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CONTINUOUS RIGID FUNCTIONS

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ is vertically [horizontally] rigid for $C \subseteq (0, \infty)$ if $graph(cf)$ $[graph(f(c\,\cdot))]$ is isometric with $graph(f)$ for every $c \in C$. f is vertically [horizontally] rigid if this applies to $C = (0, \infty)$.

Balka and Elekes have shown that a continuous function f vertically rigid for an uncountable set C must be of the form $f(x) = px + q$ or $f(x) = pe^{qx} + r$, in this way confirming Jancović's conjecture saying that a continuous f is vertically rigid if and only if it is of one of these forms. We prove that their theorem actually applies to every $C \subseteq (0, \infty)$ generating a dense subgroup of $((0, \infty), \cdot)$, but not to any smaller C.

A continuous f is shown to be horizontally rigid if and only if it is of the form $f(x) = px + q$. In fact, f is already of that kind if it is horizontally rigid for some C with $card(C \cap ((0, \infty) \setminus \{1\})) \geq 2$.

1 Introduction and Main Results.

Given a set $C \subseteq (0, \infty)$ and a set $\mathcal{I}$ of Euclidean isometries of the plane $\mathbb{R}^2$, a function $f : \mathbb{R} \to \mathbb{R}$ is called *vertically rigid for C via $\mathcal{I}$* if for every $c \in C$ there exists $\alpha \in \mathcal{I}$ such that

$$graph(cf) = \alpha(graph(f)).$$

We call f *vertically rigid for C* if $\mathcal{I}$ contains all isometries, *vertically rigid via $\mathcal{I}$* if $C = (0, \infty)$, and *vertically rigid* if $C = (0, \infty)$ and $\mathcal{I}$ consists of all isometries (see [2, 1]).

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Of course, if f is vertically rigid, then for every $c \in \mathbb{R} \setminus \{0\}$ there is an isometry α satisfying the above equation. Every $f : \mathbb{R} \to \mathbb{R}$ is vertically rigid for $c = 1$.

Functions of the form $f(x) = px + q$ and of the form $f(x) = pe^{qx} + r$, $p, q, r \in \mathbb{R}$, clearly are vertically rigid. The following central theorem from [1] confirms a conjecture of D. Janković formulated in [2] and says that all continuous vertically rigid functions are of that kind.

Theorem 1. *Let a continuous function $f : \mathbb{R} \to \mathbb{R}$ be vertically rigid for an uncountable set $C \subseteq (0, \infty)$. Then there exist $p, q, r \in \mathbb{R}$ such that $f(x) = px + q$ for all $x \in \mathbb{R}$ or $f(x) = pe^{qx} + r$ for all $x \in \mathbb{R}$.*

The authors of [1] ask for the role of C in this theorem. Does it need to be uncountable? The following two statements show that the crucial condition for C is to generate a dense subgroup of $((0, \infty), \cdot)$. They will be proved in Section 2.

Theorem 2. *Let a continuous function $f : \mathbb{R} \to \mathbb{R}$ be vertically rigid for a set $C \subseteq (0, \infty)$ generating a dense subgroup of $((0, \infty), \cdot)$. Then there exist $p, q, r \in \mathbb{R}$ such that $f(x) = px + q$ for all $x \in \mathbb{R}$ or $f(x) = pe^{qx} + r$ for all $x \in \mathbb{R}$.*

Proposition 1. *Suppose that $C \subseteq (0, \infty)$ does not generate a dense subgroup of $((0, \infty), \cdot)$. Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is vertically rigid for C via horizontal translations, but is not of the form of Theorems 1 and 2.*

Every set $C_1 = \{c_1, c_2\} \subseteq (0, 1) \cup (1, \infty)$ with $\frac{\ln c_1}{\ln c_2} \notin \mathbb{Q}$ generates a dense subgroup of $((0, \infty), \cdot)$, because $\{\ln c_1, \ln c_2\}$ generates a dense subgroup of $(\mathbb{R}, +)$.

The set $C_2 = \{e^p : p \in \mathbb{Q}\}$ is a countable dense subgroup of $((0, \infty), \cdot)$. But no finite subset of C_2 generates a dense subgroup of $((0, \infty), \cdot)$. In particular, C_2 does not contain a subset of the form C_1.

Every non-dense subgroup G of $((0, \infty), \cdot)$ is of the form $G = \{g_0^k : k \in \mathbb{Z}\}$ with some $g_0 \in (0, \infty)$, since $\bar{G} = \{\ln g : g \in G\}$ must be a non-dense subgroup of $(\mathbb{R}, +)$, that is, $\bar{G} = \{k\bar{g}_0 : k \in \mathbb{Z}\} = \bar{g}_0\mathbb{Z}$ with $\bar{g}_0 \in \mathbb{R}$.

Balka and Elekes prove Theorem 1 by reducing it to the case of vertical rigidity via translations. We shall follow a similar strategy. As an analogue of their statement on translations we shall show the following proposition.

Proposition 2. *Let $C \subseteq (0, \infty)$ generate a dense subgroup of $((0, \infty), \cdot)$ and let $f : \mathbb{R} \to \mathbb{R}$ have at least one point of continuity and be vertically rigid for C via translations. Then there exist $p, q, r \in \mathbb{R}$ such that $f(x) = pe^{qx} + r$ for all $x \in \mathbb{R}$.*

Note that the requirement on C to generate a dense group is again crucial, as Proposition 1 shows.

We define analogous concepts of *horizontal* rigidity by replacing $graph(cf)$ with $graph(f(c\ \cdot))$ in the above definition (see [1]). The following theorem from [1] characterizes all functions horizontally rigid via translations.

Theorem 3. *A function $f : \mathbb{R} \to \mathbb{R}$ is horizontally rigid via translations if and only if there exists $p \in \mathbb{R}$ such that f is constant on $(-\infty, p)$ and constant on (p, ∞).*

Consequently, every continuous function horizontally rigid via translations is constant. We shall show that in the context of continuous functions the assumption of horizontal rigidity via translations can essentially be weakened.

Proposition 3. *Let a continuous function $f : \mathbb{R} \to \mathbb{R}$ be horizontally rigid for some $c \in (0,1) \cup (1,\infty)$ via a translation. Then f is constant.*

In the previous statement it is important that the rigidity can be realized via a translation. Indeed, for every $c \in (0,1) \cup (1,\infty)$, the function

$$f_c = \begin{cases} -\frac{x}{c}, & x \geq 0, \\ -x, & x \leq 0 \end{cases}$$

is both horizontally and vertically rigid for c via the reflection with respect to the straight line "$x = y$" as well as via a rotation depending on f_c, because

$$f_c(cx) = cf_c(x) = f_c^{-1}(x) = \begin{cases} -x, & x \geq 0, \\ -cx, & x \leq 0 \end{cases}$$

and $graph(f_c^{-1})$ is obtained from $graph(f_c)$ by the reflection mentioned above. Moreover, $graph(f_c)$ is symmetric under a reflection with respect to its bisector. Composition of both reflections gives the required rotation.

Of course, every function of the form $f(x) = px + q$ is horizontally rigid. The following theorem says in particular that all continuous horizontally rigid functions are of that kind and this way answers a second question from [1].

Theorem 4. *Let a continuous function $f : \mathbb{R} \to \mathbb{R}$ be horizontally rigid for two values $c_1, c_2 \in (0,1) \cup (1,\infty)$, $c_1 \neq c_2$. Then there exist $p, q \in \mathbb{R}$ such that $f(x) = px + q$ for all $x \in \mathbb{R}$.*

The above example shows that rigidity for at least two different values c_1, c_2 is a necessary assumption in Theorem 4. Proposition 3 and Theorem 4 will be proved in Section 3.

2 Vertically Rigid Functions.

PROOF OF PROPOSITION 1. C generates a non-dense subgroup $G = \{g_0^k : k \in \mathbb{Z}\}$ of $((0,\infty),\cdot)$. Let $h_1 : \mathbb{R} \to \mathbb{R}$ be a continuous function with period 1. We define $f(x) = h_1(x)g_0^x$. Then

$$g_0^k f(x) = h_1(x)g_0^{x+k} = h_1(x+k)g_0^{x+k} = f(x+k).$$

Hence, for every $k \in \mathbb{Z}$, f is vertically rigid for g_0^k via a horizontal translation. This applies in particular to all $c = g_0^k \in C$.

If h_1 is non-constant, then f is neither of the form $f(x) = px + q$ nor of the form $f(x) = pe^{qx} + r$. This proves the claim. □

The preparation of the proof of Proposition 2 starts with a characterization of all functions f vertically rigid for some fixed c via some fixed translation.

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in (0,1) \cup (1,\infty)$, $u, v \in \mathbb{R}$. Then the following are equivalent.*

(i) $cf(x) = f(x+u) + v$ for all $x \in \mathbb{R}$.

(ii) If $u = 0$, then $f(x) \equiv \frac{v}{c-1}$ is constant. Otherwise there exists a function $h_u : \mathbb{R} \to \mathbb{R}$ with period u such that $f(x) = h_u(x)c^{\frac{x}{u}} + \frac{v}{c-1}$ for all $x \in \mathbb{R}$.

PROOF. The implication (ii)⇒(i) and the case $u = 0$ in (i)⇒(ii) are trivial. For showing (i)⇒(ii) under the assumption $u \neq 0$ we define

$$h_u(x) = \big(f(x) - \tfrac{v}{c-1}\big)c^{-\frac{x}{u}}.$$

Then $f(x) = h_u(x)c^{\frac{x}{u}} + \frac{v}{c-1}$ by definition. One easily checks by (i) that h_u has the period u. □

The following fact can be found in [1]. We present a proof to keep the present paper self-contained.

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be vertically rigid for a set $C \subseteq (0,1) \cup (1,\infty)$ via translations. Then there exists $a \in \mathbb{R}$ such that $f - a$ is vertically rigid for C via horizontal translations.*

PROOF. For every $c \in C$, there are $u_c, v_c \in \mathbb{R}$ such that $cf(x) = f(x+u_c)+v_c$ for all $x \in \mathbb{R}$. Putting $a_c = \frac{v_c}{c-1}$ we easily obtain $c(f(x)-a_c) = f(x+u_c)-a_c$. Hence the lemma is proved once it is shown that $a_c = a$ is universal for all c.

We fix $c_0 \in C$. Then

$$c_0cf(0) = c_0(f(u_c)+v_c) = c_0f(u_c) + c_0v_c = f(u_c+u_{c_0}) + v_{c_0} + c_0v_c$$

and, by reversing the order of c_0 and c, $c_0cf(0) = f(u_{c_0}+u_c) + v_c + cv_{c_0}$. So $v_{c_0} + c_0v_c = v_c + cv_{c_0}$ and $a_c = \frac{v_c}{c-1} = \frac{v_{c_0}}{c_0-1} = a_{c_0}$ does not depend on c. □

Next we generalize a statement from [1].

Lemma 3. *Let $C \subseteq (0,1) \cup (1,\infty)$ generate a dense subgroup of $((0,\infty),\cdot)$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(0) = 1$ and be vertically rigid for C via horizontal translations. Then there exists a dense subgroup $(G,+)$ of $(\mathbb{R},+)$ such that $f(G) \subseteq (0,\infty)$ and*

$$f(x+g) = f(x)f(g) \textit{ for all } x \in \mathbb{R},\, g \in G.$$

PROOF. For every $c \in C$, there is $u_c \in \mathbb{R}$ such that $cf(x) = f(x+u_c)$ and in turn $\frac{1}{c}f(x) = f(x-u_c)$ for $x \in \mathbb{R}$. Let $G = \{k_1u_{c_1} + \ldots + k_mu_{c_m} : m \geq 0, c_i \in C, k_i \in \mathbb{Z}\}$ be the subgroup of $(\mathbb{R},+)$ generated by $\{u_c : c \in C\}$. Iteration of the previous equations yields

$$c_1^{k_1} \ldots c_m^{k_m} f(x) = f(x + k_1u_{c_1} + \ldots + k_mu_{c_m}) = f(x+g)$$

for arbitrary $x \in \mathbb{R}$ and $g = k_1u_{c_1} + \ldots + k_mu_{c_m} \in G$. Application of that to $x = 0$ and the supposition $f(0) = 1$ give

$$c_1^{k_1} \ldots c_m^{k_m} = f(k_1u_{c_1} + \ldots + k_mu_{c_m}) = f(g).$$

Consequently, $f(g) > 0$ for all $g \in G$ and

$$f(x+g) = f(x)f(g) \text{ for all } x \in \mathbb{R},\, g \in G.$$

It remains to show that G is dense in $\mathbb{R}$. Let us assume the contrary; that is, $G = a\mathbb{Z}$ with some fixed $a \geq 0$. Hence, for every $c \in C$, there is $k_c \in \mathbb{Z}$ such that $u_c = k_ca$. Note that $k_c, a \neq 0$, because $u_c \neq 0$, for $f(0) \neq cf(0) = f(u_c)$.

By Lemma 1, $f(x) = h_{u_c}(x)c^{\frac{x}{u_c}}$, where h_{u_c} has the period u_c and satisfies $h_{u_c}(0) = h_{u_c}(0)c^{\frac{0}{u_c}} = f(0) = 1$.

We fix $c_0 \in C$. Then

$$\begin{aligned} f(k_{c_0}k_ca) = f(k_{c_0}u_c) &= h_{u_c}(k_{c_0}u_c)c^{\frac{k_{c_0}u_c}{u_c}} \\ &= h_{u_c}(0)c^{k_{c_0}} = c^{k_{c_0}} = e^{k_{c_0}\ln c}. \end{aligned}$$

Reversing the order of k_{c_0} and k_c we get $f(k_{c_0}k_ca) = e^{k_c \ln c_0}$. So $k_{c_0}\ln c = k_c \ln c_0$ and $\ln c = k_c\frac{\ln c_0}{k_{c_0}}$ for all $c \in C$. Hence $\{\ln c : c \in C\} \subseteq \frac{\ln c_0}{k_{c_0}}\mathbb{Z}$, which shows that $\{\ln c : c \in C\}$ does not generate a dense subgroup of $(\mathbb{R},+)$. Thus C does not generate a dense subgroup of $((0,\infty),\cdot)$, a contradiction. □

PROOF OF PROPOSITION 2. We can assume that $C \subseteq (0,1) \cup (1,\infty)$ and that f is non-constant. Lemma 2 justifies the additional assumption that f is vertically rigid for C via horizontal translations. Moreover, we suppose that $f(0)=1$. This can be obtained by horizontally translating the graph of f and by scaling f with some factor from $\mathbb{R}\setminus\{0\}$.

By the previous lemma, there is a dense subgroup G of $(\mathbb{R},+)$ such that $f(G)\subseteq(0,\infty)$ and

$$f(x+g)=f(x)f(g) \text{ for all } x\in\mathbb{R},\, g\in G. \tag{1}$$

Application of this to $x=g_1, g=g_2$ implies

$$\ln f(g_1+g_2)=\ln f(g_1)+\ln f(g_2) \text{ for all } g_1,g_2\in G.$$

So the function $\ln\circ f|_G$ is additive on the dense subgroup G of $(\mathbb{R},+)$. Since f has a point of continuity $x_0\in\mathbb{R}$, $\ln\circ f|_G$ is bounded on some interval. Therefore $\ln\circ f|_G$ is of the form $\ln f(g)=qg$ for all $g\in G$ with some fixed $q\in\mathbb{R}$ and

$$f(g)=e^{qg} \text{ for all } g\in G. \tag{2}$$

Now let $x\in\mathbb{R}$ be arbitrary. We pick a sequence $(g_i)_{i=0}^{\infty}\subseteq G$ converging to x_0-x. Then, by (1), $f(x)=\frac{f(x+g_i)}{f(g_i)}$ and, by (2) and the continuity of f at x_0,

$$f(x)=\lim_{i\to\infty}\tfrac{f(x+g_i)}{f(g_i)}=\lim_{i\to\infty}\tfrac{f(x+g_i)}{e^{qg_i}}=\tfrac{f(x_0)}{e^{q(x_0-x)}}=\tfrac{f(x_0)}{e^{qx_0}}e^{qx}.$$

This proves $f(x)=pe^{qx}$ with $p=\frac{f(x_0)}{e^{qx_0}}$ for all $x\in\mathbb{R}$. □

The proof of Theorem 2 requires additional preparation. The first observation is obvious.

Lemma 4. *Let $f:\mathbb{R}\to\mathbb{R}$ be vertically rigid for c_1 via α_1 and for c_2 via α_2. Then $c_1 f$ is vertically rigid for $\frac{c_2}{c_1}$ via $\alpha_2\alpha_1^{-1}$.*

Given an isometry $\alpha:\mathbb{R}^2\to\mathbb{R}^2$, M_α is to denote the uniquely determined orthogonal matrix satisfying $\alpha\binom{x}{y}=M_\alpha\binom{x}{y}+\binom{u}{v}$ with the universal translation vector $\binom{u}{v}=\alpha\binom{0}{0}$ for all $\binom{x}{y}\in\mathbb{R}^2$.

Lemma 5. *Let $c\in(0,1)\cup(1,\infty)$, let α be an isometry of $\mathbb{R}^2$, and let $f:\mathbb{R}\to\mathbb{R}$ be vertically rigid for c via α.*

(a) If $M_\alpha\in\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\begin{pmatrix}-1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}$ then f is vertically rigid for c^2 via a translation. If, in addition, f is continuous, then f is not bijective from $\mathbb{R}$ onto $\mathbb{R}$.

(b) If $M_\alpha\in\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\right\}$ then f is a bijection from $\mathbb{R}$ onto $\mathbb{R}$.

Proof. In all cases we shall use

$$\left\{\begin{pmatrix} x \\ cf(x) \end{pmatrix} : x \in \mathbb{R}\right\} = graph(cf) = \alpha(graph(f)) = \left\{M_\alpha \begin{pmatrix} x \\ f(x) \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} : x \in \mathbb{R}\right\}.$$

Case 1. $M_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $cf(x) = f(x-u) + v$ for all $x \in \mathbb{R}$. Hence

$$c^2 f(x) = c(f(x-u) + v) = cf(x-u) + cv = f(x-2u) + v + cv,$$

which shows that f is vertically rigid for c^2 via a translation.

Case 2. $M_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Now $cf(x) = -f(-x+u) + v$ and

$$c^2 f(x) = -cf(-x+u) + cv = f(x) - v + cv,$$

which gives the claim. In particular, $f(x) \equiv \frac{v}{c+1}$ is constant.

Case 3. $M_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. In this case $cf(x) = f(-x+u) + v$ and

$$c^2 f(x) = cf(-x+u) + cv = f(x) + v + cv.$$

In particular, $f(x) \equiv \frac{v}{c-1}$ is constant.

Case 4. $M_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $cf(x) = -f(x-u) + v$ and

$$c^2 f(x) = -cf(x-u) + cv = f(x-2u) - v + cv.$$

In the previous four cases we have obtained $c^2 f(x) = f(x + \bar{u}) + \bar{v}$. Let us assume that f is continuous. Then, by Lemma 1, $f(x) \equiv \frac{\bar{v}}{c^2-1}$ if $\bar{u} = 0$ or $f(x) = h_{\bar{u}}(x) c^{2\frac{x}{\bar{u}}} + \frac{\bar{v}}{c^2-1}$ with a continuous $h_{\bar{u}}$ with period $\bar{u}$ if $\bar{u} \neq 0$. Hence $h_{\bar{u}}$ is bounded and one of the limits $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ exists and agrees with $\frac{\bar{v}}{c^2-1}$. However, if f were a bijection from $\mathbb{R}$ onto $\mathbb{R}$, f would be monotonous with $\{\lim_{x\to\infty} f(x), \lim_{x\to-\infty} f(x)\} = \{\infty, -\infty\}$. This completes the proof of (a).

Case 5. $M_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We obtain

$$\left\{\begin{pmatrix} x \\ cf(x) \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} : x \in \mathbb{R}\right\} = \left\{M_\alpha \begin{pmatrix} x \\ f(x) \end{pmatrix} : x \in \mathbb{R}\right\} = \left\{\begin{pmatrix} f(x) \\ x \end{pmatrix} : x \in \mathbb{R}\right\}.$$

The left-hand side is a translate of the graph of $cf : \mathbb{R} \to \mathbb{R}$ and in turn a graph of a well-defined function from $\mathbb{R}$ into $\mathbb{R}$. The coincidence with the right-hand side shows that f^{-1} is a function from $\mathbb{R}$ into $\mathbb{R}$. This yields the claim.

Case 6. $M_\alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Now

$$\left\{-\begin{pmatrix} x \\ cf(x) \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} : x \in \mathbb{R}\right\} = \left\{-M_\alpha \begin{pmatrix} x \\ f(x) \end{pmatrix} : x \in \mathbb{R}\right\} = \left\{\begin{pmatrix} f(x) \\ x \end{pmatrix} : x \in \mathbb{R}\right\}.$$

The left-hand side is the graph of the function $x \mapsto -cf(-x+u) + v$ from $\mathbb{R}$ into $\mathbb{R}$. This gives the claim as in the previous case and completes the proof. □

PROOF OF THEOREM 2. For every $c \in C$, we fix an isometry α_c such that

$$graph(cf) = \alpha_c(graph(f)). \tag{3}$$

As it has been done in [1], we study the set

$$S_f = \left\{ \tfrac{\mathbf{a}-\mathbf{b}}{\|\mathbf{a}-\mathbf{b}\|} : \mathbf{a}, \mathbf{b} \in graph(f),\, \mathbf{a} \neq \mathbf{b} \right\},$$

where $\|\cdot\|$ stands for the Euclidean norm. S_f is non-empty and symmetric with respect to the origin. More precisely, S_f splits into $S_f^+ = \left\{ \binom{x}{y} \in S_f : x > 0 \right\}$ and $-S_f^+$, the components S_f^+ and $-S_f^+$ each being connected according to the intermediate value theorem.

For $c > 0$, let ψ_c be the self-map of $\mathbb{S}^1 = \{\mathbf{a} \in \mathbb{R}^2 : \|\mathbf{a}\| = 1\}$ defined by $\psi_c\big((x,y)^t\big) = \frac{(x,cy)^t}{\|(x,cy)^t\|}$. Equation (3) yields

$$\psi_c(S_f) = M_{\alpha_c}(S_f) \text{ for all } c \in C, \tag{4}$$

where $\psi_c(S_f)$ splits into two connected components $\psi_c(S_f^+)$ and $\psi_c(-S_f^+) = -\psi_c(S_f^+)$ and $M_{\alpha_c}(S_f)$ consists of two disjoint isometric copies of S_f^+. Hence

$$length(\psi_c(S_f^+)) = length(S_f^+) \text{ for all } c \in C. \tag{5}$$

Case 1. $length(S_f^+) = 0$. Then S_f^+ is a singleton, $S_f = S_f^+ \cup (-S_f^+) = \{\mathbf{s_0}, -\mathbf{s_0}\}$, and f is of the form $f(x) = px + q$.

Case 2. $length(S_f^+) > 0$. We denote the two end-points of S_f^+ by $\mathbf{e_1}, \mathbf{e_2}$. Equation (5) can be stated in terms of scalar products.

$$\langle \psi_c(\mathbf{e_1}), \psi_c(\mathbf{e_2}) \rangle = \langle \mathbf{e_1}, \mathbf{e_2} \rangle = \langle \psi_1(\mathbf{e_1}), \psi_1(\mathbf{e_2}) \rangle \text{ for all } c \in C. \tag{6}$$

Assume for a moment that $\{\mathbf{e_1}, \mathbf{e_2}\} \not\subseteq \left\{ \binom{0}{-1}, \binom{1}{0}, \binom{0}{1} \right\}$. Then elementary differential calculus shows that the map $c \mapsto \langle \psi_c(\mathbf{e_1}), \psi_c(\mathbf{e_2}) \rangle$ from $(0,\infty)$ into $\mathbb{R}$ attains every value at most twice. However, since C generates a dense subgroup of $((0,\infty), \cdot)$, C contains at least two distinct elements c_1, c_2 different from $c_0 = 1$. By (6), $\langle \psi_c(\mathbf{e_1}), \psi_c(\mathbf{e_2}) \rangle$ coincide for $c \in \{c_0, c_1, c_2\}$. This contradiction yields

$$\{\mathbf{e_1}, \mathbf{e_2}\} \subseteq \left\{ \binom{0}{-1}, \binom{1}{0}, \binom{0}{1} \right\}. \tag{7}$$

Case 2.1. $\{\mathbf{e_1}, \mathbf{e_2}\} \subseteq \left\{ \binom{0}{-1}, \binom{0}{1} \right\}$. Then S_f^+ is an open half-circle, $S_f = \mathbb{S}^1 \setminus \left\{ \binom{0}{-1}, \binom{0}{1} \right\}$, and (4) amounts to $S_f = M_{\alpha_c}(S_f)$ for all $c \in C$. Thus

$$M_{\alpha_c} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and, by Lemma 5 (a), f is vertically rigid for c^2 via a translation. So f is vertically rigid for $\bar{C} = \{c^2 : c \in C\}$ via translations. The subgroup $\bar{G}$ of $((0,\infty),\cdot)$ generated by $\bar{C}$ is $\bar{G} = \{g^2 : g \in G\}$, G denoting the group generated by C. Hence $\bar{C}$ generates a dense group, too. Now Proposition 2 shows that $f(x) = pe^{qx} + r$.

Case 2.2. $\{\mathbf{e_1}, \mathbf{e_2}\} \not\subseteq \{\binom{0}{-1}, \binom{0}{1}\}$. Then, by (7), S_f^+ is a quarter of $\mathbb{S}^1$ between $\binom{1}{0}$ and $\binom{0}{1}$ or $\binom{0}{-1}$ and $S_f = S_f^+ \cup (-S_f^+)$ is the corresponding symmetric set. (4) yields $S_f = M_{\alpha_c}(S_f)$ and in turn

$$M_{\alpha_c} \in \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right\} \text{ for all } c \in C.$$

Lemma 5 shows that, depending on whether f is bijective from $\mathbb{R}$ onto $\mathbb{R}$ or not, either

$$M_{\alpha_c} \in \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \text{ for all } c \in C, \text{ or} \tag{8}$$

$$M_{\alpha_c} \in \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right\} \text{ for all } c \in C. \tag{9}$$

The situation (8) can be treated as in Case 2.1.

Finally, we assume (9), which corresponds to the case that f is a bijection from $\mathbb{R}$ onto $\mathbb{R}$. There exist at least two distinct elements $c_1, c_2 \in C$, because C generates a dense subgroup of $((0,\infty),\cdot)$. By Lemma 4, $c_1 f$ is vertically rigid for $\frac{c_2}{c_1}$ via $\alpha_{c_2}\alpha_{c_1}^{-1}$. (9) yields

$$M_{\alpha_{c_2}\alpha_{c_1}^{-1}} = M_{\alpha_{c_2}} M_{\alpha_{c_1}}^{-1} = M_{\alpha_{c_2}} M_{\alpha_{c_1}} \in \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}.$$

So, by Lemma 5 (a), $c_1 f$ is not bijective from $\mathbb{R}$ onto $\mathbb{R}$ and in turn neither is f. This contradiction completes the proof. □

3 Horizontally Rigid Functions.

Proof of Proposition 3. There exist $u, v \in \mathbb{R}$ such that $f(cx) = f(x + u) + v$ for all $x \in \mathbb{R}$. We can assume $c > 1$, because the previous equation yields $f\big(\frac{1}{c}x\big) = f(x - uc) - v = f(x + \bar{u}) + \bar{v}$. Note that $v = 0$, since

$$f\big(c\tfrac{u}{c-1}\big) = f\big(\tfrac{u}{c-1} + u\big) + v = f\big(c\tfrac{u}{c-1}\big) + v.$$

Hence $f(cx) = f(x + u)$ and $f(x) = f\big(\frac{x}{c} + u\big)$ for all $x \in \mathbb{R}$. The last is

$$f(x) = f\big(\tfrac{1}{c}\big(x - \tfrac{cu}{c-1}\big) + \tfrac{cu}{c-1}\big).$$

Iteration of this gives

$$f(x) = f\big(\tfrac{1}{c^k}\big(x - \tfrac{cu}{c-1}\big) + \tfrac{cu}{c-1}\big)$$

for all $x \in \mathbb{R}$ and $k \in \{1, 2, \ldots\}$. The argument on the right-hand side tends to $\frac{cu}{c-1}$ as $k \to \infty$. So, by continuity, $f(x) = f\big(\frac{cu}{c-1}\big)$ for all $x \in \mathbb{R}$. $\square$

The proof of Theorem 4 as well as its preparation are close to those of Theorem 2. We start again with an obvious fact.

Lemma 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be horizontally rigid for c_1 via α_1 and for c_2 via α_2. Then $f(c_1 \cdot)$ is horizontally rigid for $\frac{c_2}{c_1}$ via $\alpha_2 \alpha_1^{-1}$.*

Lemma 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be horizontally rigid for some $c \in (0, 1) \cup (1, \infty)$ via an isometry α such that $M_\alpha = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then f is horizontally rigid for c^2 via a horizontal translation. If, moreover, f is continuous, then f is constant.*

PROOF. There exist $u, v \in \mathbb{R}$ such that

$$\left\{\left(\begin{smallmatrix} x \\ f(cx) \end{smallmatrix}\right) : x \in \mathbb{R}\right\} = graph(f(c\,\cdot)) = \alpha(graph(f)) = \left\{M_\alpha \left(\begin{smallmatrix} x \\ f(x) \end{smallmatrix}\right) + \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) : x \in \mathbb{R}\right\}.$$

Hence $f(cx) = -f(-x + u) + v$ and

$$f(c^2 x) = -f(-cx + u) + v = -f\big(c\big(-x + \tfrac{u}{c}\big)\big) + v = f\big(x - \tfrac{u}{c} + u\big)$$

for all $x \in \mathbb{R}$. So f is horizontally rigid for c^2 via a horizontal translation. Now the second claim is a consequence of Proposition 3. $\square$

PROOF OF THEOREM 4. Let α_i be the isometry corresponding to to c_i; that is,

$$graph(f(c_i \,\cdot)) = \alpha_i(graph(f)) \text{ for } i = 1, 2.$$

Using the sets S_f, S_f^+ and the maps $\psi_c\big((x, y)^t\big) = \frac{(x, cy)^t}{\|(x, cy)^t\|} = \frac{(\frac{x}{c}, y)^t}{\|(\frac{x}{c}, y)^t\|}$ from the proof of Theorem 2 we obtain the following analogues of (4) and (5).

$$\begin{aligned} \psi_{c_i}(S_f) &= M_{\alpha_i}(S_f), \text{ and} \\ length(\psi_{c_i}(S_f^+)) &= length(S_f^+) \text{ for } i = 1, 2. \end{aligned} \tag{10}$$

If $length(S_f^+) = 0$ we obtain the representation $f(x) = px + q$ as in the proof of Theorem 2. If $length(S_f^+) > 0$ we show as in the very proof that S_f^+ is either an open half-circle between $\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ or a quarter of a circle having $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ as an end-point. Then (10) yields $S_f = M_{\alpha_i}(S_f)$ and hence

$$M_{\alpha_i} \in \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\} \text{ for } i = 1, 2 \tag{11}$$

if S_f^+ is an half-circle or

$$M_{\alpha_i} \in \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)\right\} \text{ for } i = 1, 2 \tag{12}$$

if S_f^+ is a quarter of a circle.

Case 1. $\{M_{\alpha_1}, M_{\alpha_2}\} \cap \left\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right\} \neq \emptyset$. Then there is $i \in \{1,2\}$ such that either $M_{\alpha_i} = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, which means that f is horizontally rigid for c via a translation and in turn constant by Proposition 3, or $M_{\alpha_i} = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$, where f is constant according to Lemma 7.

Case 2. $\{M_{\alpha_1}, M_{\alpha_2}\} \cap \left\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right\} = \emptyset$. Now (11) and (12) yield

$$\{M_{\alpha_1}, M_{\alpha_2}\} \subseteq \left\{\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\right\} \text{ or } \{M_{\alpha_1}, M_{\alpha_2}\} \subseteq \left\{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix}\right)\right\}.$$

By Lemma 6, $f(c_1 \cdot)$ is horizontally rigid for $\frac{c_2}{c_1}$ via $\alpha_2\alpha_1^{-1}$. We obtain

$$M_{\alpha_2\alpha_1^{-1}} = M_{\alpha_2}M_{\alpha_1}^{-1} = M_{\alpha_2}M_{\alpha_1} \in \left\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\right\}.$$

Following the arguments of Case 1 we conclude that $f(c_1 \cdot)$ is constant. Hence f is constant as well and the proof is complete. □

References

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