RESEARCH

Real Analysis Exchange
Vol. 35(1), 2009/2010, pp. 167–194

Michael Bensimhoun, The Hebrew University, Jerusalem, Israel. email: m.bensimhoun@gmail.com

CHANGE OF VARIABLE THEOREMS FOR THE KH INTEGRAL

Abstract

Let $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ and $\varphi\colon [a,b] \to \mathcal{F}$, where $(\mathcal{E}, \mathcal{F}, \mathcal{G})$ is a Banach space triple. a) We prove that if φ is continuous $[c,d] \to [a,b]$ and $f \circ \psi \cdot d\varphi \circ \psi$ is Kurzweil or Henstock variationally integrable, then so is $f \cdot d\varphi$ and fulfills the well known change of variable formula. It follows that if ψ is an indefinite Henstock integral and if $f \circ \psi\, \psi' dx$ is K-H integrable, then so is $f dx$ and the change of variable formula applies. b) We produce several versions of the converse of a), that is, we give necessary and sufficient conditions in order that with ψ as above, the integrability of $f \cdot d\varphi$ implies that of $f \circ \psi \cdot d\varphi \circ \psi$ and the change of variable formula.

1 Introduction.

The problem of change of variable in the integral has a long history. It is best formulated in the context of the Kurzweil-Henstock-Stieltjes integral, and consists in finding the best conditions under which the relation

$$\int_a^b f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi) = \int_{\psi(a)}^{\psi(b)} f \cdot \mathrm{d}\varphi \tag{A}$$

holds, when one of these two integrals exists. Sometimes, the existence of the first integral is given and one wants to ensure the existence of the second one: this is what we call the *first category* change of variable theorems; and

Mathematical Reviews subject classification: Primary: 28B05; Secondary: 46G10, 26A42
Key words: integral, integration, Kurzweil, Henstock, generalized Riemann, substitution, change of variable, variational equivalence, Banach spaces
Received by the editors February 17, 2009
Communicated by: Stefan Schwabik

sometimes the existence of the second integral is given and one wants to ensure the existence of the first one: this is the *second category* change of variable theorems. Mostly, change of variable theorems replace expression (A) by

$$\int_a^b f \circ \psi \cdot \psi' \mathrm{d}x = \int_{\psi(a)}^{\psi(b)} f \cdot \mathrm{d}x; \tag{B}$$

that is, they assume that 1) $\varphi = \mathrm{Id}$, 2) ψ is absolutely continuous, and 3) (almost always) ψ is a real valued function. This is necessary in order to handle the problem in the context of the Lebesgue integral (the more general form (A) isn't relevant since the Stieltjes extension of the Lebesgue integral must assume the integrator monotonic). But even in this reduced form, the question has a long history (see [8]). Nevertheless, it is only recently that Leader succeeded in proving that the only three assumptions above are sufficient to ensure formula (B), when the existence of the first integral is assumed [6]. Actually, Leader proved a formula of the form (A) in the context of the Kurzweil-Henstock-Stieltjes integral, where he assumed ψ and φ continuous, ψ real valued of bounded variation, and $f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$ absolutely integrable (this implies easily the first category version of (B) for the Lebesgue integral). His proof relies heavily upon a nice generalization of his own of the Banach indicatrix theorem, that can be stated only in the context of the Kurzweil-Henstock integral. Moreover, Leader was able to express the absolute integral $\int_a^b f \circ \psi \cdot |\mathrm{d}(\varphi \circ \psi)|$ by mean of his generalized Banach indicatrix. The result of Leader can be seen as an achievement in the Lebesgue integration theory, for it is probably the best possible (first category) change of variable theorem one can give in this context. But from the point of view of the Kurzweil-Henstock integration theory, this result is not entirely satisfactory: Indeed, the function ψ has to be of bounded variation, and $f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$ absolutely integrable, what makes this result a theorem in absolute integration. Second, the proof of Leader is intricate and seems very difficult to generalize to Banach space valued functions f and φ, whenever there exists a product between the respective Banach spaces. Other previous works on first category theorems in the context of the KH-integration include formulae of type (A), where ψ is assumed strictly monotonic. This is not too difficult to prove and actually, is an important step in the proof of our theorems. There is also a change of variable formula for the distributional integral (see [9]).

In this paper, we give a first category theorem of the form (A) that removes all the restrictions above, valid for Banach Space valued functions f and φ: Suppose the first integral in formula (A) exists; if the function ψ is not continuous, then without very specific additional conditions on f, the above relation is easily seen to be false (even if $\varphi = \mathrm{Id}$). So, to ensure the existence of the last integral, the condition that ψ be continuous seems to be not too much exaggerated. We have found, and this is the content of Thm. 6.1 below, that this only condition IS SUFFICIENT. As a simple corollary, we deduce formula (B), which provides an alternative proof of the result of Leader. The demonstration of Thm. 6.1 is not terribly complicated, as soon as one has shown the following assertion: *Assume that ψ is continuous in some finite interval $[a,b]$, that $f\circ\psi\,\mathrm{d}(\varphi\circ\psi)$ is integrable in $[a,b]$, and that $\psi(a)=\psi(b)$. Then $\int_a^b f\circ\psi\,\mathrm{d}(\varphi\circ\psi)=0$.* The proof of this assertion is indirect, like the proof of the Cousin lemma: it consists in showing that for every given positive function δ defined in $[a,b]$, there exists a tagged division D of $[a,b]$ subordinated to δ such that the Riemann sum associated to D and to $f\circ\psi\,\mathrm{d}(\varphi\circ\psi)$ is equal to 0 (Lemma 6.5).

The second category change of variable theorems seem more difficult to handle: assuming the existence of the second integral in (A), there is no simple natural condition on ψ one can hope to ensure the existence of the first integral and formula (A) (the continuity of ψ is very unlikely to provide such a condition, despite we know no counterexample). We shall nevertheless provide theorems of this kind that are better than all we know on the subject. Let us assume in the sequel that ψ is continuous, and put $F(x)=\int_a^x f\cdot\mathrm{d}\varphi$. In Thm. 7.5, we assume furthermore the function ψ is *locally decomposable* in its domain, at all but a countable number of points; by "decomposable" at x, we mean that the slopes of ψ at x are either non-negative, or non-positive, or that x is a strict maximum or a strict minimum of ψ. In these conditions, the second category version of (A) holds. Since almost all the continuous functions that occur in practice are decomposable at all but a countable number of points, this theorem has some practical interest. In Thm. 7.2, we assume the underlying Banach triple fulfills some topological conditions, and that ψ is surjective. Then we show that the existence of a function g such that $F\circ\psi(x)=\int_c^x g\circ\psi\cdot\mathrm{d}\varphi\circ\psi$ is sufficient to ensure the second category version of (A). Finally, in Thm. 7.7, we assume that $\varphi(x)=x$ and investigate two cases: 1) ψ is of bounded variation and 2) ψ is the indefinite Henstock integral of a function. The latter case is motivated by the fact that if one defines the *Henstock derivative* of a function f as a function f' such that f is the Henstock indefinite integral of f', then most of the classical theorems of calculus hold for this derivative (including the Taylor formula). So it would be nice to include the chain rule in the list, which is precisely the motivation of the second case above. Unfortunately, it is difficult to prove or to disprove both cases (Pb. 7.6). Nevertheless, we have found a small set of points S such that the second category version of (A) holds whenever S is $F\circ\psi$-null.

2 Definitions and Preliminaries.

2.1 Metric of $\bar{\mathbb{R}}$.

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ be the extended real line, endowed with its usual topology, ordering, and operations extending the operations of $\mathbb{R}$. Define $\rho\colon [-\infty, +\infty] \to [-1, 1]$ by $\rho(x) = \arctan(x)/\pi$. Using the function ρ, we can endow $\bar{\mathbb{R}}$ with a metric that induces its topology by $\bar{d}(x, y) = |\rho(x) - \rho(y)|$.. Then the metric $\bar{d}$ induces the same topology as the usual one on $\bar{\mathbb{R}}$, and makes $\bar{\mathbb{R}}$ homeomorphic to $[0, 1]$.

In this paper, we use the metric $\bar{d}$ in the definition of the Kurzweil-Henstock integral in place of the more topological notion of gauges. The two definitions are of course equivalent because the metric $\bar{d}$ generates the topology of $\bar{\mathbb{R}}$. But the use of a metric seems to us easier and more intuitive, and furthermore allows most of the definitions usually related to compact intervals to be immediately transposed to $\bar{\mathbb{R}}$.

2.2 Banach Triples.

In the sequel, we consider integrals that involve a product between Banach spaces. Let us make the convention that $(\mathcal{E}, \| \ \|_{\mathcal{E}})$, $(\mathcal{F}, \| \ \|_{\mathcal{F}})$ and $(\mathcal{G}, \| \ \|_{\mathcal{G}})$ denote in the sequel three Banach spaces, endowed with a bilinear product $\mathcal{E} \times \mathcal{F} \to \mathcal{G}$ that fulfills (unless specified otherwise)

$$\|x \cdot y\|_{\mathcal{G}} \leq \|x\|_{\mathcal{E}} \|y\|_{\mathcal{F}}, \quad \forall\, x \in \mathcal{E},\ y \in \mathcal{F}.$$

Notice that this implies in particular the product is continuous $\mathcal{E} \times \mathcal{F} \to \mathcal{G}$. Such a triple of Banach spaces is usually called *Banach triple*, and denoted $(\mathcal{E}, \mathcal{F}, \mathcal{G})$. When $\mathcal{F} = \mathbb{R}$, one can define a bilinear product $\mathcal{E} \times \mathcal{F}$ by $x\lambda = \lambda x$. In the sequel, we use indifferently these two notations. This doesn't harm in general because the meaning to assign to the product is clear.

2.3 Tagged Intervals and Divisions.

A *tagged interval* of $[a, b] \subseteq \bar{\mathbb{R}}$ is a couple $\hat{I} = ([u, v], \xi)$, where $[u, v] \subseteq [a, b]$ and $\xi \in [u, v]$ or $\xi = \emptyset$; The interval $[u, v]$ is by definition *the interval of* $\hat{I}$. In case $\xi = \emptyset$, we identify $\hat{I}$ with $[u, v]$, so the set of tagged intervals of $[a, b]$ is a superset of the set of intervals of $[a, b]$. Two tagged intervals are said to be *non-overlapping* if the intersection of their intervals is void, or if they intersect only at their end points. If $[a, b] \in \mathbb{R}$, A *partial division* D of an interval $[a, b] \subseteq \bar{\mathbb{R}}$ is a finite set of non-overlapping tagged intervals of $[a, b]$; an *interval* of D is the interval of one of its tagged intervals, and an *anchor* point of D is one of the point ξ of the tagged intervals $([u,v],\xi)$ of D. A tagged interval $([u,v],\xi)$ is *tame* if $\xi = u$ or $\xi = v$, and a partial division is *tame* if each of its tagged intervals is tame. A partial division *anchors* in a set M if its anchor points belong to M. A *division* of $[a,b]$ is a partial division of $[a,b]$ whose intervals cover $[a,b]$.

Now, suppose that $\delta\colon [a,b] \to \mathbb{R}_+^*$ is a strictly positive function. A tagged interval $([u,v],\xi)$ of $[a,b]$ is *subordinated to* δ if $\bar{d}(\xi,v) < \delta(\xi)$ and $\bar{d}(\xi,u) < \delta(\xi)$. A partial division D of $[a,b]$ is *subordinated to* δ if each of its tagged intervals is subordinated to δ; we denote $D << \delta$ a partial division D subordinated to δ. Even if δ is not defined on the whole of $[a,b]$ but only on the anchor points of D, we say that such a partial division D is subordinated to δ. The Cousin lemma ensures that for every $\delta\colon [a,b] \to \mathbb{R}_+^*$, there exists a division of $[a,b]$ subordinated to δ. The same is true for tame divisions: to see this, it suffices to take a division subordinated to δ, and to split each of its tagged intervals $([u,v],\xi)$ into two tagged interval $([u,\xi],\xi)$ and $([\xi,v],\xi)$.

2.4 Interval Functions and Differential Elements.

Assume that $(\mathcal{G},\mathcal{F},\mathcal{G})$ is a Banach triple, and that $[a,b] \subseteq \bar{\mathbb{R}}$. If $\delta\colon [a,b] \to \mathbb{R}_+^*$, let us denote by $\hat{\mathrm{Int}}_\delta([a,b])$ the set of tagged interval of $[a,b]$ subordinated to δ. A function $\hat{\mathrm{Int}}_\delta([a,b]) \to \mathcal{G}$, for some $\delta\colon [a,b] \to \mathbb{R}_+^*$, is called a *differential element* of $[a,b]$ (or in abbreviation, a *differential* of $[a,b]$). In general, we denote such a function by a letter preceded by "*d*", like "*dh*", "*dQ*" etc. We also allow notations like $dh\colon [a,b] \to \mathcal{G}$ to signify that dh is a differential element of $[a,b]$ ranging in $\mathcal{G}$. We can add and multiply by scalars differential elements, the result being again differential elements, denoted by expressions like $\lambda dh_1 + dh_2 + dh_3$ whose meaning is clear. Indeed, if δ_1 and δ_2 are such that dh_1 and dh_2 are defined on $\hat{\mathrm{Int}}_{\delta_1}([a,b])$ and $\hat{\mathrm{Int}}_{\delta_2}([a,b])$ resp., then with $\delta = \min(\delta_1,\delta_2)$, dh_1 and dh_2 are clearly defined both on $\hat{\mathrm{Int}}_\delta([a,b])$, hence so is their sums, product with scalars etc. In the same manner, if dh_1 ranges in $\mathcal{E}$ and dh_2 ranges in $\mathcal{F}$, we can multiply dh_1 and dh_2 to get differential elements, denoted $dh_1 \cdot dh_2$, with values in $\mathcal{G}$. Also, given a function $f\colon [a,b] \to \mathcal{E}$ and a differential element $dh\colon [a,b] \to \mathcal{F}$, we can create a new differential denoted $f \cdot dh$ and defined by

$$f \cdot dh((I,\xi)) = f(\xi) \cdot dh((I,\xi))$$

(a similar definition for $dh \cdot f$, where dh is $\mathcal{E}$-valued and f is $\mathcal{F}$-valued). We denote by $\|dh\|$ the differential

$$\|dh\|(\hat{I}) = \|dh(\hat{I})\|.$$

If $\varphi\colon\ [a,b] \to \mathcal{G}$ is a function, then we denote by $d\varphi$ the differential

$$d\varphi([u,v]) = \varphi(v) - \varphi(u),$$

defined on $\hat{\mathrm{Int}}_\delta([a,b])$ for any δ. Thus, $f \cdot d\varphi$ denotes the differential

$$f \cdot d\varphi(([u,v],\xi) = f(\xi) \cdot (\varphi(v) - \varphi(u)).$$

Finally, we denote by $dx\colon\ \bar{\mathbb{R}} \to \mathbb{R}$ the differential $d\varphi$, where $\varphi(x) = x$ if $x \neq \pm\infty$, and $\varphi(\pm\infty) = 0$.

2.5 The Kurzweil Integral.

A differential element $dh\colon\ [a,b] \to \mathcal{G}$ is *Kurzweil (or K-) integrable*, with integral equal to $S \in \mathcal{G}$, if for every $\varepsilon > 0$, there exists a function $\delta\colon\ [a,b] \to \mathbb{R}_+^*$ such that, for every tame division D of $[a,b]$ subordinated to δ, D is in the domain of dh and

$$\Big\| S - \sum_{\hat{I} \in D} dh(\hat{I}) \Big\|_{\mathcal{G}} < \varepsilon. \tag{1}$$

It is not difficult to show, with the (tame version of the) Cousin lemma, that the integral of dh is well defined. Moreover, it doesn't depend on the metric $\bar{d}$, but only on the topology of $\bar{\mathbb{R}}$: any other metric of $\bar{\mathbb{R}}$ that would induce this topology would give the same result. The K-integral of dh will be denoted by (K) $\int_a^b \mathrm{d}h$, or simply $\int_a^b \mathrm{d}h$ if there is no risk of confusion. Also, if $f\colon\ [a,b] \to \mathcal{E}$, then the integral of f is by definition the integral of $f\mathrm{d}x\,(= dxf)$, where dx is defined as above.

It may look surprising that we use the notion of tame division in place of the traditional notion of division in our definition of the Kurzweil integral; there are several good reasons to do so, by far the most important of which is the fact that for any function φ of bounded variation, $\|d\varphi\|$ is tamely K-integrable (with integral equal to the variation of φ). On the contrary, this holds for (traditional) K-integration only if φ is continuous from the right or from the left at any point: consider the function φ defined by $\varphi(x) = 0$ for all $x \neq 0$ and $\varphi(x) = 1$ if $x = 0$; then it is easily seen (even directly) that $|d\varphi|$ is not (traditional) K-integrable in any interval $[a,b] \ni 0$ with $a, b \neq 0$. Nevertheless, we point out that most of the theorems in this paper hold for KH-integrals defined by mean of classic divisions. In fact, upon redefining in an obvious way the concepts of "variational equivalence" (see Sec. 3 below), most of the proofs can be adapted "as is" in the sequel.

Finally, we say that a function $F\colon\ [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{G}$ is an *indefinite integral in* $[a,b]$ if there exists $f\colon\ [a,b] \to \mathcal{G}$ such that $f(x) = \int_c^x f\,\mathrm{d}x$ for some $c \in$

$[a, b]$. It is an *indefinite integral in* $]a, b[$ if it is an indefinite integral in every $[\alpha, \beta] \subset]a, b[$.

The Kurzweil integral fulfills all the usual properties one may expect of an integral. We need the following essential theorem in the sequel.

Theorem 2.5.1 (The Henstock-Saks lemma)**.** *Let* $dh\colon [a, b] \subseteq \bar{\mathbb{R}} \to \mathcal{G}$ *be a differential element. Assume that* dh *is K-integrable in* $[a, b]$*. Then for every* $\varepsilon > 0$*, there exists a function* $\delta\colon [a, b] \to \mathbb{R}_+^*$ *such that, for every PARTIAL tame division* $D << \delta$*,*

$$\left\| \sum_{(I,\xi)\in D} \int_I \mathrm{d}h - dh((I, \xi)) \right\| < \varepsilon. \tag{2}$$

Furthermore, if $\mathcal{G}$ *has finite dimension, then condition* (2) *can be strengthened to*

$$\sum_{(I,\xi)\in D} \left\| \int_I \mathrm{d}h - dh((I, \xi)) \right\| < \varepsilon. \tag{3}$$

Proof. The real case is well known, and (2) is an immediate adaptation to Banach spaces. Assertion (3) can be derived from the more general proposition 3.1 below. □

2.6 The Henstock Integral.

It is not true, if $\mathcal{G}$ is a general Banach space that condition (3) always holds (see [3], ex. 2.1 and 3.1). The Henstock integral, that postulates this property, is less general than the Kurzweil integral, but behaves much better. The original definition of Henstock was related to additive interval functions. I see no reason not to generalize his definition to arbitrary differential elements.

Let $dh\colon [a, b] \subseteq \bar{\mathbb{R}} \to \mathcal{G}$ be a differential element. Then dh is *variationally (or Henstock) integrable* if it is Kurzweil integrable, and condition (3) holds for a suitable δ. We denote the Henstock integral of dh by $(H) \int_a^b \mathrm{d}h$. As expected again, the Henstock integral fulfills the usual properties of integrals.

We make the following convention in the sequel: If we use the terms "integrable" or "integral" without other specification in the formulation of a proposition, it has to be understood either as "Kurzweil integrable/integral", or "Henstock integrable/integral" in the whole proposition. Furthermore, if some integrals appear in such a proposition, then they are either Kurzweil integrals, or Henstock integrals according to this choice.

3 Variational Equivalence on a Set.

Let dh_1 and dh_2 be differential elements $[a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{G}$, $[\alpha,\beta] \subseteq [a,b]$ be an interval, and $M \subseteq \bar{\mathbb{R}}$ be a set. Then the differential elements dh_1 and dh_2 will be said *weakly variationally equivalent on* M *in* $[\alpha,\beta]$, if for every $\varepsilon > 0$, there exist $\delta\colon\ M \cap [\alpha,\beta] \to \mathbb{R}_+^*$ such that, for every partial tame division $D << \delta$ of $[\alpha,\beta]$ that anchors in M,

$$\Big\|\sum_{\hat{I}\in D} dh_1(\hat{I}) - dh_2(\hat{I})\Big\| < \varepsilon. \tag{4}$$

The elements dh_1 and dh_2 will be said *variationally equivalent on* M in $[\alpha,\beta]$ if condition (4) can be strengthened to

$$\sum_{\hat{I}\in D} \|dh_1(\hat{I}) - dh_2(\hat{I})\| < \varepsilon.$$

We say for convenience that dh_1 and dh_2 are (weakly) variationally equivalent on M if they are (weakly) variationally equivalent on M in $[a,b]$. Similarly, we say that dh_1 and dh_2 are (weakly) variationally equivalent if they are (weakly) variationally equivalent on $[a,b]$ in $[a,b]$. In the sequel, the notation $dh_1 \sim dh_2$ means "dh_1 is weakly variationally equivalent to dh_2", and the notation $dh_1 \approx dh_2$ means "dh_1 is variationally equivalent to dh_2". To prove the following proposition, the equivalence of the norms in $\mathbb{R}^n$, or alternatively, an argument of compacity of the unit ball, must be used.

Proposition 3.1. *Assume that $\mathcal{G}$ is Banach finite dimensional. Then the concepts of variational equivalence and of weak variational equivalence coincide in $\mathcal{G}$.*

The remaining propositions and theorems of this section will be used again and again in the sequel. The proofs present no real difficulties and are omitted (see also [6]).

Theorem 3.2. *Let dh_1 and dh_2 be differential $[a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{G}$. The following conditions are sufficient in order that dh_1 and dh_2 be variationally (resp. weakly variationally) equivalent on a set $M \subseteq [a,b]$:*

(i) dh_1 and dh_2 are variationally (resp. weakly variationally) equivalent on each set of a countable cover $\{\, M_k \subseteq [a,b],\ k \in \mathbb{N}^* \,\}$ of M.

(ii) dh_1 and dh_2 are variationally (resp. weakly variationally) equivalent on M in $[\alpha,\beta]$, for every bounded $[\alpha,\beta] \subset]a,b[$, and also on $\{a,b\} \cap M$ in $[a,b]$.

Furthermore, condition (ii) is necessary in order dh_1 and dh_2 be variationally (resp. weakly variationally) equivalent on M, and so is condition (i) if each $M_k \subseteq M$.

Proposition 3.3. *Let $[a, b] \subseteq \bar{\mathbb{R}}$ be given, and differential dh and dh' $[a, b] \to \mathcal{E}$, $d\varphi$ and $d\varphi'$, $[a, b] \to \mathcal{F}$, and dh_1, dh_1', dh_2, dh_2', $[a, b] \to \mathcal{G}$. Assume that $M \subseteq \bar{\mathbb{R}}$ is a set.*

(i) *If $dh_1 \approx dh_1'$ and $dh_2 \approx dh_2'$ on M then $dh_1 + dh_2 \approx dh_1' + dh_2'$ on M. The same assertion holds for the weak variational equivalence.*

(ii) *If $dh \approx dh'$ on M, then $\lambda dh \approx \lambda dh'$ for every $\lambda \in \mathbb{R}$. The same holds for weak variational equivalence.*

(iii) *Assume that dh is bounded at every point of M. Then if $d\varphi \approx d\varphi'$ on M, $dh \cdot d\varphi \approx dh \cdot d\varphi'$ on M. The converse is true provided the product between $\mathcal{E}$ and $\mathcal{F}$ is norm preserving.*

(iv) *If $d\varphi \approx d\varphi'$ on M, then $f \cdot d\varphi \approx f \cdot d\varphi'$ on M for every function $f\colon [a, b] \to \mathcal{E}$. Conversely, assume that $f(x) \neq 0$ for all x and that the product between $\mathcal{E}$ and $\mathcal{F}$ is norm preserving. Then $f \cdot d\varphi \approx f \cdot d\varphi'$ on M implies $d\varphi \approx d\varphi'$ on M.*

(v) *If $dh \approx dh'$ and $d\varphi \approx d\varphi'$ on M, then $dh \cdot d\varphi \approx dh' \cdot d\varphi'$ on M, provided dh and $d\varphi'$ be bounded at every point of M, or dh' and $d\varphi$ be bounded at every point of M.*

(vi) *Let $g\colon \mathcal{E} \to \mathcal{E}'$ be a Lipschitz function to another Banach space $\mathcal{E}'$. Then if $dh \approx dh'$, $d(g \circ dh) \approx d(g \circ dh')$.*

(vii) *If $dh \approx dh'$ on M, then $\|dh\| \approx \|dh'\|$ on M.*

(viii) *Let $\psi\colon [a, b] \to \mathcal{G}$ be of bounded variation in $[a, b]$, and put $V(x) = \mathrm{Var}_a^x \psi$. Then $\|d\psi\|$ is (tamely) variationally equivalent to dV. This doesn't hold for non-tame equivalence, unless ψ is $\bar{d}$-continuous from the right or from the left at any point.*

(ix) *$dh \approx 0$ on M if and only if $\|dh\| \approx 0$ on M.*

(x) *If $\|dh\| \leq \|dh'\|$ and $dh' \approx 0$, then $dh \approx 0$.*

(xi) *If dh is continuous on a countable set, then it is ≈ 0 on this set. In particular, if $f\colon [a, b] \to \mathcal{G}$ is continuous, then $df \approx 0$ on every countable set.*

(xii) If $d\varphi \approx 0$ *on* M*, then* $f \cdot d\varphi \approx 0$ *on* M *for every function* $f\colon [a,b] \to \mathcal{E}$*.*

In particular, $f dx \approx 0$ *on every negligible set* $M \subset [a,b]$*. The converse is true, provided that* $f(x) \neq 0$ *for all* $x \in M$*.*

Proposition 3.4. *Let* $F\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ *be differentiable at every point of a set* $T \subseteq]a,b[$*. Then* $dF \approx F'dx$ *and* $\|dF\| \approx \|F'dx\|$ *on* T *in* $[a,b]$*.*

Theorem 3.5. *Let* dh_1 *and* dh_2 *be differentials* $[a,b] \to \mathcal{G}$ *(*$[a,b] \subseteq \bar{\mathbb{R}}$*). Assume that* dh_1 *is K-integrable (resp. H-integrable) in* $[a,b]$*. Then in order that* dh_2 *be K-integrable (resp. H-integrable) in [a,b], it is sufficient that* dh_2 *be weakly variationally equivalent (resp. variationally equivalent) to* dh_1*. In other words, if* dh_1 *and* dh_2 *are (weakly) variationally equivalent, the integrability of either differential element implies the integrability of the other and equality of their integrals.*

4 Absolute Integrability.

Let $dh\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{G}$ be a differential element. Then dh is *absolutely K-integrable* (resp. *absolutely H-integrable*) in $[a,b]$, if dh is K-integrable (resp. H-integrable) in $[a,b]$, and $\|dh\|$ is TAMELY integrable in $[a,b]$. It is important to realize that in this definition, the integrability of dh can be thought either in the context of tame integration, or in that of non-tame integration, but the integrability of $\|dh\|$ relates to tame integration only. A function f is *absolutely K-integrable* (resp. *absolutely H-integrable*) if $f dx$ is absolutely K-integrable (resp. H-integrable).

Theorem 4.1. *Let* $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ *and* $\varphi\colon [a,b] \to \mathcal{F}$*.*
(i) The differential $\|d\varphi\|$ *is tamely H-integrable in* $[a,b]$ *if and only if* φ *is of bounded variation in* $[a,b]$*, and then* $\mathrm{Var}_a^b(\varphi) = \int_a^b \|\mathrm{d}\varphi\|$*.*
(ii) If φ *is of bounded variation in* $[a,b]$ *and* $V(x) = \mathrm{Var}_a^x(\varphi)$*, then* $f\,\|d\varphi\|$ *is tamely integrable if and only if* $f dV$ *is, and then* $\int_a^b f\,\|\mathrm{d}\varphi\| = \int_a^b f\,\mathrm{d}V$*.*

PROOF. This follows from Prop. 3.3 (viii) and (iv) and Thm. 3.5. □

Theorem 4.2. *Let* $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ *be a function and* $dh\colon [a,b] \to \mathcal{F}$ *be a differential. If* $f \cdot dh$ *is absolutely integrable in* $[a,b]$*, then the function* $F(x) = \int_a^x f \cdot \mathrm{d}h$ *is of bounded variation in* $[a,b]$*. Conversely, if* $f \cdot dh$ *is H-integrable in* $[a,b]$ *and* $F(x) = \int_a^x f \cdot \mathrm{d}h$ *is of bounded variation in* $[a,b]$*, then* $f \cdot dh$ *is absolutely H-integrable in* $[a,b]$*, with (tame)* $\int_a^b \|f \cdot \mathrm{d}h\| = \mathrm{Var}_a^b(F)$*.*

Proof. The first assertion is straightforward. Conversely, suppose $f \cdot dh$ H-integrable and F of bounded variation in $[a, b]$. By the Henstock-Saks lemma, $f \cdot dh \approx dF$ in $[a, b]$ (all the equivalence here are tame). Hence $\|f \cdot dh\| \approx \|dF\|$ in $[a, b]$ (Prop. 3.3 (vii)). Put $V(x) = \mathrm{Var}_a^x(F)$. Since $\|dF\| \approx dV$ (by Thm. 4.1 (i)), $\|f \cdot dh\| \approx dV$ in $[a, b]$. Thus, $\int_a^b \|f \cdot dh\| = \mathrm{Var}_a^b(F)$ (Thm. 3.5). □

5 Other Needed Theorems.

Theorem 5.1. *Let f and g be defined in $[a, b]$, and dh be a differential of $[a, b]$. Assume that the products between the underlying Banach spaces is associative, and that $g \cdot dh$ is Henstock integrable, with $G(x) = \int_a^x g \cdot \mathrm{d}h$. Then $f \cdot g \cdot dh \approx f \cdot dG$ in $[a, b]$. In particular, if one of the integrals $\int_a^b f \cdot \mathrm{d}G$ or $\int_a^b f \cdot g \cdot \mathrm{d}h$ exists, the other integral exists and they are equal.*

Proof. By the Henstock-Saks lemma, $dG \approx g \cdot dh$. By Prop. 3.3 (iv), $f \cdot dG \approx f \cdot g \cdot dh$. Conclude by Thm. 3.5. □

In the next theorem (second assertion), we introduce a special condition on $\mathcal{E}$ that is true in euclidean spaces, and more generally in separable spaces (that is, spaces that have a countable dense subset). We have a proof that is by no means trivial, but the limitations on the size of the paper prevent us to give it here.

Theorem 5.2. *Let $f, g\colon [a, b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ and $dh\colon [a, b] \to \mathcal{F}$ be such that $f \cdot dh$ is integrable. Then if f and g differ only on a dh-negligible set of $[a, b]$, $g \cdot dh$ is integrable and $\int_a^b f \cdot \mathrm{d}h = \int_a^b g \cdot \mathrm{d}h$. Conversely, assume that for every $\varepsilon > 0$, there exists a countable cover of $\mathcal{E}$ by balls of radius less than ε, that the product between $\mathcal{E}$ and $\mathcal{F}$ is norm-preserving, and that $f \cdot dh$ is H-integrable. Then if $\int_\alpha^\beta f \cdot \mathrm{d}h = 0$ for all $[\alpha, \beta] \subseteq [a, b]$, $f(x) = 0$ dh-almost everywhere in $[a, b]$.*

Theorem 5.3 ([2], Thm. 5.9)**.** *Let $F\colon [a, b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ be a function. Then F is the Henstock integral of a function $f\colon [a, b] \to \mathcal{E}$ if and only if F is continuous in $[a, b]$ (relative to $\bar{\mathrm{d}}$), and differentiates to f in $]a, b[$, at the possible exception of a negligible set of points S on which $dF \approx 0$. When this occurs, $dF \approx 0$ on every negligible set in $[a, b]$.*

Theorem 5.4 ([7])**.** *Let $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$. Assume that ψ is differentiable at each point of a set $M \subseteq [c,d]$. Then $\psi(M)$ is negligible if and only if $\psi'(x) = 0$ for almost every point x of M. In particular, if $\psi'(x) \neq 0$ for every $x \in M$, then M is negligible if and only if $\psi(M)$ is negligible.*

Corollary 5.5. *Let $S \subseteq [a,b]$ be a negligible set, and $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b] \subseteq \mathbb{R}$ be an indefinite H-integral in $[c,d]$. Then $d\psi \approx 0$ on $\psi^{-1}(S)$.*

Proof. Let $\varepsilon > 0$, and let ψ' such that $\psi(x) = \int_c^x \psi' \mathrm{d}x$. Since $d\psi \approx 0$ on the set of points where it is not differentiable (Thm. 3.3 (xii) and 5.3), we can suppose w.l.g. that ψ is differentiable at every point of $\psi^{-1}(S)$. Since $\psi(\psi^{-1}(S)) \subseteq S$, Thm. 5.4 implies that $\psi'(x) = 0$ for a.e. point $x \in \psi^{-1}(S)$. Let S' be the set of points $x \in \psi^{-1}(S)$ such that $\psi'(x) \neq 0$. Since $dx \approx 0$ on every negligible set, so is $\psi' dx$, and hence $d\psi$. Thus $d\psi \approx 0$ on S'. On the other hand, since $\psi'(x) = 0$ for all $x \in S \smallsetminus S'$, $\psi' dx \approx 0$ on $S \smallsetminus S'$; hence $d\psi \approx 0$ on $S \smallsetminus S'$ (Prop. 3.4). In conclusion, $d\psi \approx 0$ on $\psi^{-1}(S)$. $\square$

Theorem 5.6 ([5], Chap. 9 Thm. 3)**.** *Let $S \subseteq [a,b]$ be a negligible set, and $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$ be continuous and of bounded variation in $[a,b]$. Then $d\psi \approx 0$ on $\psi^{-1}(S)$.*

6 Change of Variables Theorems, First Category.

Theorem 6.1 (Main theorem)**.** *Let the functions $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$, $\varphi\colon [a,b] \to \mathcal{F}$, and $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$ be given. Assume that ψ is continuous relatively to $\bar{d}$ as a metric of both $[a,b]$ and $[c,d]$. Then if $f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$ is integrable in $[c,d]$, $f \cdot \mathrm{d}\varphi$ is integrable in $\psi([c,d])$, and*

$$\int_c^d f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi) = \int_{\psi(c)}^{\psi(d)} f \cdot \mathrm{d}\varphi.$$

Furthermore, if $f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$ is absolutely integrable in $[c,d]$, then $f \cdot \mathrm{d}\varphi$ is absolutely integrable in $\psi([c,d])$, with

$$\int_c^d \|f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)\| = \int_{\psi(c)}^{\psi(d)} \|f \cdot \mathrm{d}\varphi\|.$$

Corollary 6.2. *Let* $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}$*, and* ϕ *be absolutely continuous* $[c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$*. If* $(f \circ \phi)\,\phi'$ *is Lebesgue integrable in* $[c,d]$*, then* f *is Lebesgue integrable in* $\phi([c,d])$ *and* $\int_{\phi(c)}^{\phi(d)} f\,\mathrm{d}x = \int_c^d (f \circ \phi)\,\phi'\mathrm{d}x$*.*

Proof. Since ϕ is absolutely continuous in $[c,d]$, it is the absolute integral of its derivative defined a.e. and Thm. 5.1 shows that

$$\int_c^d (f \circ \phi)\,\phi'\mathrm{d}x = \int_c^d (f \circ \phi)\,\mathrm{d}\phi.$$

Since $f \circ \phi\,\phi' dx$ is Lebesgue integrable, it is absolutely integrable. Furthermore,

$$f \circ \phi\,\phi' dx \approx f \circ \phi\, d\phi,$$

hence $f \circ \phi\, d\phi$ is absolutely integrable (Prop. 3.3 (vii)). Thm. 6.1 says that f is absolutely integrable, hence Lebesgue integrable in $\phi([c,d])$, with

$$\int_{\phi(c)}^{\phi(d)} f\,\mathrm{d}x = (K)\int_c^d (f \circ \phi)\,\mathrm{d}\phi = \int_c^d f \circ \phi\,\phi'\mathrm{d}x. \qquad \square$$

Proof of Thm. 6.1. We will prove the theorem in the context of the H-integral, the proof being almost the same (and even easier) for K-integrals, upon replacing expressions like $\sum\|\cdots\|$ by $\|\sum\cdots\|$, and variational equivalence by weak variational equivalence.

Given a continuous function $\mu\colon [c,d] \to [a,b]$, such that $\mu(c) = a$ and $\mu(d) = b$, we define a function $\hat{\mu}\colon [a,b] \to [c,d]$, that we call the *upper invert of* μ, by

$$\hat{\mu}(x) = \inf\{\, \alpha \in [c,d]\colon \mu(\alpha) = x \,\}.$$

We define also $\hat{\mu}^+$ by $\hat{\mu}^+(x) = \lim_{t\to x+} \hat{\mu}(t)$ for every $x < b$, and $\hat{\mu}^+(b) = d$.

Lemma 6.3. $\hat{\mu}$ *fulfills the following properties:*

(i) $\mu(\hat{\mu}(x)) = x$ *for all* x *in* $[a,b]$*;*

(ii) $\hat{\mu}$ *is strictly increasing* $[a,b] \to [c,d]$*,* $\hat{\mu}(a) = c$ *and* $\hat{\mu}(b) \le d$*;*

(iii) $\hat{\mu}$ *is continuous from the left in* $]c,d]$*;*

(iv) $\hat{\mu}^+$ *is strictly increasing, continuous from the right in* $[a,b[$*, and* $\hat{\mu}^+(x) \ge \hat{\mu}(x)$ *for all* x*; but if* $x < y$*,* $\hat{\mu}^+(x) < \hat{\mu}(y)$*;*

(v) For all $x \in [a,b[$, μ is constant in $[\hat{\mu}(x), \hat{\mu}^+(x)]$. It follows in particular that $\mu(\hat{\mu}^+(x)) = x$ for all $x \in [a,b]$.

PROOF. Straightforward □

Lemma 6.4. *With the hypothesis of Thm. 6.1, if ψ is continuous and monotonic $([c,d],\bar{d}) \to ([a,b],\bar{d})$, then the integrability of $f \cdot \mathrm{d}\varphi$ in $\psi([a,b])$ implies that of $f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$ in $[c,d]$, and then*

$$\int_c^d f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi) = \int_{\psi(c)}^{\psi(d)} f \cdot \mathrm{d}\varphi.$$

If ψ is a homeomorphism (that is, ψ is continuous and strictly monotonic), then the converse is true, and the above relation holds again.

PROOF. For the first assertion, see [5], Thm. 1. Now assume $f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$ is H-integrable and ψ is an homeomorphism. Then ψ is invertible and ψ^{-1} is continuous and monotonic. But

$$f \cdot \mathrm{d}\varphi = f \circ \psi \circ \psi^{-1} \cdot d\varphi \circ \psi \circ \psi^{-1},$$

hence the second assertion is deduced by applying the first assertion to the differential element $(f \circ \psi) \cdot d\varphi \circ \psi$ and to the function ψ^{-1}. □

The following lemma is the kernel of the proof. It shows in particular that if $f \circ \psi \cdot d\varphi \circ \psi$ is integrable in $[c,d]$ and $\psi(c) = \psi(d)$, then $\int_c^d f \circ \psi \cdot \mathrm{d}\varphi \circ \psi = 0$.

Lemma 6.5. *Let $[c,d] \subseteq \mathbb{R}$ be a bounded interval, and $\psi\colon [c,d] \to [a,b]$ be continuous and such that $\psi(c) = \psi(d)$. Then given any $\delta'\colon [c,d] \to \mathbb{R}_+$ and any two functions f and φ defined on $[a,b]$, there exists a division of $[c,d]$, subordinated to δ', such that*

$$\sum_{([u,v],\xi) \in D} f(\psi(\xi)) \cdot (\varphi(\psi(v)) - \varphi(\psi(u))) = 0.$$

PROOF. We fix $\delta'\colon [c,d] \to \mathbb{R}_+$ and for $[x,y] \subseteq [c,d]$, we consider the property (P)=(P($[x,y]$)): there exists a division $D << \delta'$ of $[x,y]$, such that, for every f and φ defined in $[a,b]$,

$$\sum_{([u,v],\xi) \in D} f(\psi(\xi)) \cdot (\varphi(\psi(v)) - \varphi(\psi(u))) = 0.$$

The proof of the lemma is indirect: we suppose for a contradiction that P$([a,b])$ is false. Define an increasing sequence $(c_n)_n$ and a decreasing sequence $(d_n)_n$ by induction in the following way: $c_0 = c$ and $d_0 = d$. For all $n \geq 1$, there exists

$$\alpha_n = \inf\{\, |x-y| \colon x, y \in [c_{n-1}, d_{n-1}],\ \psi(x) = \psi(y),\ \mathrm{P}([x,y]) \text{ is false}\,\}.$$

Then define c_n and d_n such that $c_n \leq d_n$, $[c_n, d_n] \subseteq [c_{n-1}, d_{n-1}]$, $\psi(c_n) = \psi(d_n)$, P$([c_n, d_n])$ is false and $d_n - c_n < \alpha_n + 1/n$ (notice that the existence of α_n is ensured by induction). Let $\bar{c} = \lim c_n$ and $\bar{d} = \lim d_n$. Notice that c_n tends increasingly to $\bar{c}$ and d_n tends decreasingly to $\bar{d}$. By the continuity of ψ, $\psi(\bar{c}) = \psi(\bar{d})$. We now consider several cases:
case 1: $\bar{c} = \bar{d}$; let $n \geq 1$ such that $[c_n, d_n] \subseteq]\bar{c} - \delta'(\bar{c}), \bar{d} + \delta'(\bar{d})[$. Then the division $\{([c_n, d_n], \bar{c})\}$ is a division of $[c_n, d_n]$ that fulfills condition (P), a contradiction.
case 2: $\bar{c} < \bar{d}$ and P$([\bar{c}, \bar{d}])$ is true; we can choose $n \geq 1$ such that $[c_n, d_n] \subseteq]\bar{c} - \delta'(\bar{c}), \bar{d} + \delta'(\bar{d})[$, and a division D of $[\bar{c}, \bar{d}]$ that fulfills condition (P). Then the division

$$\{\, ([c_n, \bar{c}], \bar{c}), ([\bar{d}, d_n], \bar{d}) \,\} \cup D$$

is a division of $[c_n, d_n]$ that fulfills the conditions of (P), a contradiction.
case 3: $\bar{c} < \bar{d}$ and P$([\bar{c}, \bar{d}])$ is false; we have to examine two subcases:

first subcase: for every $\varepsilon > 0$, there exists $s > \bar{c}$ and $t < \bar{d}$ such that $s - \bar{c} < \varepsilon$, $\bar{d} - t < \varepsilon$, and $\psi(s) = \psi(t)$.

Choose such an s and t in $]\bar{c}, \bar{c} + \delta'(\bar{c})[$ and $]\bar{d} - \delta'(\bar{b}), \bar{d}[$ resp. In particular, $([\bar{c}, s], \bar{c}) << \delta$ and $([t, \bar{d}]) << \delta$ (recall that $\bar{d}(x,y) < |x-y|$)). Then P([s,t]) is false, else we could choose a division D of $[s,t]$ such that P$([s,t])$ is true, and the division

$$\{\, ([\bar{c}, s], \bar{c}), ([t, \bar{d}], \bar{d}) \,\} \cup D$$

would be a division of $[\bar{c}, \bar{d}]$ that fulfills condition (P), a contradiction. Hence, P$([s,t])$ is false. But this also leads to a contradiction with the definition of $\bar{c}$ and $\bar{d}$: indeed, put $\varepsilon = s - \bar{c} + \bar{d} - t\,(>0)$. Let n be such that $1/n < \varepsilon$. Then

$$t - s = \bar{d} - \bar{c} - \varepsilon < d_n - c_n - \varepsilon < d_n - c_n - \frac{1}{n} < \alpha_{n+1}.$$

This contradicts the fact that

$$\alpha_{n+1} = \inf\{\, |x-y| \colon x, y \in [c_n, d_n],\ \psi(x) = \psi(y),\ \mathrm{P}([x,y]) \text{ is false}\,\}.$$

second subcase: there exists $\varepsilon > 0$ such that, for every $s > \bar{c}$ and $t < \bar{d}$ that fulfills $s - \bar{c} < \varepsilon$ and $\bar{d} - t < \varepsilon$, $\psi(s) \neq \psi(t)$ (notice that this is the exact converse of the first subcase).

From the intermediate value theorem, there can not exist $s > \bar{c}$ and $t < \bar{d}$ that fulfill $|\bar{c} - s| < \varepsilon$ and $|\bar{d} - t| < \varepsilon$, and such that ($\psi(s) \geq \psi(\bar{c})$ and $\psi(t) \geq \psi(\bar{d})$) or ($\psi(s) < \psi(\bar{c})$ and $\psi(t) < \psi(\bar{d})$), else we could choose suitable s' and t' in $]\bar{c}, s]$ and $[t, \bar{d}[$ such that $\psi(s') = \psi(t')$. Hence, for all $s > \bar{c}$ and $t < \bar{d}$ such that $|\bar{c} - s| < \varepsilon$ and $|t - \bar{d}| < \varepsilon$, ($\psi(s) \geq \psi(\bar{c})$ and $\psi(t) < \psi(\bar{d})$), or ($\psi(s) < \psi(\bar{c})$ and $\psi(t) \geq \psi(\bar{d})$). Suppose for example that $\psi(s) \geq \psi(\bar{c})$ for all $s \in [\bar{c}, \bar{c} + \varepsilon[$, the end of the proof being similar in the other case. Then $\psi(t) < \psi(\bar{d}) = \psi(\bar{c})$ by what has been just said; hence there exists $e \in]\bar{c}, t[$ such that $\psi(e) = \psi(\bar{c}) = \psi(\bar{d})$ (intermediate value theorem again). If $\mathrm{P}([\bar{c}, e])$ and $\mathrm{P}([e, \bar{d}])$ were true, then concatenating the corresponding divisions, $\mathrm{P}([\bar{c}, \bar{d}])$ would also be true, in contradiction with the hypothesis. Hence $\mathrm{P}([\bar{c}, e])$ or $\mathrm{P}([e, \bar{d}])$ is false; but this leads again to a contradiction because $|\bar{c} - e| < |\bar{c} - \bar{d}|$ and $|e - \bar{d}| < |\bar{c} - \bar{d}|$, and we can use the same argument as that at the end of the first subcase. □

REDUCTION. *we can suppose without loss of generality that* $[c, d]$ *is finite,* $\psi(c) = a$, *and* $\psi(d) = b$.

PROOF. Assume the theorem has been proved under the conditions of the reduction. If ψ is any continuous function $[c, d] \to [a, b]$, and $f \circ \psi \cdot d(\varphi \circ \psi)$ is integrable in $[c, d]$, we have to prove that $f \cdot d\varphi$ is integrable in $\psi([c, d])$, and fulfills the change of variable formula. Nothing is lost if we suppose that $[a, b] = \psi([c, d])$. By the continuity of ψ, there exists γ and δ in $[c, d]$ such that $\psi(\gamma) = a$ and $\psi(\delta) = b$; thus, the reduction hypothesis implies that $f \cdot d\varphi$ is integrable in $[a, b]$, and if $f \circ \psi \cdot d(\varphi \circ \psi)$ is absolutely integrable, then so is $f \cdot d\varphi$. Thus, the only thing we have to prove is that

$$\int_{\psi(c)}^{\psi(d)} f \cdot \mathrm{d}\varphi = \int_c^d f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi). \tag{a}$$

Let ρ be the homeomorphism $[-\infty, +\infty] \to [-1, 1]$ defined in Sec. 2.1. Then $\rho([c, d])$ is bounded, say $\rho([c, d]) = [c', d']$. Now choose $-\infty < c'' < c'$ and $d' < d'' < +\infty$. Define a function $\nu \colon [c'', d''] \to [a, b]$ by

$$\nu(x) = \begin{cases} \psi \circ \rho^{-1}(x), & x \in [c', d']; \\ a, & x = c''; \\ b, & x = d''; \\ \nu(x) \text{ linear}, & x \in [c'', c'] \text{ and } x \in [d', d'']. \end{cases}$$

Of course, ν is a continuous function that fulfills the conditions of the reduction, that is, $[c'', d'']$ is bounded and $[\nu(c''), \nu(d'')] = \nu([c'', d'']) = [a, b]$.

Furthermore Lemma 6.4 implies that $f \circ \nu \cdot \mathrm{d}\varphi \circ \nu$ is integrable in $[c'', d'']$ because 1) $f \cdot \mathrm{d}\varphi$ is integrable in $[a, b]$ and ν is continuous and monotonic on each of the intervals $[c'', c']$ and $[d', d'']$, and 2) $f \circ \psi \cdot \mathrm{d}\varphi \circ \psi$ is integrable in $[c, d]$ and ρ^{-1} is homeomorphic $[c', d'] \to [c, d]$. Dealing with $f \circ \nu \cdot \mathrm{d}\varphi \circ \nu$ and making use of the reduction hypothesis, it is now not difficult to prove (a). $\square$

Now, let ψ satisfy the conditions of the reduction, and $\hat{\psi}$ be the upper invert of ψ (see definition before Lemma 6.3). Let $\varepsilon > 0$. We define $\delta\colon [a, b] \to \mathbb{R}_+$ in the following way: Let $\delta'\colon [c, d] \to \mathbb{R}_+$ be such that for every division $D' << \delta'$ of $[c, d]$,

$$\sum_{([u,v],\xi)\in D'} \left\| \int_u^v f \circ \psi \cdot \mathrm{d}\varphi \circ \psi - f(\varphi(x)) \cdot (\psi(\varphi(v)) - \psi(\varphi(u))) \right\| < \varepsilon.$$

Because of the continuity from the left of $\hat{\psi}$ in $]a, b]$, there exists for all $x \in]a, b]$ a real $\eta^-(x)$ such that $y < x$ and $x - y < \eta^-(x)$ imply

$$|\hat{\psi}(y) - \hat{\psi}(x)| < \delta'(\hat{\psi}(x)).$$

Define also $\eta^-(a) > 0$ in any manner. Similarly, for all $x \in [a, b[$, there exists $\eta^+ > 0$ such that $y > x$ and $y - x < \eta^+$ imply $|\hat{\psi}^+(y) - \hat{\psi}^+(x)| < \delta'(\hat{\psi}^+(x))$. We define $\eta^+(b) > 0$ in any manner, and put

$$\delta(x) = \min(\eta^+(x), \eta^-(x)).$$

Diminish δ in such a way that every division $D << \delta$ anchor in $\{a, b\}$. Let D be a division of $[a, b]$ subordinated to δ. Since D anchors in $\{a, b\}$, the tagged interval of D that contains the point a as its end point is of the form $([a, v_a], a)$ where $v_a > a$, and the tagged interval of D that contains the point b as its end point is of the form $([u_b, b], b)$ where $u_b < a$.

We shall match to D a special division D' of $[c, d]$ subordinated to δ': to each $([u, v], \xi) \in D$ such that $v > \xi$, attach the partial division

$$D'_\xi = \{\, ([\hat{\psi}(u), \hat{\psi}(\xi)], \hat{\psi}(\xi)), ([\hat{\psi}^+(\xi), \hat{\psi}(v)], \hat{\psi}^+(\xi)) \,\}$$

of $[c, d]$. Furthermore, if $\hat{\psi}(\xi) < \hat{\psi}^+(\xi)$, insert inside D'_ξ a division of the interval $[\hat{\psi}(\xi), \hat{\psi}^+(\xi)]$ subordinated to δ' that fulfills the assertion of Lemma 6.5 (recall that $\psi(\hat{\psi}(\xi)) = \xi = \psi(\hat{\psi}^+(\xi))$). In particular, D'_a is a total division of $[c = \hat{\psi}(a), \hat{\psi}(v_a)]$. Next, to each $([u, v], \xi) \in D$ such that $v = \xi$, attach the partial division

$$D'_\xi = \{\, ([\hat{\psi}(u), \hat{\psi}(\xi)], \hat{\psi}(\xi)) \,\}$$

of $[c,d]$. In particular, D'_b is of the form $([\hat{\psi}(u_b),\hat{\psi}(b)],\hat{\psi}(b))$. If $\hat{\psi}(b)<\hat{\psi}^+(b)=d$, we concatenate to D'_b a division of $[\hat{\psi}(b),d]$ that fulfills the assertion of Lemma 6.5. In such a way, we obtain a total division $D'=\bigcup_\xi D'_\xi$ of $[c,d]$ subordinated to δ'. It is easily seen that

$$\sum_{([u,v],\xi)\in D}\Big\| f(\xi)\cdot(\varphi(v)-\varphi(u))-\sum_{([u',v'],\xi')\in D'_\xi} f(\psi(\xi'))\cdot\big(\varphi(\psi(v')-\varphi(\psi(u'))\Big\|=0.$$

Since $D'<<\delta'$, the Henstock-Saks lemma gives

$$\sum_{([u,v],\xi)\in D}\Big\|\sum_{([u',v'],\xi')\in D'_\xi} f(\psi(\xi'))\cdot(\varphi(\psi(v')-\varphi(\psi(u'))-\int_{u'}^{v'} f\circ\psi\cdot\mathrm{d}\varphi\circ\psi\Big\|<\varepsilon.$$

Denote by D^* the partial division $D\smallsetminus\{([u_b,b],b)\}$. Then it follows from the two previous inequalities that

$$\sum_{([u,v],\xi)\in D^*}\Big\| f(\xi)\cdot(\varphi(v)-\varphi(u))-\int_{\hat{\psi}(u)}^{\hat{\psi}(v)} f\circ\psi\cdot\mathrm{d}\varphi\circ\psi\Big\|$$
$$+\Big\| f(b)\cdot(\varphi(u_b)-\varphi(b))-\int_{\hat{\psi}(u_b)}^{d} f\circ\psi\cdot\mathrm{d}\varphi\circ\psi\Big\|<\varepsilon.$$

Taking into account that $\hat{\psi}(a)=c$, this shows first that f is K-integrable in $[a,b]$, with integral equal to $\int_c^d f\circ\psi\cdot\mathrm{d}\varphi\circ\psi$, and then that it is Henstock integrable, because $(K)\ \int_u^v f\cdot\mathrm{d}\varphi=(K)\ \int_{\hat{\psi}(u)}^{\hat{\psi}(v)} f\circ\psi\cdot\mathrm{d}\varphi\circ\psi$ by what we have just shown, and $(K)\ \int_{u_b}^{b} f\cdot\mathrm{d}\varphi=(K)\ \int_{\hat{\psi}(u_b)}^{d} f\circ\psi\cdot\mathrm{d}\varphi\circ\psi$ since $\psi(\hat{\psi}(b)=b$ and $\psi(d)=b$.

Finally, let us show the last assertion of the theorem. Put $G(x)=\int_c^x f\circ\psi\cdot\mathrm{d}(\varphi\circ\psi)$ and $F(x)=\int_a^x f\cdot\mathrm{d}\varphi$. Since $f\circ\psi\cdot d(\varphi\circ\psi)$ is absolutely integrable, $G=F\circ\psi$ is of bounded variation in $[c',d']$, say $\mathrm{Var}_c^d(G)=M<\infty$ (Thm. 4.2). If F were not of bounded variation in $[a,b]$, there would exist a finite set π of non-overlapping closed intervals of $[a,b]$ such that

$$\sum_{[u,v]\in\pi}\|F(v)-F(u)\|>M.$$

We can suppose w.l.g. that the intervals of π cover $[a,b]$. Denote by $a=y_0<y_1<\cdots<y_n=b$ their end-points in the increasing order. Define

$$x_0=\hat{\psi}(y_0)=a<x_1=\hat{\psi}(y_1)<\cdots<x_n=\hat{\psi}(y_n)=b;$$

in particular, $\psi(x_i) = y_i$ for all $0 \le i \le n$. Then

$$\sum_{[u,v]\in\pi} \|F(v) - F(u)\| = \sum_{i=0}^{n-1} \|F \circ \psi(x_{i+1}) - F \circ \psi(x_i)\| = \sum_{i=0}^{n-1} \|G(x_{i+1}) - G(x_i)\|$$

by the first assertion of the theorem again. Therefore, $\mathrm{Var}_c^d(G) > M$, a contradiction. Thus, F is of bounded variation in $[a,b]$. It follows from Thm. 4.2 that $f \cdot d\varphi$ is absolutely integrable. □

Corollary 6.6. *Let $\psi\colon [c,d] \to [a,b]$ be the indefinite Henstock integral of a function ψ', and assume that $f \circ \psi\, \psi' dx$ is integrable in $[c,d]$. Then $f\, dx$ is integrable in $[a,b]$ and fulfills*

$$\int_a^b f\, dx = \int_c^d f \circ \psi\, \psi' dx.$$

Proof. This follows immediately from Thms. 6.1 and 5.1. □

7 Change of Variable Theorems, Second Category.

Now, let us investigate the converse of Thm. 6.1. The more general question one can ask is:

Problem 7.1. *Let $f \cdot d\varphi$ be K-integrable (resp. H-integrable) in $[a,b]$, and $\psi\colon [c,d] \to [a,b]$ be continuous. To prove or to disprove:*

The Kurzweil (resp. Henstock) integral $\int_c^d f \circ \psi \cdot \mathrm{d}\varphi \circ \psi$ exists and is equal to $\int_{\psi(c)}^{\psi(d)} f \cdot \mathrm{d}\varphi$.

For the moment, we have enough tools to show the following proposition.

Proposition 7.2. *Let $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ and $\varphi\colon [a,b] \to \mathcal{F}$ such that $f \cdot d\varphi$ be integrable in $[a,b]$. Let $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$ be continuous (relative to $\bar{d}$) and surjective. Put $F(x) = \int_{\psi(c)}^x f \cdot \mathrm{d}\varphi$. If there exists an element $g \cdot \mathrm{d}\varphi\colon [a,b] \to \mathcal{G}$ such that $g \circ \psi \cdot d(\varphi \circ \psi)$ is integrable in $[c,d]$, with $F \circ \psi(x) = \int_c^x g \circ \psi \cdot \mathrm{d}(\varphi \circ \psi)$, then $g \cdot d\varphi$ is integrable in $[a,b]$, and $F(x) = \int_{\psi(c)}^x g \cdot \mathrm{d}\varphi$. In particular,*

(i) if for every $\varepsilon > 0$, there exists a countable cover of $\mathcal{E}$ by balls of radius less than ε, and if the product between $\mathcal{E}$ and $\mathcal{F}$ is norm-preserving, then $f = g$ $d\varphi$-a.e., hence $f \circ \psi \cdot d(\varphi \circ \psi)$ is integrable in $[c,d]$, with $\int_c^x f \circ \psi \cdot \mathrm{d}(\varphi \circ \psi) = F \circ \psi(x)$.

(ii) if $\varphi = x$ and $f dx$ is H-integrable in $[a,b]$, then $f = g$ a.e., hence again, $f \circ \psi \cdot \mathrm{d}\psi$ is integrable in $[c,d]$, with $\int_c^x f \circ \psi \, \mathrm{d}\psi = F \circ \psi(x)$.

PROOF. By Thm. 6.1, we know that $g \cdot d\varphi$ is integrable in $[a,b]$. Put $G(x) = \int_{\psi(c)}^x g \cdot \mathrm{d}\varphi$. Then $G \circ \psi(x) = F \circ \psi(x)$ by Thm. 6.1 again. Since ψ is surjective, $F(x) = G(x)$ for all $x \in [a,b]$. This proves the first assertion. To prove (i), remark that this implies $\int_{\psi(c)}^x (f-g) \cdot d\varphi = 0$ for all x. Hence, $f = g$ $d\varphi$-a.e. Conclude by Thm. 5.2. The proof of (ii) follows from Thm. 5.3, F and G being differentiable a.e., with $F' = f = G' = g$ a.e. □

From (ii), if F is an indefinite H-integral and ψ is continuous and surjective, we see that $F \circ \psi$ can be an indefinite integral only if $f \circ \psi \, d\psi$ is integrable in $[c,d]$. So, either $F \circ \psi$ is not an indefinite H-integral, or the change of variable formula applies.

It is possible to produce another version of (ii), where ψ is not necessarily continuous, using existing theorems. This is the object of the following theorem, that is a Kurzweil-Henstock version of Thm. (6.95) in [8], due to J. Serrin and D. E. Varberg in the Lebesgue case.

Theorem 7.3. *Let $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ and $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$ such that $f\,dx$ be H-integrable in $[a,b]$, and ψ be differentiable a.e. in $[c,d]$. Put $F(x) = \int_a^x f \, \mathrm{d}x$. Then if $F \circ \psi$ is continuous, differentiable a.e., and ≈ 0 on every negligible set, $f \circ \psi \, \psi' \, dx$ is integrable in $[c,d]$, with $\int_c^d f \circ \psi \, \psi' \mathrm{d}x = F \circ \psi(d) - F \circ \psi(c)$.*

PROOF. This follows from Thm. (6.93) of [8], Thm. 5.3, and the well known fact that an indefinite Henstock integral maps negligible sets to negligible sets. □

The next theorem is perhaps the most general and useful second category theorem we offer. To state it, we need some definitions.

We say that a function $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$ is *increasing* at a point x, if

$$\begin{cases} \psi(x+h) \geq \psi(x), & h \geq 0, \\ \psi(x+h) \leq \psi(x), & h \leq 0 \end{cases}$$

whenever $x+h$ is in the domain of ψ. We define similarly the notion of being *decreasing* at x. Finally, we say that ψ is *decomposable* at x if ψ is either increasing at x, or decreasing at x, or admits a STRICT maximum at x, or admits a STRICT minimum at x. Also, ψ is increasing (decreasing, decomposable) at x *in* an interval $I \ni x$ if $\psi|_I$ is, and ψ is *locally* increasing (decreasing, decomposable) at x if there exits an open interval $I \ni x$ of $[c,d]$ in which ψ is decomposable at x. For example, if ψ has an order n non-zero derivative at x, then ψ is locally decomposable at x by the Taylor formula. Also, if ψ is constant in a neighborhood of x, then ψ is locally increasing, hence locally decomposable at x.

Lemma 7.4. *Let $f\colon [a,b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ and $\varphi\colon [a,b] \to \mathcal{F}$. Assume that $f \cdot \mathrm{d}\varphi$ is H-integrable (resp. K-integrable) in $[a,b]$, with $F(x) = \int_a^x f \cdot \mathrm{d}\varphi$. Let $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$ be continuous (with respect to $\bar{d}$).*

(i) $f \circ \psi \cdot d\varphi \circ \psi$ is variationally equivalent to $d(F \circ \psi)$ on every countable set $S \in [a,b]$.

(ii) $f \circ \psi \cdot d\varphi \circ \psi$ is (resp. weakly) variationally equivalent to $d(F \circ \psi)$ on the set of points where ψ is locally decomposable.

Proof. We make the proof for H-integrals, the argument being the same for K-integrals, upon replacing variational equivalence by weak variational equivalence.
(i) It suffices to show that $f \circ \psi \cdot d\varphi \circ \psi \approx d(F \circ \psi)$ on every point $s \in S$ (Prop. 3.2). If D is a tame division that anchors in $\{s\}$, D is of the form $\{([u,v],s)\}$ where $u = s$ or $v = s$. By the Henstock-Saks lemma, $dF \approx f \cdot d\varphi$. In particular, for every $\varepsilon > 0$, there exists δ such that

$$\|F(\psi(v)) - F(\psi(u)) - f(\psi(s)) \cdot (\varphi(\psi(v)) - \varphi(\psi(u)))\| < \varepsilon \qquad (a)$$

whenever $\bar{d}(\psi(v), \psi(u)) < \delta(\psi(s))$. But by the continuity of ψ, it is always possible to find δ' such that $\bar{d}(u,v) < \delta'(s)$ implies $\bar{d}(\psi(u), \psi(v)) < \delta(s)$. This shows that (a) holds whenever $\bar{d}(u,v) < \delta'(s)$.
(ii) By (i), $d(F \circ \psi) \approx f \circ \psi \cdot d\varphi \circ \psi$ on $\{c\}$ and $\{d\}$, hence we can suppose w.l.g. that $[c,d]$ is finite (Prop. 3.2 (ii)). Let A be the set of points where ψ is locally decomposable. For each $x \in A$, there exists $1 \geq \alpha_x > 0$ such that ψ is decomposable at x in $[x - \alpha_x, x + \alpha_x] \cap [c,d]$. For $n \in \mathbb{N}^*$, define $A_n \subseteq A$ by

$$A_n = \{\, x \in A\colon \alpha_x \in \left]\tfrac{1}{n+1}, \tfrac{1}{n}\right] \}$$

Then A is the countable union of the sets A_n, $n \in \mathbb{N}^*$. So, it suffices to prove that $d(F \circ \psi) \approx f \circ \psi \cdot d\varphi \circ \psi$ on A_n for every $n \in \mathbb{N}^*$ (Prop. 3.2). Fix such an n to suppose w.l.g. that $A = A_n$. In particular, $\alpha_x = \alpha$ is now the same for all $x \in A$.

By the Henstock-Saks lemma, there exists δ such that every division $D << \delta$ that anchors in $\psi(A)$ fulfills

$$\sum_{([u,v],\xi)\in D} \|F(v) - F(u) - f(\xi)\cdot(\varphi(v) - \varphi(u))\| < \frac{\varepsilon}{2}. \tag{b}$$

From the continuity of ψ, we can define $\delta' \colon [c,d] \to \mathbb{R}_+$ such that $\bar{d}(x,y) < \delta'(x)$ implies $\bar{d}(\psi(x),\psi(y)) < \delta(x)$. Now, let A_1 be the set of all $x \in A$ such that ψ is increasing at x, A_2 the set of all $x \in A$ such that ψ is decreasing at x, A_3 the set of all x that are strict local minimum of ψ, and A_4 the set of all x that are strict local maximum of ψ. Then $A = A_1 \cup A_2 \cup A_3 \cup A_4$, so, it suffices to prove that $d(F\circ\psi) \approx f\circ\psi\cdot d\varphi\circ\psi$ on these four sets. The proof for A_3 and A_4 is easy because the sets A_3 and A_4 are countable (in fact finite): indeed, each point ξ of A_3 (resp. A_4) is the (unique) strict minimum (resp. maximum) of ψ in $]\xi-\alpha,\xi+\alpha[$; so, the points of A_3 (resp. A_4) are separated by a length of at least α, showing that A_3 (resp. A_4) is countable. Since $A_3 \cup A_4$ is countable, $d(F\circ\psi) \approx f\circ\psi\cdot d\varphi\circ\psi$ on $A_3 \cup A_4$ by (i). Now, let us prove that $d(F\circ\psi) \approx f\circ\psi\cdot d\varphi\circ\psi$ on A_1, the proof being similar for A_2. Each point x of A_1 fulfills

$$\begin{cases} \psi(x+h) \geq \psi(x), & h \geq 0, \\ \psi(x+h) \leq \psi(x), & h \leq 0 \end{cases} \tag{c}$$

for all $0 \leq h \leq \alpha$. Divide the interval $[c,d]$ into a finite number of interval I_k of equal length, but less than $\alpha/2$. It suffices to prove that $d(F\circ\psi) \approx f\circ\psi\cdot d\varphi\circ\psi$ on $A_1 \cap I_k$ (Prop. 3.2). Fix $k \in \mathbb{N}$. For each point $x \in A_1 \cap I_k$, (c) surely obtains with $h < \alpha/2$. Diminish δ' at each x in such a way that $\delta'(x) < \alpha/2$. Let $D' << \delta'$ be a partial division of $[c,d]$ anchoring in $A_1 \cap I_k$. Order the tagged intervals of D' in the increasing order, say

$$([u_1,v_1],\xi_1),([u_2,v_2],\xi_2),\ldots,([u_s,v_s],\xi_s).$$

Consider two tagged intervals of D'

$$([u_i,v_i],\xi_i) \quad \text{and} \quad ([u_{i+j},v_{i+j}],\xi_{i+j}), \quad j \geq 1.$$

From (c), it follows that

$$\psi(u_i) \leq \psi(\xi_i) \leq \psi(v_1), \psi(u_{i+j}), \psi(\xi_{i+j}), \psi(v_{i+j}),$$

and also

$$\psi(v_{i+j}) \geq \psi(\xi_{i+j}) \geq \psi(u_{i+j}), \psi(v_i), \psi(u_i).$$

Thus,

$$([\psi(u_i), \psi(v_i)], \psi(\xi_i)) \quad \text{and} \quad ([\psi(u_{i+j}), \psi(v_{i+j})], \psi(\xi_{i+j}))$$

are both tagged intervals, that overlap at most on their half $[\psi(\xi_i), \psi(v_i)]$ and $[\psi(u_{i+j}), \psi(\xi_{i+j})]$ resp. Therefore, it is clear that the tagged intervals

$$([\psi(u_1), \psi(v_1)], \psi(\xi_1)), ([\psi(u_3), \psi(v_3)], \psi(\xi_3)), ([\psi(u_5), \psi(v_5)], \psi(\xi_5)), \ldots$$

are non-overlapping, and so are

$$([\psi(u_2), \psi(v_2)], \psi(\xi_2)), ([\psi(u_4), \psi(v_4)], \psi(\xi_4)), ([\psi(u_6), \psi(v_6)], \psi(\xi_6)), \ldots.$$

Theses two lists of tagged intervals give rise partial divisions

$$D_1' = \{\, ([\psi(u_1), \psi(v_1)], \psi(\xi_1)), ([\psi(u_3), \psi(v_3)], \psi(\xi_3))], \ldots \}$$

and

$$D_2' = \{\, ([\psi(u_2), \psi(v_2)], \psi(\xi_2)), ([\psi(u_4), \psi(v_4)], \psi(\xi_4)), \ldots \}$$

of $[a, b]$, that anchor in A_1, and subordinated to δ. By (b),

$$\sum_{i \in \{1,2\}} \sum_{([u,v],\xi) \in D_i'} \| F \circ \psi(v) - F \circ \psi(u) - f \circ \psi(\xi) \cdot (\varphi \circ \psi(v) - \varphi \circ \psi(u)) \| < \varepsilon,$$

or

$$\sum_{([u,v],\xi) \in D'} \| F \circ \psi(v) - F \circ \psi(u) - f \circ (\xi) \cdot (\varphi \circ \psi(v) - \varphi \circ \psi(u)) \| < \varepsilon,$$

showing that $d(F \circ \psi) \approx f \circ \psi \cdot d\varphi \circ \psi$ on $A_1 \cap I_k$, as desired. $\square$

Most of the continuous functions that occur in practice are decomposable at all but a countable number of points, and it is in fact not so easy to produce a function that is not (such examples can be constructed with the Cantor set). So, the promised following theorem seems to be useful in practice.

Theorem 7.5. *Let $f\colon [a, b] \subseteq \bar{\mathbb{R}} \to \mathcal{E}$ and $\varphi\colon [a, b] \to \mathcal{F}$. Assume that $\psi\colon [c, d] \subseteq \bar{\mathbb{R}} \to [a, b]$ is continuous, and locally decomposable at all but a countable number of points. Then $f \cdot d\varphi$ is integrable in $[a, b]$ if and only if $f \circ \psi \cdot d\varphi \circ \psi$ is integrable in $[c, d]$, and the substitution formula holds:*

$$\int_c^d f \circ \psi \cdot \mathrm{d}\varphi \circ \psi = \int_{\psi(c)}^{\psi(d)} f \cdot \mathrm{d}\varphi.$$

PROOF. The "if" part of the theorem follows from Thm. 6.1. The "only if" part follows from Lemma 7.4 (i) and (ii). □

The next problem appears to be true in the context of the Riemann integral, (this was first proved in [1], and a simpler and more general proof was given in [4]). But in the context of the KH integral, it remains open.

Problem 7.6. *Let $f\colon [a,b] \to \mathbb{R}$ be H-integrable, and $\psi'\colon [c,d] \to [a,b]$ be H-integrable, with $\psi(x) = \int_c^x \psi'(t)\,\mathrm{d}t$. To prove or to disprove: $f \circ \psi\,\mathrm{d}\psi$ is H-integrable in $[c,d]$, and $\int_c^d f \circ \psi\,\mathrm{d}\psi = \int_{\psi(c)}^{\psi(d)} f\,\mathrm{d}x$.*

Of course, a counterexample to this problem would be a counterexample to Problem 7.1. Thm. 7.7 below reflects our attempts to answer as well as possible Problem 7.6. It generalizes some known theorems of the Lebesgue theory. We need one more definition.

We say that a function $F\colon [a,b] \to \mathcal{E}$ is *Lipschitz at x* if for every $[y,z] \ni x$ sufficiently small, there exists $k_x \geq 0$ such that

$$\|F(y) - F(z)\| \leq k_x \|y - z\|.$$

In particular, if F is differentiable at x, then it is Lipschitz at x. Also, if F is the integral of a function f locally almost everywhere bounded, then F is Lipschitz at every $x \in [a,b]$.

Theorem 7.7. *Let $f\colon [a,b] \subseteq \mathbb{R} \to \mathcal{E}$ be Henstock integrable, with $F(x) = \int_a^x f(t)\,\mathrm{d}t$, and $\psi\colon [c,d] \subseteq \overline{\mathbb{R}} \to [a,b]$ be an indefinite H-integral in $[c,d]$. Let S be the (negligible) set of points $x \in [a,b]$ such that F is not Lipschitz at x, T be the set of points $x \in [c,d]$ such that ψ is locally decomposable at x, and T' be the set of points $x \in [c,d]$ such that $F \circ \psi$ is differentiable at x with $(F \circ \psi)'(x) = 0$.*

(i) The set $\psi^{-1}(S) \smallsetminus (T \cup T')$ is negligible

(ii) In order that $f \circ \psi\, d\psi$ be H-integrable in $[c,d]$, it is necessary and sufficient that $F \circ \psi \approx 0$ on $\psi^{-1}(S) \smallsetminus (T \cup T')$. When this occurs, $\int_c^d f \circ \psi\,\mathrm{d}\psi = \int_{\psi(c)}^{\psi(d)} f\,\mathrm{d}x$.

(iii) The same conclusions holds if in place of assuming that ψ is an indefinite H-integral in $[c,d]$, we assume that ψ is of bounded variation in $[c,d]$.

(iv) If $\psi^{-1}(S) \smallsetminus (T \cup T')$ is countable, then $d(F \circ \psi) \approx 0$ on this set, and assertions (i) and (ii) hold.

Proof. (i) By Thm. 5.4, and taking into account that indefinite integrals are N-functions, $\psi^{-1}(S)\setminus(T \cup T')$ is negligible because 1) its image by $F \circ \psi$ is negligible (since S is negligible), and 2) $F \circ \psi$ is differentiable a.e. by Thm. 5.3 and Thm. 6.93 of [8].
(ii) Let S_2 be the set of points where F is not differentiable, and T_2 be the set of points where ψ is not differentiable; then S_2 and T_2 are negligible by Thm. 5.3, and $S \subseteq S_2$. Put $S_1 = [a, b]\setminus S_2$ and $T_1 = [c, d]\setminus T_2$. Let ψ' be the derivative of ψ at each point of T_1, and define ψ' in any manner at each point of T_2. Consider the three sets: $M_{11} = \psi^{-1}(S_1) \cap T_1$, $M_{12} = \psi^{-1}(S_1) \cap T_2$, and $M_2 = \psi^{-1}(S_2)$ (so, $[c, d] = M_{11} \cup M_{12} \cup M_2$). Of course, $F \circ \psi$ is continuous because F and ψ are. We first examine variational equivalence between $f \circ \psi\, d\psi$ and $d(F \circ \psi)$ on each of these sets. Notice that $f \circ \psi\, d\psi \approx 0$ on $\{c, d\}$ in $[c, d]$, because of the continuity of ψ, and the same is true for $d(F \circ \psi)$ since it is continuous. Hence, by Lemma 3.2 (ii), we can suppose w.l.g. that $[c, d]$ is finite. Let $\varepsilon > 0$.
Claim 1: *$f \circ \psi\, d\psi \approx d(F \circ \psi)$ on M_{11}.*

By Prop. 3.4, $d\psi \approx \psi' dx$ on M_{11}, hence $f \circ \psi\, d\psi \approx f \circ \psi\, \psi' dx$ on M_{11} (see Prop. 3.3 (iv)). On the other hand, $d(F \circ \psi) \approx F \circ \psi\, \psi' dx$ on M_{11} by Prop. 3.4 again. Hence $f \circ \psi\, d\psi \approx d(F \circ \psi)$ on M_{11}.
Claim 2: *$f \circ \psi\, d\psi \approx d(F \circ \psi)$ on M_{12}.*

Since T_2 is negligible, $d\psi \approx 0 \approx f \circ \psi\, d\psi$ on M_{12} (Thm. 3.3 (xii) and 5.3). Let us show that $d(F \circ \psi)$ is also variationally equivalent to 0 on M_{12}: indeed, for each $x \in M_{12}$, there exists, by the continuity of ψ and the differentiability of F at $\psi(x)$, $\delta(x) > 0$ such that $\bar{d}(x, y) < \delta(x)$ imply

$$\|F(\psi(y)) - F(\psi(x))\| \leq \|f(\psi(x))(\psi(y) - \psi(x))\| + |\psi(y) - \psi(x)|.$$

In other words,

$$\|d(F \circ \psi)\| \leq \|F \circ \psi\, d\psi\| + |d\psi|.$$

By Prop. 3.3 (ix) and (x), this shows that $d(F \circ \psi) \approx 0$ on M_{12}.
Claim 3: *$f \circ \psi\, d\psi \approx 0$ on M_2*

It suffices to show that $d\psi \approx 0$ on $M_2 = \psi^{-1}(S_2)$. But this is the content of Cor. 5.5, since S_2 is negligible by Thm. 5.3.
Claim 4: *$d(F \circ \psi) \approx 0$ on the set $M_2\setminus\psi^{-1}(S)$.*

For every $x \notin S$, there exists k_x such that $[y, z] \ni x$ and $|z - y|$ sufficiently small imply

$$\|F(y) - F(z)\| \leq k_x |y - z|.$$

From the continuity of ψ, there exists $\delta\colon [c, d] \to \mathbb{R}_+$ such that $|x - y| < \delta(x)$ imply

$$\|F \circ \psi(y) - F \circ \psi(x)\| < k_{\psi(x)} |\psi(y) - \psi(x)|.$$

Since $d\psi \approx 0$ on $\psi^{-1}(S)$ (Cor. 5.5), so is the element $k_{\psi(x)} d\psi$ (Prop. 3.3 (iv)). Therefore $d(F \circ \psi) \approx 0$ on $\psi^{-1}(S)$ (Prop. 3.3 (x)).
CLAIM 5: *$d(F \circ \psi) \approx 0$ on $M_2 \cap T$.*

This follows immediately from Claim 3 and Lemma 7.4.
CLAIM 6: *$d(F \circ \psi) \approx 0$ on T'*

This follows from Prop. 3.4 since $F \circ \psi$ has a null derivative at each point of T'.

Let us sum up our results: we have shown that $F \circ \psi \approx f \circ \psi\, d\psi$ on M_{11} and M_{12}, that $f \circ \psi\, \mathrm{d}\psi \approx 0$ on M_2, and that $d(F \circ \psi) \approx 0$ on $M_2 \smallsetminus \psi^{-1}(S)$, on $M_2 \cap T$ and on T'. Since $\psi^{-1}(S) \subseteq M_2$, $d(F \circ \psi) \approx 0$ at each point of $[c, d]$ that do not belong to $\psi^{-1}(S) \smallsetminus (T \cup T')$. Thus, if $d(F \circ \psi)$ is variationally equivalent to 0 on $\psi^{-1}(S) \smallsetminus (T \cup T')$, it is variationally equivalent to $f \circ \psi\, d\psi$ in $[c, d]$. Hence $f \circ \psi\, d\psi$ is H-integrable, with

$$\int_c^d f \circ \psi\, \mathrm{d}\psi = F \circ \psi(d) - F \circ \psi(c).$$

Conversely, if $f \circ \psi\, d\psi$ is integrable, then by Thm. 6.1,

$$\int_c^x f \circ \psi\, \mathrm{d}\psi = F(\psi(x)) - F(\psi(c));$$

Therefore $d(F \circ \psi)$ is variationally equivalent to $f \circ \psi\, d\psi$ on M_2, hence to 0 on M_2, and hence to 0 on $\psi^{-1}(S)$. This ends the proof of (ii).
(iii) As in (ii), we suppose w.l.g. that $[c, d]$ is bounded, and define as previously S_2 to be the set of points where F is not differentiable. Let $M_1 = [c, d] \smallsetminus \psi^{-1}(S_2)$ and $M_2 = \psi^{-1}(S_2)$ (so, $[c, d] = M_1 \cup M_2$). Let $\varepsilon > 0$.
CLAIM 1': *$d(F \circ \psi) \approx f \circ \psi\, d\psi$ on M_1.*

For $x \in M_1$, define $\delta(x)$ in such a way that $\bar{d}(x, y) < \delta(x)$ imply

$$\|F \circ \psi(y) - F \circ \psi(x) - f \circ \psi(x)(\psi(y) - \psi(x))\| < \frac{\varepsilon}{\mathrm{Var}_c^d \psi} |\psi(y) - \psi(x)|.$$

Then if $D << \delta$ is a tame division that anchors in M_1,

$$\sum_{([u,v],\xi) \in D} \|F \circ \psi(v) - F \circ \psi(u) - f \circ \psi(\xi)(\psi(v) - \psi(u))\| < \varepsilon;$$

thus, $d(F \circ \psi) \approx f \circ \psi\, d\psi$ on M_1.
CLAIM 2': *$f \circ \psi\, d\psi \approx 0$ on M_2.*

It suffices to prove that $d\psi \approx 0$ on M_2. This is the content of Thm. 5.6.

From this point, the rest of the proof of (iii) goes exactly like the proof of (ii), from Claim 5 to the end. □

Corollary 7.8. *Let* $f\colon [a,b] \subseteq \mathbb{R} \to \mathcal{E}$ *be H-integrable,* $F(x) = \int_a^x f(t)\,\mathrm{d}t$*, and* $\psi\colon [c,d] \subseteq \bar{\mathbb{R}} \to [a,b]$*. Then* $f \circ \psi\, d\psi$ *is integrable and the change of variable formula applies in the following cases:*
(i) f *is locally bounded in* $[a,b]$*, and* ψ *is either an indefinite integral, or is continuous of bounded variation in* $[c,d]$*;*
(ii) ψ *is either an indefinite integral, or is continuous of bounded variation in* $[c,d]$*, and the set* $\psi^{-1}(S)\smallsetminus(T \cup T')$ *is countable (*S*,* T *and* T' *defined as in Thm. 7.7).*

Proof. It is easy to see that F is Lipschitz at every point x in the neighborhood of which f is bounded, hence assertion (i) follows from Thm. 7.7. Since $F \circ \psi$ is continuous in $[c,d]$, it is ≈ 0 on every countable set. Therefore, assertion (ii) follows from Thm. 7.7 again. □

Notice that one can infer from these theorems corresponding theorems in the Lebesgue integral theory, as we did for Thm. 6.2, assuming that the functions playing role are Lebesgue integrable, and using Thm. 5.1.

Acknowledgment. The author wish to thank Pr. Schwabik for his useful and constructive critique of the first draft.

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