Real Analysis Exchange
Vol. 24(2), 1998/9, pp. 589–598

Jan Jastrzębski,* Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland. e:mail `jjas@ksinet.univ.gda.pl`

Tomasz Natkaniec,* Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland. e:mail: `mattn@ksinet.univ.gda.pl`

ON SUMS AND PRODUCTS OF EXTENDABLE FUNCTIONS

Abstract

We study the maximal additive, multiplicative and lattice-like families for the class of all extendable functions. This article is a continuation of earlier papers, in which the same questions concerning other Darboux-like functions have been studied.

1 Preliminaries

Our terminology is standard. By $\mathbb{R}$ and $\mathbb{I}$ we denote the set of all reals and the interval $[0,1]$, respectively. The letters X, Y and Z will denote topological spaces. The symbols $\overline{A}$, $\mathrm{int}(A)$ and $\mathrm{bd}(A)$ denote the closure, interior and boundary of a set A, respectively.

No distinction is made between a function and its graph. For functions $f\colon X \to Y$ and $g\colon X \to Z$, the symbol (f,g) denotes the diagonal of f and g, i.e., $(f,g)\colon X \to Y \times Z$, $(f,g)(x) = (f(x), g(x))$ for every $x \in X$.

For a function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ the symbols $C^-(f,x)$, $C^+(f,x)$ denote the cluster sets from the left and right, respectively, of the function f at the point x.

By $\mathrm{C}(X,Y)$ and $\mathrm{Const}(X,Y)$ we denote the family of all continuous functions and constant functions from X to Y, respectively. We shall write C and Const when X and Y are clear from the context.

We shall consider the following classes of functions from X to Y.

Key Words: extendable functions, peripherally continuous functions, family of peripheral intervals, maximal additive family, maximal multiplicative family.

Mathematical Reviews subject classification: Primary 26A15; Secondary 54C08.

Received by the editors April 16, 1998

*Both authors are supported by UG Research Grant No BW/5100-5-0282-7 and by NSF Cooperative Research Grant INT-9600548 with its Polish part financed by KBN.

PC(X, Y) — the class of all peripherally continuous functions. A function $f\colon X \to Y$ is *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset $W \subset U$ such that $x \in W$ and $f(\mathrm{bd}(W)) \subset V$.

E(X, Y) — the class of all functions with property $\mathcal{E}$. A function $f\colon X \to Y$ has *property* $\mathcal{E}$ if it is extendable to a peripherally continuous function, i.e., if there exists a peripherally continuous function $F\colon X \times \mathbb{I} \to Y$ such that $f(x) = F(x, 0)$ for all $x \in X$.

Conn(X, Y) — the class of all connectivity functions. A function $f\colon X \to Y$ is *connectivity*, if the restriction $f\restriction C$ is a connected subset of $X \times Y$ whenever C is a connected subset of X.

Ext(X, Y) — the family of *extendable* functions, i.e., functions $f\colon X \to Y$ for which there exists a connectivity function $F\colon X \times \mathbb{I} \to Y$ with property that $F(x, 0) = f(x)$ for every $x \in X$.

It is well-known that $\mathrm{PC}(\mathbb{R}^2, \mathbb{R}) = \mathrm{Conn}(\mathbb{R}^2, \mathbb{R})$. (The inclusion "$\subset$" was proved by Hamilton [OH] and by Stallings [JS], and the inclusion "$\supset$" by Hagan [H].) Therefore, $\mathrm{E}(\mathbb{R}, \mathbb{R}) = \mathrm{Ext}(\mathbb{R}, \mathbb{R})$.

Moreover, we shall consider the following class of real functions of one real variable that was introduced by R. Fleissner [RF].

M – the class of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that if x_0 is a right (left) point of discontinuity of f, then $f(x_0) = 0$ and there is a sequence (x_n) converging to x_0 such that $x_n > x_0$ ($x_n < x_0$) and $f(x_n) = 0$.

Recall that if $f \in \mathrm{M}$, then the set D of all points at which f is not continuous is nowhere dense, $D \subset f^{-1}(0)$, the set $f^{-1}(0)$ is closed, and f is continuous on the closure of every component of $\mathbb{R} \setminus \mathrm{bd}(f^{-1}(0))$. Consequently, f is a Darboux Baire one function. Thus it is also an extendable function. (See [BHL].)

Let $\mathcal{X}$ be a class of real functions. The *maximal additive (multiplicative, lattice-like,* respectively) *class* for $\mathcal{X}$ is defined to be the class of all $f \in \mathcal{X}$ for which $f + g \in \mathcal{X}$ ($fg \in \mathcal{X}$, $\max(f, g) \in \mathcal{X}$ and $\min(f, g) \in \mathcal{X}$, respectively) whenever $g \in \mathcal{X}$. The respective classes are denoted by $\mathcal{M}_a(\mathcal{X})$, $\mathcal{M}_m(\mathcal{X})$ and $\mathcal{M}_l(\mathcal{X})$. Those classes for some families of Darboux-like functions from $\mathbb{R}$ to $\mathbb{R}$ were studied by several authors. (See, e.g., [Ra] and [Fa] for the class of all Darboux functions, [AB] and [RF] for the class of all Darboux, Baire one functions, [KB] for the class of all perfect road functions and for the class of all peripherally continuous functions and [JJN] for the class of almost continuous functions, for the class of connectivity functions and for the class of functionally connected functions. See also the survey [GN] for definitions and relations between those properties.) The first systematic study of the operators $\mathcal{M}_a(\_)$, $\mathcal{M}_m(\_)$ and $\mathcal{M}_l(\_)$ was done by Jastrzębski, Jędrzejewski and Natkaniec in [JJN]. In particular, they proved the following two basic lemmas.

Lemma 1.1. ([JJN, Lemma 2.1]) *Let Φ be a property of functions and X be a topological space. For $i = 1, 2$ let $\mathcal{X}_i$ be the class of all functions $f\colon X \to \mathbb{R}^i$ with property Φ. Suppose the classes $\mathcal{X}_1$ and $\mathcal{X}_2$ fulfill the following conditions.*

(1.1) If $g \in \mathcal{X}_2$ and $h \in \mathrm{C}(\mathbb{R}^2, \mathbb{R})$, then $h \circ g \in \mathcal{X}_1$.

(1.2) If $f \in \mathcal{X}_1$ and $g \in \mathrm{C}(X, \mathbb{R})$, then $(f, g) \in \mathcal{X}_2$.

Then $\mathrm{C}(X, \mathbb{R}) \subset \mathcal{M}_a(\mathcal{X}_1) \cap \mathcal{M}_m(\mathcal{X}_1) \cap \mathcal{M}_l(\mathcal{X}_1)$.

Lemma 1.2. ([JJN, Lemma 2.2]) *Let $\mathcal{X}$ be a family of real functions defined on intervals that fulfills the following conditions.*

(2.0) If $f \in \mathcal{X}$ and x belongs to the domain of f, then the sets $C^+(f, x)$, $C^-(f, x)$ are connected and $f(x) \in C^+(f, x) \cap C^-(f, x)$.

(2.1) If $f\colon I \to \mathbb{R}$, $f \in \mathcal{X}$ and J is a subinterval of an interval I, then $f\restriction J \in \mathcal{X}$.

(2.2) If $h\colon (a, b) \to \mathbb{R}$, $h \in \mathcal{X}$, $y \in C^+(h, a)$, $z \in C^-(h, b)$, then the functions $h_1\colon [a, b) \to \mathbb{R}$, $h_2\colon (a, b] \to \mathbb{R}$ and $h_3\colon [a, b] \to \mathbb{R}$ belong to $\mathcal{X}$, where $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, z)\}$, $h_3 = h_1 \cup h_2$.

(2.3) If $I \subset \mathbb{R}$ is an interval, $a \in I$ and $f\restriction(I \cap (-\infty, a]) \in \mathcal{X}$, $f\restriction(I \cap [a, +\infty)) \in \mathcal{X}$, then $f \in \mathcal{X}$.

(2.4) $\mathrm{Const} \subset \mathcal{M}_a(\mathcal{X})$ *and* $-1 \in \mathcal{M}_m(\mathcal{X})$.

(2.5) If $f\colon I \to (0, \infty)$ and $f \in \mathcal{X}$, then $1/f \in \mathcal{X}$.

Then

(i) $\mathcal{M}_a(\mathcal{X}) \cup \mathcal{M}_l(\mathcal{X}) \subset \mathrm{C}$;

(ii) $\mathcal{M}_m(\mathcal{X}) \subset \mathrm{M}$.

We shall employ those lemmas for description of the maximal additive, multiplicative and lattice-like classes for the family of all extendable functions. In our study we shall use the characterization of extendable functions via families of peripheral intervals. (See [GR].)

Definition 1. *Let $f\colon \mathbb{I} \to \mathbb{I}$ be a function. A family of peripheral intervals (or simply, a PI family) for f consists of a sequence of ordered pairs (I_n, J_n) of subintervals of $\mathbb{I}$ such that*

(1) I_n is open in $\mathbb{I}$ and the length of I_n converges to 0;

(2) for each $x \in \mathbb{I}$ and for any $\varepsilon > 0$ there exists (I_n, J_n) such that $x \in I_n$, $f(x) \in J_n$ and the length of I_n and J_n are less than ε;

(3) both endpoints of I_n map into J_n;

(4) if I_n and I_m have points in common but neither is a subset of the other, then J_n and J_m have points in common.

Gibson and Roush in [GR, Theorems 1 and 2] proved the following theorem.

Theorem 1.1. *If $f\colon \mathbb{I} \to \mathbb{I}$ is an extendable function, then there exists a PI family for f. On the other hand, if for $f\colon \mathbb{I} \to \mathbb{I}$ there exists a PI family, then f is an extendable function. Moreover, then f is the restriction of connectivity function $F\colon \mathbb{I}^2 \to \mathbb{I}$ such that F is continuous on the complement of $\mathbb{I} \times \{0\}$.*

It is easy to observe that the analogous characterization is valid for any real function defined on an interval.

2 Extendable Functions

Lemma 2.1. *If $g \in \mathrm{PC}(X, Y)$ and $h \in \mathrm{C}(Y, Z)$, then $h \circ g \in \mathrm{PC}(X, Z)$.*

PROOF. Let $x \in X$, U and V be open neighborhoods of x and $h(g(x))$, respectively. Then there exists an open neighborhood $W \subset Y$ of the point $g(x)$ such that $h(W) \subset V$ and there exists an open neighborhood $U_0 \subset U$ of the point x such that $g(\operatorname{bd} U_0) \subset W$. Hence $h(g(\operatorname{bd} U_0)) \subset V$. □

Corollary 2.1. *If $g \in \mathrm{E}(X, Y)$ and $h \in \mathrm{C}(Y, Z)$, then $h \circ g \in \mathrm{E}(X, Z)$.*

PROOF. There exists a peripherally continuous function $G\colon X \times \mathbb{I} \to Y$ such that $g(x) = G(x, 0)$ for all $x \in X$. Then, by Lemma 2.1, the function $h \circ G\colon X \times \mathbb{I} \to Z$ is peripherally continuous, and $h \circ g(x) = h \circ G(x, 0)$ for all $x \in X$. So $h \circ g$ has property $\mathcal{E}$. □

Lemma 2.2. *Assume that X is a regular topological space. If $f \in \mathrm{PC}(X, Y)$ and $g \in \mathrm{C}(X, Z)$, then $(f, g) \in \mathrm{PC}(X, Y \times Z)$.*

PROOF. Fix $x_0 \in X$. Let U be an open neighborhood of x_0 and V be an open neighborhood of $(f(x_0), g(x_0))$. There exist open neighborhoods $V_1 \subset Y$ and $V_2 \subset Z$ of $f(x_0)$ and $g(x_0)$, respectively, such that $V_1 \times V_2 \subset V$. Let U_1 be an open subset of U such that $x_0 \in U_1$, $g(U_1) \subset V_2$ and let U_2 be an open set such that $\overline{U_2} \subset U_1$, $x_0 \in U_2$ and $f(\operatorname{bd} U_2) \subset V_1$. Hence

$$(f,g)(\operatorname{bd} U_2) \subset f(\operatorname{bd} U_2) \times g(\operatorname{bd} U_2) \subset V_1 \times V_2 \subset V.$$

Thus (f,g) is a peripherally continuous function at x_0. □

Corollary 2.2. *Assume that X is a regular topological space. If $f \in \mathrm{E}(X,Y)$ and $g \in \mathrm{C}(X,Z)$, then $(f,g) \in \mathrm{E}(X, Y \times Z)$.* □

Now, Corollaries 2.1, 2.2 and Lemma 1.1 imply the following inclusions.

Corollary 2.3.

$$\mathrm{C}(\mathbb{R},\mathbb{R}) \subset \mathcal{M}_a(\mathrm{E}(\mathbb{R},\mathbb{R})) \cap \mathcal{M}_m(\mathrm{E}(\mathbb{R},\mathbb{R})) \cap \mathcal{M}_l(\mathrm{E}(\mathbb{R},\mathbb{R}))$$

Corollary 2.4.

$$\mathrm{C}(\mathbb{R},\mathbb{R}) \subset \mathcal{M}_a(\mathrm{Ext}(\mathbb{R},\mathbb{R})) \cap \mathcal{M}_m(\mathrm{Ext}(\mathbb{R},\mathbb{R})) \cap \mathcal{M}_l(\mathrm{Ext}(\mathbb{R},\mathbb{R}))$$

Now we shall verify that the class of all extendable real functions satisfies all assumptions of Lemma 1.2.

Lemma 2.3. *Assume that $g\colon (c,\infty) \to \mathbb{R}$ is an extendable function and $y \in C^+(g,c) \cap \mathbb{R}$. Then $f = g \cup \{(c,y)\}$ is also an extendable function.*

PROOF. Let $\mathcal{J}_0$ be a PI family for g. For every $n \in \mathbb{N}$ choose $c_n \in (c, c+1]$ such that

(a) $c_0 = c+1$;

(b) $c_n < \min(c + \frac{1}{n}, c_{n-1})$;

(c) $|g(c_n) - y| < \frac{1}{n}$.

Put $C = \{c_n \colon n \in \mathbb{N}\} \cup \{c\}$. Now we shall construct a PI family $\mathcal{J}$ for f. A pair (I,J) belongs to $\mathcal{J}$ iff either

(i) $I = [c, c_n)$ and $J = (y - \frac{1}{n}, y + \frac{1}{n})$ for some $n \in \mathbb{N}$; or

(ii) $(I,J) \in \mathcal{J}_0$, $I \cap C = \{c_m\}$ and $g(c_m) \in J$; or

(iii) $(I,J) \in \mathcal{J}_0$ and $C \cap I = \emptyset$.

We shall verify that $\mathcal{J}$ is a PI family for f. Arrange all elements of $\mathcal{J}$ in a sequence $(I_n, J_n)_{n\in N}$. Then all I_n are open in $[c,\infty)$ and the lengths of I_n converge to 0; so condition (1) from Definition 1 is satisfied.

For $x = c$ condition (2) is clear by (i). For $x \neq c$, (2) follows easily from the fact that $\mathcal{J}_0$ is a PI family for g.

Condition (3) is also obvious. Thus we have to verify only condition (4). Fix n, m such that $I_n \cap I_m \neq \emptyset$ and neither is a subset of the other. Note that either $c \notin I_n$ or $c \notin I_m$. If $c \notin I_n \cup I_m$, then $(I_n, J_n), (I_m, J_m) \in \mathcal{J}_0$ and therefore $J_n \cap J_m \neq \emptyset$. So, suppose that $c \in I_n$ and $c \notin I_m$. Then $I_n = [c, c_{k_n})$ for some $k_n \in N$ and $c_{k_n} \in I_m$. By (ii), $g(c_{k_n}) \in J_m$ and, by (i) and (c), $g(c_{k_n}) \in J_n$. Thus $J_n \cap J_m \neq \emptyset$. □

Lemma 2.4. *If $c \in \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ and $f\restriction(-\infty, c]$, $f\restriction[c,\infty)$ are extendable functions, then f is also an extendable function.*

Proof. Let $\mathcal{J}_0$ and $\mathcal{J}_1$ denote PI families for $f\restriction(-\infty, c]$ and $f\restriction[c,\infty)$ respectively. For every $n \in \mathbb{N}$ choose $(I_n^-, J_n^-) \in \mathcal{J}_0$, $(I_n^+, J_n^+) \in \mathcal{J}_1$ such that $c \in I_n^- \cap I_n^+$ and the lengths of $I_n^- \cup I_n^+$ and $J_n^- \cup J_n^+$ are less than $\frac{1}{n}$. Now we shall define a PI family $\mathcal{J}$ for f. A pair (I, J) belongs to $\mathcal{J}$ iff either

(i) $(I, J) \in \mathcal{J}_0 \cup \mathcal{J}_1$ and $c \notin I$; or

(ii) $I = I_n^- \cup I_n^+$ and $J = J_n^- \cup J_n^+$ for some $n \in \mathbb{N}$.

It is easy to verify that $\mathcal{J}$ is a PI family for f. □

From Lemmas 1.2, 2.3 and 2.4 we obtain the following inclusions.

Corollary 2.5.

$$\mathcal{M}_a(\mathrm{Ext}(\mathbb{R},\mathbb{R})) \cup \mathcal{M}_l(\mathrm{Ext}(\mathbb{R},\mathbb{R})) \subset \mathrm{C}(\mathbb{R},\mathbb{R})$$
$$\mathcal{M}_m(\mathrm{Ext}(\mathbb{R},\mathbb{R})) \subset \mathrm{M}$$

Thus, Corollaries 2.4 and 2.5 yield the following equalities:

Theorem 2.1.

$$\mathcal{M}_a(\mathrm{Ext}(\mathbb{R},\mathbb{R})) = \mathrm{C}(\mathbb{R},\mathbb{R}) = \mathcal{M}_l(\mathrm{Ext}(\mathbb{R},\mathbb{R}))$$

Lemma 2.5. *Assume that C is a nowhere dense closed subset of $\mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfies*

(a) $g(x) = 0$ for $x \in C$;

(b) if $J \subset \mathbb{R}$ is a component of $\mathbb{R} \setminus C$, then $g\restriction\overline{J}$ is an extendable function.

Then g is an extendable function.

Proof. Let $\{(a_n,b_n)\}_n$ be the sequence of all components of $\mathbb{R}\setminus C$. For each n let $\mathcal{K}_n$ be a PI family for $g\restriction\overline{(a_n,b_n)}$. For every positive integer n choose a finite family $\mathcal{I}_n$ of open intervals such that

- $C\subset\bigcup\mathcal{I}_n$;
- the length of each $I\in\mathcal{I}_n$ is less than $1/n$;
- the end-points of every $I\in\mathcal{I}_n$ belong to $I\setminus C$;
- if $I\in\mathcal{I}_n$, $\inf(I)\in(a_i,b_i)$, $\sup(I)\in(a_j,b_j)$, $I^-=I\cap(a_i,b_i]$ and $I^+=I\cap[a_j,b_j)$, then there are intervals J^-, J^+ such that $(I^-,J^-)\in\mathcal{K}_i$, $(I^+,J^+)\in\mathcal{K}_j$ and the length of $J^-\cup J^+$ is less than $1/n$.

We shall define a PI family $\mathcal{J}$ for g. A pair (I,J) belongs to $\mathcal{J}$ iff either

(i) there exists n such that $(I,J)\in\mathcal{K}_n$ and $I\cap C=\emptyset$; or

(ii) $I\in\mathcal{I}_n$ for some n and $J=J^-\cup J^+$, where J^-, J^+ are described above.

Now we shall verify that $\mathcal{J}$ is a PI family for g. We can arrange all elements of $\mathcal{J}$ in a sequence $\{(I_n,J_n)\}_n$ such that the lengths of I_n converge to 0; so condition (1) from Definition 1 is satisfied.

To prove condition (2), fix $x\in\mathbb{R}$ and $\varepsilon>0$. If $x\notin C$, then $x\in(a_m,b_m)$ for some m. There exists $(I,J)\in\mathcal{K}_m$ that fulfills (2) and, by (i), $(I,J)\in\mathcal{J}$. If $x\in C$, then there exist $n\in\mathbb{N}$ and $(I,J)\in\mathcal{J}$ such that $I\in\mathcal{I}_n$, $1/n<\varepsilon$ and $x\in I$. Then $g(x)=0\in J$ and the lengths of I and J are less than $1/n$; so the pair (I,J) satisfies (2).

Statement (3) is obvious by the definition of $\mathcal{J}$.

To verify (4), fix (I_n,J_n) and (I_m,J_m) such that $I_n\cap I_m\neq\emptyset$ and neither is a subset of the other. Note that $0\in J_k$ whenever $I_k\cap C\neq\emptyset$. Thus, if $I_n\cap C\neq\emptyset\neq I_m\cap C$, then $0\in J_n\cap J_m$. If $(I_n\cup I_m)\cap C=\emptyset$, then $I_n\cup I_m\subset(a_i,b_i)$ for some $i\in\mathbb{N}$ and therefore $(I_n,J_n),(I_m,J_m)\in\mathcal{K}_i$. Thus $J_n\cap J_m\neq\emptyset$. So, assume that $I_n\cap C\neq\emptyset$ and $I_m\cap C=\emptyset$. Then $I_m\subset(a_i,b_i)$ for some $i\in\mathbb{N}$ and either $a_i\in I_n$ or $b_i\in I_n$. Suppose that $b_i\in I_n$. By the definition of $\mathcal{J}$ there exist $(I_0,J_0)\in\mathcal{K}_i$ such that $I_0=I_n\cap(a_i,b_i]$ and $J_0\subset J_n$. Because $(I_m,J_m),(I_0,J_0)\in\mathcal{K}_i$ and $I_m\cap I_0\neq\emptyset$, we get $J_m\cap J_0\neq\emptyset$. Consequently, $J_m\cap J_n\neq\emptyset$; so $\mathcal{J}$ satisfies condition (4). $\square$

Theorem 2.2.

$$\mathcal{M}_m(\mathrm{Ext}(\mathbb{R},\mathbb{R}))=\mathrm{M}\,.$$

PROOF. By Corollary 2.5, it is enough to prove that $\mathrm{M} \subset \mathcal{M}_m(\mathrm{Ext}(\mathbb{R},\mathbb{R}))$. Assume that $f \in \mathrm{Ext}(\mathbb{R},\mathbb{R})$ and $g \in \mathrm{M}$. Then $D = \mathrm{bd}(f^{-1}(0))$ is a closed and nowhere dense set. Let J be a component of the complement of D. Then g is continuous on $\overline{J}$ and, by Corollary 2.4, fg is extendable on $\overline{J}$. Moreover, $fg(x) = 0$ for $x \in D$, and according to Lemma 2.5, fg is an extendable function. □

3 Applications

The next lemma shows that in the definition of extendability (for real functions) we can replace the compact interval $\mathbb{I}$ by whole real line.

Lemma 3.1. *For a function $f\colon \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent.*

(i) $f \in \mathrm{Ext}(\mathbb{R},\mathbb{R})$.

(ii) there is $F \in \mathrm{PC}(\mathbb{R}^2,\mathbb{R})$ such that $f(x) = F(x,0)$ for each $x \in \mathbb{R}$.

PROOF. Let $f \in \mathrm{Ext}(\mathbb{R},\mathbb{R})$. Then there exists $F_0 \in \mathrm{PC}(\mathbb{R}\times\mathbb{I},\mathbb{R}))$ such that $f(x) = F_0(x,0)$ for $x \in \mathbb{R}$. According to [GR, Theorem 2], we can assume that F_0 is continuous on $\mathbb{R}\times(0,1]$. Thus $F_+\colon \mathbb{R}\times[0,\infty) \to \mathbb{R}$ defined by

$$F_+(x,y) = \begin{cases} F_0(x,y) & \text{for } (x,y) \in \mathbb{R}\times\mathbb{I} \\ F_0(x,1) & \text{for } (x,y) \in \mathbb{R}\times[1,\infty) \end{cases}$$

is a peripherally continuous function and consequently $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x,y) = \begin{cases} F_+(x,y) & \text{for } (x,y) \in \mathbb{R}\times[0,\infty) \\ F_+(x,-y) & \text{for } (x,y) \in \mathbb{R}\times(-\infty,0] \end{cases}$$

is also a peripherally continuous function with $f(x) = F(x,0)$ for every $x \in \mathbb{R}$.

Now, if $F \in \mathrm{PC}(\mathbb{R}^2,\mathbb{R})$ and $f(x) = F(x,0)$ for each $x \in \mathbb{R}$, then $F\restriction(\mathbb{R}\times\mathbb{I})$ is also a peripherally continuous extension of f. Thus $f \in \mathrm{Ext}(\mathbb{R},\mathbb{R})$. □

Theorem 3.1.

$$\mathcal{M}_a(\mathrm{PC}((\mathbb{R}^2,\mathbb{R})) = \mathrm{C}(\mathbb{R}^2,\mathbb{R}) = \mathcal{M}_l(\mathrm{PC}(\mathbb{R}^2,\mathbb{R}))$$

PROOF. By Lemmas 2.1 and 2.2, $\mathrm{C}(\mathbb{R}^2,\mathbb{R}) \subset \mathcal{M}_a(\mathrm{PC}(\mathbb{R}^2,\mathbb{R}))$. On the other hand, $\mathcal{M}_a(\mathrm{PC}(\mathbb{R}^2,\mathbb{R})) \subset \mathrm{PC}(\mathbb{R}^2,\mathbb{R})$ because $f \equiv 0$ belongs to $\mathrm{PC}(\mathbb{R}^2,\mathbb{R})$. Suppose that $g \in \mathrm{PC}(\mathbb{R}^2,\mathbb{R})$ is discontinuous at $x_0 \in \mathbb{R}^2$. Let $h\colon \mathbb{R} \to \mathbb{R}^2$ be a homeomorphic injection of $\mathbb{R}$ into $\mathbb{R}^2$ such that

- $h(0) = x_0$ and $g \circ h$ is discontinuous at 0;
- there is a homeomorphism $h_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h_1(x, 0) = h(x)$ for $x \in \mathbb{R}$.

By Corollary 2.5, there is $f_0 \in \mathrm{Ext}(\mathbb{R}, \mathbb{R})$ such that $f_0 + g \circ h \notin \mathrm{Ext}(\mathbb{R}, \mathbb{R})$. Let $f_1 : \mathbb{R}^2 \to \mathbb{R}$ be a peripherally continuous extension of f_0, i.e., $f_1(x, 0) = f_0(x)$ for $x \in \mathbb{R}$. Then $f = f_1 \circ h_1^{-1} \in \mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. Suppose that $f + g \in \mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. Then

$$f_1 + g \circ h_1 = (f + g) \circ h_1 \in \mathrm{PC}(\mathbb{R}^2, \mathbb{R}).$$

On the other hand, for each $x \in \mathbb{R}$ we have

$$(f_1 + g \circ h_1)(x, 0) = f_1(x, 0) + g(h_1(x, 0)) = f_0(x) + g \circ h(x).$$

Thus $f_0 + g \circ h \in \mathrm{Ext}(\mathbb{R}, \mathbb{R})$, contrary to the choice of f_0. Consequently,

$$\mathcal{M}_a(\mathrm{PC}(\mathbb{R}^2, \mathbb{R})) \subset \mathrm{C}(\mathbb{R}^2, \mathbb{R}).$$

In a similar way we can prove that

$$\mathcal{M}_l(\mathrm{PC}(\mathbb{R}^2, \mathbb{R})) = \mathrm{C}(\mathbb{R}^2, \mathbb{R}).$$

To see it, note that $\mathcal{M}_l(\mathrm{PC}(\mathbb{R}^2, \mathbb{R})) \subset \mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. Indeed, suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is not peripherally continuous at $x_0 \in \mathbb{R}^2$. Then there is an open neighborhood W of x_0 and $\varepsilon > 0$ such that $f(\mathrm{bd}(U)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ for no open neighborhood U of x_0 with $U \subset W$. Then the function $\max(f, f(x_0) - \varepsilon)$ is not peripherally continuous at x_0 and the constant function $f(x_0) - \varepsilon$ is peripherally continuous. Thus $f \notin \mathcal{M}_l(\mathrm{PC}(\mathbb{R}^2, \mathbb{R}))$. $\square$

Finally, note that Lemmas 2.1 and 2.2 yield $\mathrm{C}(\mathbb{R}^2, \mathbb{R}) \subset \mathcal{M}_m(\mathrm{PC}(\mathbb{R}^2, \mathbb{R}))$. On the other hand, it is easy to verify that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(||x||^{-1}) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 \end{cases}$$

is a discontinuous function that belongs to $\mathcal{M}_m(\mathrm{PC}(\mathbb{R}^2, \mathbb{R}))$. Thus we finish with the following problem:

Problem 1. *Characterize the class* $\mathcal{M}_m(\mathrm{PC}(\mathbb{R}^2, \mathbb{R}))$.

References

[KB] K. Banaszewski, *Algebraic properties of $\mathcal{E}$-continuous functions*, Real Anal. Exchange **18** (1992–93), 153–168.

[BHL] J. Brown, P. Humke and M. Laczkovich, *Measurable Darboux functions*, Proc. Amer. Math. Soc. **102** (1988), 603–610.

[AB] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, Vol. **659** Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[Fa] J. Farková, *About the maximum and the minimum of Darboux functions*, Mat. Čas. Slov. Akad. Vied **21** (1971), 110–116.

[RF] R. J. Fleissner, *A note on Baire 1 Darboux functions*, Real Anal. Exchange **3** (1977–78), 104–106.

[GN] R. G. Gibson and T. Natkaniec, *Darboux like functions*, Real Anal. Exchange **22** (1996–97), 492–534.

[GR] R. G. Gibson and F. Roush, *A characterization of extendable connectivity functions*, Real Anal. Exchange **13** (1987–88), 214–222.

[H] M. R. Hagan, *Equivalence of connectivity maps and peripherally continuous transformations*, Proc. Amer. Math. Soc. **17** (1966), 175–177.

[OH] O. H. Hamilton, *Fixed points for certain noncontinuous transformations*, Proc. Amer. Math. Soc. **8** (1957), 750–756.

[JJN] J. Jastrzębski, J. Jędrzejewski and T. Natkaniec, *On some subclasses of Darboux functions*, Fund. Math. **138** (1991), 165–173.

[Ra] T. Radaković, *Über Darbouxsche und stetige Funktionen*, Monat. Math. Phys. **38** (1931), 111–122.

[JS] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.