha\in\mathfrak{c}}$ be an enumeration of $\mathbb{R}$ and $\{B_\alpha\}_{\alpha\in\mathfrak{c}}$ be a partition of $\mathbb{R}^n$ into Bernstein sets. Recall that a Bernstein set has the property that both it and its complement have nonempty intersection with every perfect set. Notice also, that $|C \cap B_\alpha| = \mathfrak{c}$ for any non-degenerate continuum $C \subseteq \mathbb{R}^n$. Define $k\colon \mathbb{R}^n \to \mathbb{R}$ so that $k^{-1}(\{r_\alpha\}) = B_\alpha$ for each $\alpha \in \mathfrak{c}$. Also, for each $q \in \mathbb{Q}$ let $k_q\colon \mathbb{R}^n \to \mathbb{R}$ be such that $k_q[\mathbb{R}^n] = \{q\}$.

Now, let $H = \{k_q - g\colon q \in \mathbb{Q}\} \cup \{k - g\}$. By (15) there is an $f \in F$ such that $|[f = h]| < \mathfrak{c}$ for all $h \in H$. We now show that $f + g \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$ using Lemma 3.10. We first claim that $f + g$ is constant on no non-degenerate continuum in $C \in \mathbb{R}^n$. By way of contradiction, assume there is a non-degenerate continuum $C \in \mathbb{R}^n$ such that $(f + g)|_C$ is constant. Then, there is an $\alpha \in \mathfrak{c}$ such that $(f + g)[C] = \{r_\alpha\}$. Since B_α is a Bernstein set, $|C \cap B_\alpha| = \mathfrak{c}$. But $f(x) = r_\alpha - g(x) = (k-g)(x)$ for each $x \in C \cap B_\alpha$, which contradicts our choice of f. So the claim is established. Next, we claim that $|(f+g)^{-1}(\{q\})| < \mathfrak{c}$ for each $q \in \mathbb{Q}$. This follows from our choosing f so that $|[f = k_q - g]| < \mathfrak{c}$ for each $q \in \mathbb{Q}$. Thus, $(f + g)$ is constant on no non-degenerate continuum and $|(f + g)^{-1}(q)| < \mathfrak{c}$ for every $q \in \mathbb{Q}$. So, by Lemma 3.10, $f + g \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$. Therefore, F satisfies (16), which completes the proof. □

Finally, we generalize Lemma 3.12 to include more general range spaces.

Lemma 3.13. *If $n, m \geq 1$, then* $\mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq e_\mathfrak{c}$.

Proof. By Lemma 3.12 there is an $F \subseteq \mathbb{R}^{\mathbb{R}^n}$ such that $|F| = e_\mathfrak{c}$ and F satisfies the definition of $\mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}))$, i.e.,

$$\left(\forall g \in (\mathbb{R})^{\mathbb{R}^n}\right)(\exists f \in F)(f + g \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})). \tag{17}$$

Let $\pi\colon \mathbb{R}^m \to \mathbb{R}$ be the projection function of $\mathbb{R}^m$ onto some fixed coordinate. Since π is onto, for every $f \in F$ there is an $f^*\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $f = \pi \circ f^*$. Let $F^* = \{f^*\colon f \in F\}$ and note that $|F^*| \leq |F| = e_\mathfrak{c}$. We will be done if we show that F^* satisfies

$$\left(\forall g \in (\mathbb{R}^m)^{\mathbb{R}^n}\right)(\exists f^* \in F^*)(f^* + g \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)). \tag{18}$$

So let $g \in (\mathbb{R}^m)^{\mathbb{R}^n}$ be arbitrary and put $g_1 = \pi \circ g$. By (17), there is an $f \in F$ such that $(f + g_1) \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$. We claim that $(f^* + g) \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$, which will complete the proof. Assume that $(f^* + g) \in n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. Then there exist $\{d_j \in \mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)\}_{j=1}^n$ such that

$$f^* + g = d_1 + \ldots + d_n.$$

Since the projection onto a coordinate is additive and continuous, we now have

$$\begin{aligned} f + g_1 &= (\pi \circ f^*) + (\pi \circ g) \\ &= \pi \circ (f^* + g) \\ &= (\pi \circ d_1) + \ldots + (\pi \circ d_n) \in n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}), \end{aligned}$$

which contradicts our choice of f. Thus, F^* satisfies (18). $\square$

4 Example 2.1

In this section, we construct the Baire class 1 function of Example 2.1 and use Lemmas 3.9 and 3.10 to show it has the required properties. We denote the linear span of a collection of real numbers A over $\mathbb{Q}$ by $LIN_{\mathbb{Q}}(A)$. Recall that if A is countable, then so is $LIN_{\mathbb{Q}}(A)$.

Lemma 4.1. *Let $n \geq 1$. There is a Baire class 1 function $f \colon \mathbb{R}^n \to \mathbb{R}$ such that $|f[\mathbb{R}^n]| = \omega$ and $f[C]$ is an interval for no non-trivial connected subset C of $\mathbb{R}^n$.*

Proof. Let $A = \{\alpha_i \colon i \in n\}$ be a collection of distinct real numbers which are linearly independent over the rationals. Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q}$. For each $i \in n$, define $g_i \colon \mathbb{R} \to \mathbb{R}$ by

$$g_i(x) = \begin{cases} \alpha_i/k & \text{if } x = q_k \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Notice that each g_i is in Baire class 1. Define $f \colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(\langle x_1, \ldots, x_n \rangle) = \sum_{i=1}^{n} g_i(x_i).$$

It is easy to check that f is in Baire class 1. We show that f has the desired properties. We first notice that $f[\mathbb{R}^n] \subseteq LIN_{\mathbb{Q}}(A)$ so $|f[\mathbb{R}^n]| = \omega$. We now check the other property. Let $C \subseteq \mathbb{R}^n$ be a non-trivial connected set. Since C is non-trivial, there is some $1 \leq i \leq n$ such that the projection $\pi_i[C]$ of C onto the i^{th} coordinate is an interval with non-empty interior. Let $t \in \pi_i[C] \cap \mathbb{Q}$, $s \in \pi_i[C] \setminus \mathbb{Q}$, and $p, q \in C$ be such that $\pi_i(q) = t$ and $\pi_i(p) = s$. By definition of f we have $f(q) \notin LIN_{\mathbb{Q}}(A \setminus \{\alpha_i\})$ and $f(p) \in LIN_{\mathbb{Q}}(A \setminus \{\alpha_i\})$. In particular, $f(q) \neq f(p)$ so $f|_C$ is not constant. Since $f|_C$ is not constant and $|f[\mathbb{R}^n]| = \omega$, it is clear that $f[C]$ is not an interval. $\square$

It is immediate by Lemma 3.9 that $f \notin n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$ for all $n \geq 1$. We now show that f is not the sum of an almost continuous function, h, and $n-1$ Darboux functions when $n > 1$. By Lemma 3.9, for any $g \in (n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$ there is a non-degenerate continuum C upon which g is constant. Thus, $(h + g)|_C$ would be almost continuous (Restrictions of almost continuous functions to closed sets are almost continuous [16].); in which case $(h + g)[C]$ is an interval [14, Theorem 1.7]. Therefore, $h + g \neq f$.

5 Proof of Theorem 2.9

We first show that the containment $\mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m) \subseteq n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$ holds.

Lemma 5.1. $\mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m) \subseteq n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$.

PROOF. The lemma is known for the case $n = 1$; see [2]. We proceed by induction on n. Assume $n-1 \geq 1$ and $\mathrm{AC}(\mathbb{R}^{n-1}, \mathbb{R}^m) \subseteq (n-1)\mathrm{Dar}(\mathbb{R}^{n-1}, \mathbb{R}^m)$. Let $g \in \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)$. We show $g \in n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$.

Let $\{A_k\}_{k\in n}$ be a partition of $\mathbb{R}$ into n $\mathfrak{c}$-dense sets. Define $h \colon \mathbb{R} \to \mathbb{R}^m$ so that for each $p \in \mathbb{R}^m$ and $k \in n$

$$\{y \in A_k \colon h(y) = p\} \text{ is dense in } \mathbb{R}.$$

Let $\pi \colon \mathbb{R}^n \to \mathbb{R}$ denote the projection of $\mathbb{R}^n$ onto a fixed coordinate. Since restrictions of almost continuous functions to closed sets are almost continuous [16], $g|_{\pi^{-1}(y)}$ is almost continuous for every $y \in \mathbb{R}$. Note that $\pi^{-1}(r)$ is homeomorphic to $\mathbb{R}^{n-1}$ for every $r \in \mathbb{R}$. By inductive hypothesis, for each $l \in n$ and $y \in A_l$ we may find Darboux functions $\{g_k^y \colon \pi^{-1}(y) \to \mathbb{R}^m \colon k \in n \setminus \{l\}\}$ such that

$$\sum_{\{k\in n \colon k\neq l\}} g_k^y = g|_{\pi^{-1}(y)} - h(y).$$

Now for each $k \in n$, define $g_k \colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$g_k(p) = \begin{cases} h(y) & \text{if } \pi(p) = y \in A_k, \\ g_k^y(p) & \text{if } \pi(p) = y \notin A_k. \end{cases} \tag{19}$$

We claim that the functions of (19) are as desired. Let $k \in n$. To see that g_k is Darboux put $A = A_k$ and $B = \cup\{A_i \colon i \in n \text{ and } i \neq k\}$, then apply Lemma 3.1. We now show that

$$\sum_{k=0}^{n-1} g_k = g. \tag{20}$$

If $p \in \mathbb{R}^n$, then there is an $l \in n$ such that $y \in A_l$, $p \in \pi^{-1}(y)$, and

$$\begin{aligned} \sum_{k=0}^{n-1} g_k(p) =& h(y) + \sum_{\{k\in n \colon k\neq l\}} g_k^y(p) \\ =& h(y) + (g(p) - h(y)) = g(p). \end{aligned}$$

So (20) holds, completing the inductive step. □

We now show that the containment of Lemma 5.1 is proper. In fact, we show more. We note that the lemma below has been shown for the case $n = 2$ in [14, Example 1.6] and for $n = 1$ in [2].

Lemma 5.2. $\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m) \setminus \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m) \neq \emptyset$.

PROOF. Since $\mathbb{R}$ can be embedded in $\mathbb{R}^m$ for any $m \geq 1$, it is enough to show that

$$\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}) \setminus \mathrm{AC}(\mathbb{R}^n, \mathbb{R}) \neq \emptyset. \tag{21}$$

Take $f \in \mathrm{Dar}(\mathbb{R}, \mathbb{R}) \setminus \mathrm{AC}(\mathbb{R}, \mathbb{R})$, $n > 1$ and let $\pi \colon \mathbb{R}^n \to \mathbb{R}$ be the projection of $\mathbb{R}^n$ onto the first coordinate, i.e., $\pi\langle r_0, \ldots, r_{n-1}\rangle = r_0$. Define $g \colon \mathbb{R}^n \to \mathbb{R}$ by $g = f \circ \pi$. Since f is Darboux and π is continuous, $g \in \mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$. We show that $g \notin \mathrm{AC}(\mathbb{R}^n, \mathbb{R})$. Consider the set $S = \{p \in \mathbb{R}^n \colon p = \langle r, 0, \ldots, 0\rangle \;\&\; r \in \mathbb{R}\}$. Since $g|_S$ is an exact copy of f, it follows that $g|_S \notin \mathrm{AC}(S, \mathbb{R})$. However, restrictions of almost continuous functions to closed sets are almost continuous [16], so we must conclude that g is not almost continuous. $\square$

We now prove the last part of Theorem 2.9. We will need the following fact which may be found in [14].

Proposition 5.3. *Let $n, m \in \omega \setminus \{0\}$. There exists a family $\mathcal{B}$ of closed sets in $\mathbb{R}^n \times \mathbb{R}^m$, a blocking family, with the following properties:*

(i) *$f \in \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)$ if and only if $f \cap B \neq \emptyset$ for each $B \in \mathcal{B}$ and*

(ii) *for every $B \in \mathcal{B}$ the projection of B onto $\mathbb{R}^n$ is a non-degenerate connected set.*

Lemma 5.4. *If $n \geq 2$ and $m \geq 1$, then there is an almost continuous function f such that $f \notin (n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$.*

PROOF. Let $\pi \colon \mathbb{R}^m \to \mathbb{R}$ be the projection function of $\mathbb{R}^m$ onto some fixed coordinate. Since π is additive and continuous, if $f \in (n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$, then $(\pi \circ f) \in (n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$. It then follows from Lemma 3.9 that $(\pi \circ f)$ is constant on some non-degenerate continuum in $\mathbb{R}^n$. So, if we show there is a $g \in \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)$ such that $g_1 = \pi \circ g$ is constant on no non-degenerate continuum in $\mathbb{R}^n$, we will be done. We now construct g. Let $\{P_{\langle\alpha,i\rangle}\}_{\langle\alpha,i\rangle \in \mathfrak{c}\times 2}$ be a partition of $\mathbb{R}^n$ into Bernstein sets. Let $\{B_\alpha\}_{\alpha\in\mathfrak{c}}$ be an enumeration of the elements of the blocking family of Proposition 5.3, and for each $\alpha \in \mathfrak{c}$ let B^*_α denote the projection of B_α onto $\mathbb{R}^n$. Take an enumeration $\{r_\alpha\}_{\alpha\in\mathfrak{c}}$ of $\mathbb{R}^n$. By Proposition 5.3, $|B^*_\alpha| = \mathfrak{c}$ and B^*_α is an F_σ-set. It follows that $|P_{\langle\alpha,i\rangle} \cap B^*_\alpha| = \mathfrak{c}$ for every $\langle\alpha, i\rangle \in \mathfrak{c} \times 2$. For each $\alpha \in \mathfrak{c}$, define $h^\alpha \colon \mathbb{R}^n \to \mathbb{R}^m$

so that $h^\alpha|_{P_{\langle\alpha,0\rangle}\cap B^*_\alpha} \subseteq B_\alpha$, $h^\alpha[P_{\langle\alpha,1\rangle}\cap B^*_\alpha] = \{r_\alpha\}$, and let h^α be arbitrary elsewhere. Let $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$g(x) = \begin{cases} h^\alpha(x) & \text{if } x \in B^*_\alpha \cap (P_{\alpha,0} \cup P_{\alpha,1}) \\ 0 & \text{otherwise.} \end{cases}$$

We claim g is as desired. Since $g|_{P_{\alpha,0}\cap B^*_\alpha} = h^\alpha|_{P_{\alpha,0}\cap B^*_\alpha} \subseteq B_\alpha$ for every $\alpha \in \mathfrak{c}$, $g \in \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)$. We now show that $g_1 = \pi \circ g$ is constant on no continuum. Since Bernstein sets intersect all perfect sets, they intersect all non-degenerate continua. It follows from the way we defined g that if $C \subseteq \mathbb{R}^n$ is a non-degenerate continuum, then $(\pi \circ g)[C] = \pi[\mathbb{R}^m] = \mathbb{R}$. Thus, g_1 is constant on no non-degenerate subcontinuum of $\mathbb{R}^n$. □

6 Proof of Theorem 2.3

By Theorem 2.9 $\mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m) \subseteq n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. By Proposition 1.1(iii), we have, using Theorem 2.2, $\mathrm{A}(\mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)) \leq \mathrm{A}(n\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) = e_{\mathfrak{c}}$. So we only need to prove that $\mathrm{A}(\mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)) \geq e_{\mathfrak{c}}$. To see this, let $F \subseteq (\mathbb{R}^m)^{\mathbb{R}^n}$ and $|F| < e_{\mathfrak{c}}$. We must find a $g\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $g + F \subseteq \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)$. Let $\{P_\alpha\}_{\alpha\in\mathfrak{c}}$ be a partition of $\mathbb{R}^n$ into Bernstein sets, $\{B_\alpha\}_{\alpha\in\mathfrak{c}}$ be an enumeration of the elements of the blocking family of Proposition 5.3; and for each $\alpha \in \mathfrak{c}$ let B^*_α denote the projection of B_α onto $\mathbb{R}^n$. By Proposition 5.3 $|B^*_\alpha| = \mathfrak{c}$ and B^*_α is an F_σ-set. It follows that $|P_\alpha \cap B^*_\alpha| = \mathfrak{c}$ for each $\alpha \in \mathfrak{c}$. For each $\alpha \in \mathfrak{c}$, define $h^\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ so that $h^\alpha|_{P_\alpha\cap B^*_\alpha} \subseteq B_\alpha$ and let h^α be arbitrary elsewhere. Put

$$F^* = \{h^\alpha - f\colon f \in F \text{ and } \alpha \in \mathfrak{c}\}.$$

Since $\mathfrak{c} < e_{\mathfrak{c}}$, it follows that $|F^*| < e_{\mathfrak{c}}$. So, for every $\alpha \in \mathfrak{c}$ there is a function $g_\alpha\colon (P_\alpha \cap B^*_\alpha) \to \mathbb{R}^m$ such that $|\{x \in P_\alpha \cap B^*_\alpha\colon g_\alpha(x) = (h^\alpha - f)(x)\}| = \mathfrak{c}$. Let $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$g(x) = \begin{cases} g_\alpha(x) & \text{if } x \in P_\alpha \cap B^*_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

We claim g is as desired. Let $f \in F$ and $B \in \mathcal{B}$. There is an $\alpha \in \mathfrak{c}$ such that $B_\alpha = B$. By the way we defined g, we have

$$\begin{aligned} \left|(f+g)|_{P_\alpha\cap B^*_\alpha} = h^\alpha|_{P_\alpha\cap B^*_\alpha}\right| &= \left|f|_{P_\alpha\cap B^*_\alpha} + g_\alpha = h^\alpha|_{P_\alpha\cap B^*_\alpha}\right| \\ &= \left|g_\alpha = (h^\alpha - f)|_{P_\alpha\cap B^*_\alpha}\right| \\ &= \mathfrak{c}. \end{aligned}$$

Thus, $|(g+f)|_{P_\alpha\cap B^*_\alpha} = h^\alpha| = \mathfrak{c}$. In particular, $(f+g) \cap B_\alpha \neq \emptyset$. Since B was arbitrary, we conclude that $f + g \in \mathrm{AC}(\mathbb{R}^n, \mathbb{R}^m)$. □

7 Proof of Theorem 2.1

Our first goal is to show that $\mathrm{A}(n\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) \le \mathfrak{c}^+$ for $n > 1$. The following proposition can be found in [6, Lemma 3.1] where it is stated for $\mathbb{R}^{\mathbb{R}}$. The proof is essentially the same for $(\mathbb{R})^{\mathbb{R}^n}$.

Proposition 7.1. *There is a family $F \subseteq (\mathbb{R})^{\mathbb{R}^n}$ of cardinality $\mathfrak{c}^+$ such that for every distinct $f, h \in F$, every perfect set $P \subseteq \mathbb{R}^n$, and every $k < \omega$ there exists an $x \in P$ such that $|f(x) - h(x)| \ge k$.*

The next proposition follows immediately from [7, Proposition 2.8] and the fact that $\mathrm{Con}(\mathbb{R}^n,\mathbb{R}) = \mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$ for $n > 1$.

Proposition 7.2. *If $n > 1$ and $g \in n\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$, then there exists a perfect set P such that $g|_P$ is continuous.*

Lemma 7.3. *If $n > 1$, then $\mathrm{A}(n\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) \le \mathfrak{c}^+$.*

Proof. Let $F \subseteq (\mathbb{R})^{\mathbb{R}^n}$ be as in Proposition 7.1. We claim that $g + F \subseteq n\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})$ for no $g\colon \mathbb{R}^n \to \mathbb{R}$. By way of contradiction, assume that such a g exists. By Proposition 7.2, for every $f \in F$ there is a perfect set P_f such that the restriction of $g + f$ to P_f is continuous. Since $|F| = \mathfrak{c}^+$ and there are only $\mathfrak{c}$-many perfect sets, there are $f, h \in F$ such that $P_f = P_h$. It follows that $f - h = (g + f) - (g + h)$ is continuous on P_f, which contradicts our choice of F. $\square$

Since $\mathrm{A}_{n,n-1}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R})) \le \mathrm{A}(n\mathrm{Con}(\mathbb{R}^n,\mathbb{R}))$ by Proposition 1.7 (viii), the proof of Theorem 2.1 is completed by showing that $\mathfrak{c}^+ \le \mathrm{A}_{n,n-1}(\mathrm{Ext}(\mathbb{R}^n,\mathbb{R}))$.

For what follows we will use the notation of [7, Sec. 6]. For sets $\{A_i : i \in n\}$ and $\{B_i : i \in n\}$ and for $f\colon n \to 2$ we let

$$A_i \vee_f B_i = \begin{cases} A_i & \text{if } f(i) = 0 \\ B_i & \text{if } f(i) = 1. \end{cases}$$

If $j \in n$ and C is a set, we define

$$A_i \vee_f B_i \vee_j C = \begin{cases} C & \text{if } i = j \\ A_i & \text{if } i \ne j \text{ and } f(i) = 0 \\ B_i & \text{if } i \ne j \text{ and } f(i) = 1. \end{cases}$$

We call $M \subseteq \mathbb{R}$ a thick meager set provided that M is dense and is a countable union of nowhere dense perfect sets. Notice that any thick meager set is $\mathfrak{c}$-dense in $\mathbb{R}$. We will also need the following four propositions from [7].

Proposition 7.4. ([7, Lemma 4.1])*If G is a dense G_δ-set in $\mathbb{R}^n$, then for each $i \in n$ there is a countable dense set $B_i \subseteq \mathbb{R}$ and a thick meager set $Y_i \subseteq \mathbb{R}$ such that $B_i \cap Y_i = \emptyset$ and*

$$\prod_{i=0}^{n-1}(B_i \cup Y_i) \subseteq G.$$

Proposition 7.5. ([7, Proposition 2.3])*Let $n \geq 1$. There is a dense G_δ-set $G \subseteq \mathbb{R}^n$ and a function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that for any $g\colon \mathbb{R}^n \to \mathbb{R}$ if $g(x) = f(x)$ for every $x \notin G$, then $g \in \mathrm{Con}(\mathbb{R}^n, \mathbb{R})$.*

Since $\mathrm{Con}(\mathbb{R}^n, \mathbb{R}) = \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ for $n > 1$, we have an obvious corollary of Proposition 7.5.

Corollary 7.6. *Let $n \geq 1$. There is a dense G_δ-set $G \subseteq \mathbb{R}^n$ and a function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that for any $g\colon \mathbb{R}^n \to \mathbb{R}$ if $g(x) = f(x)$ for every $x \notin G$, then $g \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$.*

PROOF. When $n > 1$ the proposition follows from Proposition 7.5 and the fact that $\mathrm{Con}(\mathbb{R}^n, \mathbb{R}) = \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$. When $n = 1$ then the proposition reduces to [6, Corollary 3.4]. $\square$

Proposition 7.7. ([7, Lemma 4.3]) *Let $n > 0$ and $G \subseteq \mathbb{R}^n$ be a G_δ-set. If $f\colon n \to 2$ is a function, $i \in n$, and $\langle b_0, \ldots, b_{n-1}\rangle \in \mathbb{R}^n$, then the set*

$$\left\{x \in \mathbb{R}\colon \prod_{j=0}^{n-1}(\{b_j\} \vee_f \mathbb{R} \vee_i \{x\}) \subseteq G\right\}$$

is a G_δ-subset of $\mathbb{R}$.

The next proposition is a more detailed statement of [7, Proposition 2.4].

Proposition 7.8. *Let G be a dense G_δ-subset of $\mathbb{R}^n$. Then there exist countable dense sets $\{B_i\}_{i=0}^{n-1}$ of $\mathbb{R}$ and homeomorphisms $h_1, \ldots, h_n\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\prod_{i=0}^{n-1}(B_i \vee_f \mathbb{R}) \subseteq G \cup \left(\bigcup_{i=1}^{k} h_i(G)\right)$$

for each $f \in 2^n$ with $|f^{-1}(1)| = k$. In particular, if $k = n$ then,

$$\mathbb{R}^n = G \cup \left(\bigcup_{i=1}^{n} h_i(G)\right).$$

PROOF. The statement of the proposition follows directly from consideration of the inductive step of the proof of Proposition 2.4 in [7]. $\square$

Lemma 7.9. *Let $n \geq 1$ and $G \subseteq \mathbb{R}^n$ be a dense G_δ. There exist homeomorphisms $h_1, \dots, h_{n-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ and meager subsets $\{M_i\}_{i=0}^{n-1}$ of $\mathbb{R}$ such that*

$$\mathbb{R}^n \setminus \left(G \cup \left(\bigcup_{j=1}^{n-1} h_j[G] \right) \right) \subseteq \prod_{j=0}^{n-1} M_j. \tag{22}$$

PROOF. Let $\{B_i\}_{i=0}^{n-1}$ be the countable dense subsets of $\mathbb{R}$ from Proposition 7.8 and $h_1, \dots, h_{n-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ be the first $n-1$ homeomorphisms of Proposition 7.8. Then

$$\prod_{j=0}^{n-1} (B_j \vee_f \mathbb{R}) \subseteq G \cup \left(\bigcup_{j=1}^{n-1} h_j(G) \right) \tag{23}$$

for each $f \in 2^n$ such that $|f^{-1}(1)| = n-1$. Let $i \in n$ and $f_i \colon n \to 2$ be such that $f_i^{-1}(0) = \{i\}$. By Proposition 7.7, for each $b = \langle b_0, \dots, b_{n-1} \rangle \in B_0 \times \cdots \times B_{n-1}$ there is a G_δ-subset K_i^b of $\mathbb{R}$ such that

$$\prod_{j=0}^{n-1} (\{b_j\} \vee_{f_i} \mathbb{R} \vee_i K_i^b) \subseteq G \cup \left(\bigcup_{j=1}^{n-1} h_j[G] \right).$$

By (23), we know that $B_i \subseteq K_i^b$; so, K_i^b is a dense G_δ-subset of $\mathbb{R}$. So the set

$$K_i = \bigcap \{K_i^b \colon b \in B_0 \times \cdots \times B_{n-1}\}$$

is a dense G_δ-subset of $\mathbb{R}$, satisfying

$$\prod_{j=0}^{n-1} (B_j \vee_{f_i} \mathbb{R} \vee_i K_i) \subseteq G \cup \left(\bigcup_{j=1}^{n-1} h_j[G] \right).$$

Since $f_i[n \setminus \{i\}] = \{1\}$, we have

$$\prod_{j=0}^{n-1} (K_j \vee_{f_i} \mathbb{R}) = \prod_{j=0}^{n-1} (B_j \vee_{f_i} \mathbb{R} \vee_i K_i) \subseteq G \cup \left(\bigcup_{j=1}^{n-1} h_j[G] \right).$$

Letting $M_i = \mathbb{R} \setminus K_i$ for each $i \in n$, we have

$$\mathbb{R}^n \setminus \left(G \cup \left(\bigcup_{j=1}^{n-1} h_j[G] \right) \right) \subseteq \mathbb{R}^n \setminus \bigcup_{i=0}^{n-1} \left(\prod_{j=0}^{n-1} (K_j \vee_{f_i} \mathbb{R}) \right) = \prod_{i=0}^{n-1} M_i.$$

Thus, (22) holds. $\square$

Lemma 7.10. *Let $\{M_i\}_{i=0}^{n-1}$ be meager subsets of $\mathbb{R}$ and $M = \prod_{i=0}^{n-1} M_i \subseteq \mathbb{R}^n$. For any dense G_δ-set $G \subseteq \mathbb{R}^n$ there exist homeomorphisms $\{h_\xi \colon \mathbb{R}^n \to \mathbb{R}^n\}_{\xi\in\mathfrak{c}}$ such that*

(i) *$h_\xi[M] \cap h_\zeta[M] = \emptyset$ for all $\zeta < \xi < \mathfrak{c}$ and*

(ii) *$h_\xi[M] \subseteq G$ for all $\xi < \mathfrak{c}$.*

PROOF. By Proposition 7.4 there exist thick meager subsets $\{N_i\}_{i=0}^{n-1}$ of $\mathbb{R}$ such that

$$\prod_{i=0}^{n-1} N_i \subseteq G. \tag{24}$$

Let $\{N_{i,\xi}\}_{\xi\in\mathfrak{c}}$ be a partition of N_i into thick meager sets. By [10, Lemma 3.2], for each $i \in n$ and $\xi \in \mathfrak{c}$ there exists a homeomorphism $h_{i,\xi} \colon \mathbb{R} \to \mathbb{R}$ such that

$$h_{i,\xi}[M_i] \subseteq N_{i,\xi}. \tag{25}$$

For each $\xi \in \mathfrak{c}$ let $h_\xi = h_{0,\xi} \times \cdots \times h_{n-1,\xi} \colon \mathbb{R}^n \to \mathbb{R}^n$. The homeomorphisms $\{h_\xi\}_{\xi\in\mathfrak{c}}$ satisfy (i) and (ii). Indeed, $\{N_{i,\xi}\}_{\xi\in\mathfrak{c}}$ partitions N_i for each $i \in n$ so (i) follows from (25). Using (24) and (25), we conclude (ii). $\square$

Lemma 7.11. *Let $n \geq 1$. There exist a meager subset M of $\mathbb{R}^n$, meager subsets $\{M_i \colon i \in n\}$ of $\mathbb{R}$, and a function $f \colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$M = \prod_{i=0}^{n-1} M_i$$

and for any $g \colon \mathbb{R}^n \to \mathbb{R}$, if $g|_M = f|_M$, then $g \in n\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$.

PROOF. By Corollary 7.6 there is an extendable function $l \colon \mathbb{R}^n \to \mathbb{R}$ and a dense G_δ-set $G \subseteq \mathbb{R}^n$ such that for any function $k \colon \mathbb{R}^n \to \mathbb{R}$ if $k(x) = l(x)$ for every $x \notin G$, then $k \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$. Let $h_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ be the identity homeomorphism. Pick $h_1, \ldots, h_{n-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ and $\{M_i\}_{i=0}^{k-1}$, as in Lemma 7.9 for G. Put

$$M = \prod_{i=0}^{n-1} M_i$$

and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sum_{i=0}^{n-1} (l \circ h_i^{-1})(x) & \text{if } x \in M \\ 0 & \text{otherwise.} \end{cases}$$

We show f is as desired. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be such that $g|_M = f|_M$. We show $g \in n\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$. Let $G_0 = G$, and for each $0 < i \leq n$ let

$$G_i = h_i[G] \setminus \left(\bigcup_{j=0}^{i-1} h_j[G] \right).$$

Note that $\{G_i \colon i \in n\}$ is a collection of pairwise disjoint sets such that

$$\mathbb{R}^n \setminus M \subseteq \bigcup_{i=0}^{n-1} G_i.$$

For each $i \in n$ we define $g_i \colon \mathbb{R}^n \to \mathbb{R}$ by

$$g_i(x) = \begin{cases} g(x) - \sum_{\{j \in n \colon j \neq i\}} (l \circ h_j^{-1})(x) & \text{if } x \in G_i \\ (l \circ h_i^{-1})(x) & \text{if } x \notin G_i. \end{cases}$$

Now $g = g_0 + \cdots + g_{n-1}$. We show that each g_i is in $\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$. Fix an $i \in n$. If $x \in \mathbb{R}^n \setminus G$, we have $g_i(h_i(x)) = (l \circ h_i^{-1})(h_i(x)) = l(x)$ since $h_i(x) \notin G_i$. Thus, $(g_i \circ h_i) \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ by our choice of l. Since the composition of a extendable function and a homeomorphism is extendable [15, Lemma 1], it then follows that $g_i = ((g_i \circ h_i) \circ h_i^{-1}) \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$. $\square$

Lemma 7.12. $\mathrm{A}_{n,n-1}(\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})) \geq \mathfrak{c}^+$.

Proof. Let $\{g_\xi\}_{\xi \in \mathfrak{c}}$ be a collection of functions from $\mathbb{R}^n$ into $\mathbb{R}$. Suppose M and $f\colon \mathbb{R}^n \to \mathbb{R}$ are as in Lemma 7.11. Take an extendable function $f^*\colon \mathbb{R}^n \to \mathbb{R}$ and a dense G_δ-set $G \subseteq \mathbb{R}^n$, as in Corollary 7.6. By Lemma 7.10 there exist homeomorphisms $\{h_\xi \colon \mathbb{R}^n \to \mathbb{R}^n\}_{\xi \in \mathfrak{c}}$ such that

(i) $h_\xi[M] \cap h_\zeta[M] = \emptyset$ for all $\zeta < \xi < \mathfrak{c}$.

(ii) $h_\xi[M] \subseteq G$ for all $\xi \in \mathfrak{c}$.

Define $l\colon \mathbb{R}^n \to \mathbb{R}$ so that

$$l(x) = \begin{cases} g_\xi(x) - (f \circ h_\xi^{-1})(x) & \text{if } x \in h_\xi[M] \text{ for some } \xi \in \mathfrak{c} \\ f^*(x) & \text{otherwise.} \end{cases}$$

Note that $l \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ by our choice of f^* and G. We will be done if we show that $g_\xi - l \in n\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ for each $\xi \in \mathfrak{c}$. So fix $\xi \in \mathfrak{c}$. If $x \in M$, then $(g_\xi - l)(h_\xi(x)) = g_\xi(h_\xi(x)) - (g_\xi(h_\xi(x)) - (f \circ h_\xi^{-1})(h_\xi(x))) = (f \circ h_\xi^{-1})(h_\xi(x)) =$

$f(x)$. So for each $x \in M$ we have $(g_\xi - l)(h_\xi(x)) = f(x)$. By our choice of M and f it follows that there exist $f_{0,\xi}, \dots, f_{n-1,\xi} \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ such that

$$(g_\xi - l) \circ h_\xi = \sum_{i=0}^{n-1} f_{i,\xi}.$$

So,

$$(g_\xi - l) = \left(\sum_{i=0}^{n-1} f_{i,\xi} \right) \circ h_\xi^{-1} = \sum_{i=0}^{n-1} (f_{i,\xi} \circ h_\xi^{-1}).$$

Since the composition of an extendable function and a homeomorphism is extendable, $(f_{i,\xi} \circ h_\xi^{-1}) \in \mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$ for each $i \in n$. Thus, $(g_\xi - l) \in n\mathrm{Ext}(\mathbb{R}^n, \mathbb{R})$. The proof is complete. $\square$

8 Proof of Theorem 2.4

In this section we will use the notation of the section above. Additionally for $k \leq n$, we make the definition $F_k^n = \{f \in 2^n : |f^{-1}(0)| = k\}$. We first show that $\mathrm{A}_{n,\lfloor n/2 \rfloor}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \geq \mathfrak{c}^+$.

Lemma 8.1. *Let $n \geq k \geq 1$, $\{M_i \subseteq \mathbb{R} : i \in n\}$ be a collection of thick meager sets and*

$$M = \bigcup_{f \in F_k^n} \prod_{i=0}^{n-1} (M_i \vee_f \mathbb{R}).$$

Then there is a function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that for any $g : \mathbb{R}^n \to \mathbb{R}^m$ if $g|_M = f|_M$, then $g \in k\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$.

PROOF. Throughout this proof we assume that $\mathbb{R}^n$ is written in the form $\mathbb{R}^n = \{\langle r_0, \dots, r_{n-1} \rangle : r_i \in \mathbb{R} \text{ for each } 0 \leq i \leq n-1\}$. We do an induction on n.

Suppose n=k=1. In this case $\{M_i : i \in 1\} = \{M_0\}$ and $M = M_0$. Since M is a thick meager set, $|(a, b) \cap M| = \mathfrak{c}$ for all $a < b \in \mathbb{R}$. So, using transfinite induction, we may easily construct a function f which maps each $M \cap (a, b)$ onto $\mathbb{R}^m$ for all $a < b \in \mathbb{R}$. Clearly, f has the desired property.

So, let $n \geq k \geq 1$ and assume the lemma holds for $n - 1 \geq 1$. We now show that the lemma holds for n. Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be the projection of $\mathbb{R}^n$ onto the last coordinate; and note that $\pi^{-1}(r)$ is homeomorphic to $\mathbb{R}^{n-1}$ for each $r \in \mathbb{R}$. For every $r \in \mathbb{R}$, put $M^r = M \cap \pi^{-1}(r)$. Let $\{B_i\}_{i=0}^{k-1}$ be a partition of M_{n-1} into $\mathfrak{c}$-dense sets and $d : \mathbb{R} \to \mathbb{R}$ be a function such that for each $y \in \mathbb{R}^m$ and $i \in k$

$$\{x \in B_i : d(x) = y\} \text{ is dense in } \mathbb{R}.$$

Notice that for each $r \in M_{n-1}$ we have

$$M^r = \left(\bigcup_{f \in F^{n-1}_{k-1}} \prod_{i=0}^{n-2} (M_i \vee_f \mathbb{R}) \right) \times \{r\}. \tag{26}$$

So if $k > 1$, we may use the inductive hypothesis to find for each $r \in M_{n-1}$ a function $f^r \colon \pi^{-1}(r) \to \mathbb{R}^m$ such that for every $g \colon \pi^{-1}(r) \to \mathbb{R}^m$ if $g|_{M^r} = f^r|_{M^r}$, then $g \in (k-1)\mathrm{Dar}(\pi^{-1}(r), \mathbb{R}^m)$. When $k = 1$, then $M^r = \pi^{-1}(r)$, and we define $f^r \colon \pi^{-1}(r) \to \mathbb{R}^m$ so that $f^r[\pi^{-1}(r)] = \{d(r)\}$.

When $r \in \mathbb{R} \setminus M_{n-1}$, we must again consider two cases. If $k < n$, then

$$M^r = \left(\bigcup_{f \in F^{n-1}_{k}} \prod_{i=0}^{n-1} (M_i \vee_f \mathbb{R}) \right) \times \{r\}. \tag{27}$$

So, by the inductive hypothesis, there is a function $f^r \colon \pi^{-1}(r) \to \mathbb{R}^m$ such that for every $g \colon \pi^{-1}(r) \to \mathbb{R}^m$ if $g|_{M_r} = f^r|_{M_r}$, then $g \in k\mathrm{Dar}(\pi^{-1}(r), \mathbb{R}^m)$. In the case when $n = k$, we have $M_r = \emptyset$. It follows from Corollary 2.6 that $g \in k\mathrm{Dar}(\pi^{-1}(r), \mathbb{R}^m)$ for every $g \colon \pi^{-1}(r) \to \mathbb{R}^m$. So, in this case, we may define $f^r \colon \pi^{-1}(r) \to \mathbb{R}^m$ as we please.

Let $f = \bigcup_{r \in \mathbb{R}} f^r \colon \mathbb{R}^n \to \mathbb{R}^m$. We show f is as desired. Let $g \colon \mathbb{R}^n \to \mathbb{R}^m$ be a function such that $g|_M = f|_M$. We must show that $g \in k\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. For each $r \in \mathbb{R}$, let $g^r = g|_{\pi^{-1}(r)}$. For $i \in k$ and $r \in B_i \subseteq M_{n-1}$, there exist $\{g^r_j \in \mathrm{Dar}(\pi^{-1}(r), \mathbb{R}^m) \colon j \in k \setminus \{i\}\}$ such that

$$\sum_{j \in k \setminus \{i\}} g^r_j = g^r$$

since $g^r|_{M^r} = f^r|_{M^r}$. The above sum does not make sense when $k = 1$. In this case, (26) implies that $M^r = \pi^{-1}(r)$, so $g^r = f^r \in \mathrm{Dar}(\pi^{-1}(r), \mathbb{R}^m)$.

If $r \in \mathbb{R} \setminus M_{n-1}$ there exist $g^r_0, \ldots, g^r_{k-1} \in \mathrm{Dar}(\pi^{-1}(r), \mathbb{R}^m)$ such that

$$g^r_0 + \cdots + g^r_{k-1} = g^r$$

since $g^r|_{M^r} = f^r|_{M^r}$ or $k = n$.

For each $i \in k$, let $g_i \colon \mathbb{R}^n \to \mathbb{R}^m$ be defined by,

$$g_i(x) = \begin{cases} d(r) & \text{if } x \in \pi^{-1}(r) \text{ and } r \in B_i \\ g^r_i(x) - (d(r)/(k-1)) & \text{if } x \in \pi^{-1}(r) \text{ and } r \in M_{n-1} \setminus B_i \\ g^r_i(x) & \text{if } x \in \pi^{-1}(r) \text{ and } r \notin M_{n-1}. \end{cases}$$

Notice there is no problem with division by zero when $k = 1$ since in this case $M_{n-1} = B_0$. We claim that $g_i \in \mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$ for each $i \in k$. To see this let $A = B_i$ and $B = \mathbb{R} \setminus B_i$ and apply Lemma 3.1. It is easily checked that,

$$\sum_{i=0}^{k-1} g_i = g.$$

So $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is as desired, completing the inductive step. □

We may now prove one of the inequalities of Theorem 2.4

Lemma 8.2. *If $n \geq 1$, then* $\mathrm{A}_{n,\lfloor n/2 \rfloor}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \geq \mathfrak{c}^+$.

PROOF. Let $\{g_\xi \colon \xi \in \mathfrak{c}\}$ be a collection of functions from $\mathbb{R}^n$ into $\mathbb{R}^m$. We must find $f \in (n - \lfloor n/2 \rfloor)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$ such that $g_\xi - f \in (\lfloor n/2 \rfloor + 1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$ for every $\xi \in \mathfrak{c}$.

Let $\{M_{i,\xi} \subseteq \mathbb{R} \colon i \in n \text{ and } \xi \in \mathfrak{c}\}$ and $\{K_i \subseteq \mathbb{R} \colon i \in n\}$ be thick meager sets such that for every $i \in n$

(i) $M_{i,\xi} \cap M_{i,\zeta} = \emptyset$ for $\xi < \zeta < \mathfrak{c}$ and

(ii) $K_i \cap (\bigcup_{\xi \in \mathfrak{c}} M_{i,\xi}) = \emptyset$.

For every $\xi \in \mathfrak{c}$, let

$$M_\xi = \bigcup_{f \in F^n_{\lfloor n/2 \rfloor + 1}} \prod_{i=0}^{n-1} (M_{i,\xi} \vee_f \mathbb{R}). \tag{28}$$

We let

$$K = \bigcup_{f \in F^n_{n - \lfloor n/2 \rfloor}} \prod_{i=0}^{n-1} (K_i \vee_f \mathbb{R}). \tag{29}$$

We now claim that

(a) $M_\xi \cap M_\zeta = \emptyset$ for $\zeta < \xi < \mathfrak{c}$ and

(b) $K \cap (\bigcup_{\xi \in \mathfrak{c}} M_\xi) = \emptyset$.

We show (b). Fix $\xi \in \mathfrak{c}$. Let

$$x = \langle x_0, \ldots, x_{n-1} \rangle \in M_\xi \text{ and } y = \langle y_0, \ldots, y_{n-1} \rangle \in K.$$

We claim that $x \neq y$. By (28) we have

$$|\{i \in n \colon x_i \in M_{i,\xi}\}| \geq \lfloor n/2 \rfloor + 1.$$

By (29) we have

$$|\{i \in n\colon y_i \in K_i\}| \geq n - \lfloor n/2 \rfloor.$$

Since $n - \lfloor n/2\rfloor + \lfloor n/2 \rfloor + 1 = n+1$, by the Pigeonhole Principle there is an $i \in n$ such that $x_i \in M_{i,\xi}$ and $y_i \in K_i$. So, by (ii), $x_i \neq y_i$ which implies that $x \neq y$. A similar argument together with the fact that $2(\lfloor n/2\rfloor + 1) \geq n+1$ shows that (a) holds.

Using Lemma 8.1, we may find for each $\xi \in \mathfrak{c}$ a function $f_\xi\colon \mathbb{R}^n \to \mathbb{R}^m$ such that for any $g\colon \mathbb{R}^n \to \mathbb{R}^m$ if $g|_{M_\xi} = f_\xi|_{M_\xi}$, then $g \in (\lfloor n/2\rfloor + 1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. Again applying Lemma 8.1, we may find a function $f^*\colon \mathbb{R}^n \to \mathbb{R}$ such for any $g\colon \mathbb{R}^n \to \mathbb{R}^m$ if $g|_K = f^*|_K$ then, $g \in (n - \lfloor n/2 \rfloor)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. Define $f\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(x) = \begin{cases} f^*(x) & \text{if } x \notin \bigcup_{\xi\in\mathfrak{c}} M_\xi \\ g_\xi(x) - f_\xi(x) & \text{if } x \in M_\xi. \end{cases}$$

We claim that f has the desired property. Since $f|_K = f^*|_K$, it follows from (b) that $f \in (n - \lfloor n/2\rfloor)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. Now, we only need to show that for each $\xi \in \mathfrak{c}$ we have $g_\xi - f \in (\lfloor n/2 \rfloor + 1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. Fix $\xi \in \mathfrak{c}$. Let $x \in M_\xi$. Then, $(g_\xi - f)(x) = g_\xi(x) - (g_\xi(x) - f_\xi(x)) = f_\xi(x)$. Since $(g_\xi - f)|_{M_\xi} = f_\xi|_{M_\xi}$, we have $g_\xi - f \in (\lfloor n/2\rfloor + 1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)$. □

To complete the proof it is enough to show that $\mathrm{A}((n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq \mathfrak{c}^+$ for $n > 2$. To see this, notice that by (iii) of Proposition 1.7 we have

$$\mathrm{A}_{n,\lfloor n/2 \rfloor}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq \mathrm{A}_{n,n-2}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)).$$

By Proposition 1.7(viii), we have

$$\mathrm{A}_{n,n-2}(\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq \mathrm{A}((n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)).$$

Finally, notice that $\lfloor n/2 \rfloor < n-1$ only if $n > 2$.

Lemma 8.3. *If* $n > 2$, *then* $\mathrm{A}((n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})) \leq \mathfrak{c}^+$.

Proof. Let $F \subseteq (\mathbb{R})^{\mathbb{R}^n}$ be as in Proposition 7.1. We claim that there is no $g\colon \mathbb{R}^n \to \mathbb{R}$ such that $g + F \subseteq (n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R})$. By way of contradiction assume that there is such a g. Since $n - 1 < n$, Lemma 3.10 implies that for every $f \in F$ there is a perfect set P_f such that the restriction of $g + f$ to P_f is continuous. Since $|F| = \mathfrak{c}^+$ and there are only $\mathfrak{c}$-many perfect sets, there exist $f, h \in F$ such that $P_f = P_h$. It follows that $f - h = (g+f) - (g+h)$ is continuous on P_f, which contradicts our choice of F. □

We now work to generalize the range space of Lemma 8.3 to $\mathbb{R}^m$.

Lemma 8.4. *If* $n, m \geq 1$ *and* $n > 2$, *then* $\mathrm{A}((n-1)\mathrm{Dar}(\mathbb{R}^n, \mathbb{R}^m)) \leq \mathfrak{c}^+$.

Proof. The proof essentially follows that of Lemma 3.13.

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