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Martin G. Grigoryan, Department of Physics, Yerevan State University, 0025 Yerevan, Armenia. email: gmarting@ysu.am

# ON THE FOURIER-WALSH COEFFICIENTS

**Abstract**

For any $0 < \epsilon < 1$, $p \geq 1$ and each function $f \in L^p[0,1]$ one can find a function $g \in L^p[0,1]$, $mes\{x \in [0,1];\ g \neq f\} < \epsilon$, such that the sequence $\{|c_k(g)|,\ k \in spec(g)\}$ is monotonically decreasing, where $\{c_k(g)\}$ is the sequence of Fourier-Walsh coefficients of the function $g(x)$.

## 1 Introduction.

We will consider the behavior of Fourier-Walsh coefficients after modification of functions. Note that Luzin's idea of modification of a function improving its properties (see [1]) was substantially developed later on. In 1939, Men'shov [2] proved the following fundamental theorem.

**Theorem 1** (Men'shov's $C$-strong property)**.** *Let $f(x)$ be an a.e. finite measurable function on $[0, 2\pi]$. Then for each $\epsilon > 0$ one can define a continuous function $g(x)$ coinciding with $f(x)$ on a subset $E$ of measure $|E| > 2\pi - \epsilon$ such that its Fourier series with respect to the trigonometric system converges uniformly on $[0, 2\pi]$.*

Further interesting results in this direction were obtained by many famous mathematicians (see for example [3]-[7]).We mention also our papers [8]-[10]. Here we present results having a direct bearing on the present work.

In 1977 A. M. Olevskii [6] established that there exists a function $g(x) \in C[0, 2\pi]$, such that for any function $f(x)$ with

$$|\{x \in [0, 2\pi]\ ;\ f(x) = g(x)\}| > 0$$

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the sequence of trigonometric Fourier coefficients $\{a_n(f), b_n(f)\}$ fail to belong to $l_p$ for any $p \in (0, 2)$.

In 1990, [8] proved that for any $\epsilon > 0$ there exists a measurable set $E \subset [0, 1]$, with measure $|E| > 1 - \epsilon$, such that for any function $f(x) \in L^1[0, 1]$ there exists a function $g(x) \in L^1[0, 1]$ coinciding with $f(x)$ on $E$ and such that the sequence of Fourier coefficients $\{c_k(g)\}$ of the function $g(x)$ in the trigonometric system belongs to $l_p$ for all $p > 2$.

The Walsh system, an extension of the Rademacher system, may be obtained in the following manner. Let $r$ be the periodic function, of least period 1, defined on $[0, 1)$ by

$$r = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

The Rademacher system, $R = r_n : n = 0, 1, ...$, is defined by the conditions

$$r_n(x) = r(2^n x),\ \forall x \in R, n = 0, 1, ...,$$

and, in the ordering employed by Payley (see [11] and [12]), the $n$th element of the Walsh system $\{\varphi_n\}$ is given by

$$\varphi_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{1}$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of $n$, with each $n_k$ either 0 or 1.

Let$\{\varphi_k(x)\}$ be the Walsh system and let $f(x) \in L^p,\ p \geq 1$. We denote by $c_k(f)$ the Fourier-Walsh coefficients of $f$; i.e.

$$c_k(f) = \int_0^1 f(x)\varphi_k(x)\ dx.$$

The spectrum of $f(x)$ (denoted by $spec(f)$) is the support of $c_k(f)$; i.e. the set of integers where $c_k(f)$ is non-zero.

In the present work we prove the following theorem:

**Theorem 2.** *For any $0 < \epsilon < 1$ , $p \geq 1$ and each function $f \in L^p[0, 1]$ one can find a function $g \in L^p[0, 1]$, $mes\{x \in [0, 1];\ g \neq f\} < \epsilon$, such that the sequence*

$$\{|c_k(g)|,\ k \in spec(g)\}, \textit{is monotonically decreasing.}$$

**Remark 1.** It must be pointed out that in this theorem the "exceptional" set on which the function $f$ is modified depends on $f$.

The following problem remains open:

**Question 1.** Is it possible to construct in Theorem 2 the "exceptional" set independent from $f$?

**Question 2.** Is Theorem 2 true for the trigonometric system?

## 2 Proofs of Main Lemmas.

We put

$$I_k^{(j)}(x) = \begin{cases} 1 & \text{if } x \in [0,1] \setminus \Delta_k^{(j)} , \\ 1 - 2^k & \text{if } x \in \Delta_k^{(j)} = (\frac{j-1}{2^k}, \frac{j}{2^k}) , \end{cases} \tag{2}$$

for $k = 1, 2, \ldots$ , $1 \le j \le 2^k$, and periodically extend these functions on $R^1$ with period 1.

By $\chi_E(x)$ we denote the characteristic function of the set $E$; i.e.

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases} \tag{3}$$

Then, clearly

$$I_k^{(j)}(x) = \varphi_0(x) - 2^k \cdot \chi_{\Delta_k^{(j)}}(x), \tag{4}$$

and let for the natural numbers $k \ge 1$ and $j \in [1, 2^k]$

$$b_i(\chi_{\Delta_k^{(j)}}) = \int_0^1 \chi_{\Delta_k^{(j)}}(x)\varphi_i(x)\, dx = \pm\frac{1}{2^k} \ , \ 0 \le i < 2^k, \tag{5}$$

$$a_i(I_k^{(j)}) = \int_0^1 I_k^{(j)}(x)\varphi_i(x)\, dx = \begin{cases} 0 & \text{if } i = 0 \text{ and } i \ge 2^k, \\ \pm 1 & \text{if } 1 \le i < 2^k. \end{cases} \tag{6}$$

Hence

$$\chi_{\Delta_k^{(j)}}(x) = \sum_{i=0}^{2^k-1} b_i(\chi_{\Delta_k^{(j)}})\varphi_i(x), \tag{7}$$

$$I_k^{(j)}(x) = \sum_{i=1}^{2^k-1} a_i(I_k^{(j)})\varphi_i(x). \tag{8}$$

**Lemma 1.** *Let dyadic interval* $\Delta = \Delta_m^{(k)} = ((k-1)/2^m; k/2^m),\ k \in [1, 2^m]$ *and numbers* $N_0 \in \mathbb{N},\ \gamma \neq 0,\ \epsilon \in (0,1), p \geq 1$ *be given. Then there exists a measurable set* $E \subset [0,1]$ *and a polynomial* $Q$ *in the Walsh system* $\{\varphi_k\}$ *of the following form:*

$$Q = \sum_{k=N_0}^{N} c_k \varphi_k$$

*which satisfy the following conditions:*

1. *the coefficients* $\{c_k\}_{k=N_0}^{N}$ *are* $0$ *or* $\pm\gamma|\Delta|$,
2. $|E| > (1-\varepsilon)|\Delta|$,
3. $Q(x) = \begin{cases} \gamma & \text{if } x \in E, \\ 0 & \text{if } x \notin \ \Delta, \end{cases}$
4. $\left(\int_0^1 |Q(x)|^p dx\right)^{\frac{1}{p}} \leq 3|\gamma||\Delta|^{\frac{1}{p}} \epsilon^{-\frac{1}{q}}, \frac{1}{p} + \frac{1}{q} = 1.$

PROOF. Let

$$\nu_0 = \left[\log_2 \frac{1}{\epsilon}\right] + 1; s = [\log_2 N_0] + m. \tag{9}$$

We define the polynomial $Q(x)$ and the numbers $c_n$, $a_i$ and $b_j$ in the following form:

$$Q(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(2^s x),\ x \in [0,1], \tag{10}$$

$$c_n = c_n(Q) = \int_0^1 Q(x)\varphi_n(x)\ dx,\ \forall n \geq 0, \tag{11}$$

$$b_i = b_i(\chi_{\Delta_m^{(k)}})\ ,\ 0 \leq i < 2^m, a_j = a_j(I_{\nu_0}^{(1)}),\ 0 < j < 2^{\nu_0}. \tag{12}$$

Taking into consideration the following equation

$$\varphi_i(x) \cdot \varphi_j(2^s x) = \varphi_{j\cdot 2^s + i}(x),\ \text{if } 0 \leq i,\ j < 2^s (\text{see (1)}),$$

and having the relations (5)-(8) and (10)-(12), we obtain that the polynomial $Q(x)$ has the following form:

$$Q(x) = \gamma \cdot \sum_{i=0}^{2^m-1} b_i \varphi_i(x) \cdot \sum_{j=1}^{2^{\nu_0}-1} a_j \varphi_j(2^s x)$$
$$= \gamma \cdot \sum_{j=1}^{2^{\nu_0}-1} a_j \cdot \sum_{i=0}^{2^m-1} b_i \varphi_{j \cdot 2^s + i}(x) = \sum_{k=N_0}^{\bar{N}} c_k \varphi_k(x), \tag{13}$$

where

$$c_k = c_k(Q) = \begin{cases} \pm \frac{\gamma}{2^m} \text{ or } 0 & \text{if } k \in [N_0, \bar{N}] \\ 0 & \text{if } k \notin [N_0, \bar{N}] \end{cases}, \ \bar{N} = 2^{s+\nu_0} + 2^m - 2^s - 1. \tag{14}$$

Then let

$$E = \{x; Q(x) = \gamma\}.$$

Clearly that (see (2) and (10)),

$$|E| = 2^{-m}(1 - 2^{-\nu_0}) > (1 - \epsilon)|\Delta|, \tag{15}$$

$$Q(x) = \begin{cases} \gamma & \text{if } x \in E \\ \gamma(1 - 2^{\nu_0}) & \text{if } x \in \Delta \setminus E \\ 0 & \text{if } x \notin \Delta. \end{cases} \tag{16}$$

Thus, for $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$, then $q = \infty$)

$$\left( \int_0^1 |Q(x)|^p \, dx \right)^{\frac{1}{p}} \leq 3|\gamma||\Delta|^{\frac{1}{p}} \epsilon^{-\frac{1}{q}} .$$

□

**Lemma 2.** *Let numbers $p \geq 1$, $m_0 > 1$, positive $\varepsilon$ and $\delta$ and Walsh polynomial $f(x)$ are given. Then one can find a set $E \subset [0, 1]$, $|E| > 1 - \epsilon$ and a polynomial in the Walsh system*

$$Q(x) = \sum_{k=m_0}^{N} a_k \varphi_k(x),$$

*satisfying the following conditions:*

1. *$0 \leq |a_k| < \delta$ and the non-zero coefficients in $\{|a_k|\}_{k=m_0}^N$ are in decreasing order,*

*2.* $Q(x) = f(x)$, *for all* $x \in E$,

*3.* $\|Q\|_p < \frac{3}{\epsilon^{\frac{1}{q}}}\|f\|_p \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

PROOF. Let

$$f(x) = \sum_{k=0}^{M} b_k \varphi_k(x) = \sum_{\nu=1}^{\nu_0} \gamma_\nu \cdot \chi_{\Delta_\nu}(x), \ \sum_{\nu=1}^{\nu_0} |\Delta_\nu| = 1, \tag{17}$$

where $\Delta_\nu$ are dyadic intervals of the form $\Delta_m^{(k)} = ((k-1)/2^m; k/2^m)$, $k \in [1, 2^m]$.

Without loss of generality, one may assume that

$$0 < |\gamma_1||\Delta_1| < ... < |\gamma_\nu||\Delta_\nu| < ... < |\gamma_{\nu_0}||\Delta_{\nu_0}| < \delta. \tag{18}$$

Successively applying Lemma 1, we determine some sets $E_\nu \subset [0,1]$ and polynomials

$$Q_\nu = \sum_{j=m_{\nu-1}}^{m_\nu - 1} a_j \varphi_j \ , \ a_j = 0 \text{ or } \pm\gamma_j |\Delta_j|, \text{ if } j \in [m_{\nu-1}, m_\nu), \ \nu = 1, ..., \nu_0, \tag{19}$$

which satisfy the following conditions:

$$|E_\nu| > (1 - \epsilon) \cdot |\Delta_\nu|, \tag{20}$$

$$Q_\nu = \begin{cases} \gamma_\nu & \text{if } x \in \ E_\nu, \\ 0 & \text{if } x \notin \ \Delta_\nu, \end{cases} \tag{21}$$

$$||Q_\nu||_p = \left(\int_0^1 \mid Q_\nu(x) \mid^p dx\right)^{1/p} < \frac{3 \mid \gamma_\nu \mid}{\epsilon^{1-1/p}} \cdot \mid \Delta_\nu \mid^{1/p} . \tag{22}$$

We define

$$Q = \sum_{\nu=1}^{\nu_0} Q_\nu = \sum_{k=m_0}^{N} a_k \varphi_k, \ N = m_{\nu_0} - 1, \tag{23}$$

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu. \tag{24}$$

By (18)–(24) we obtain

$$Q(x) = f(x), \text{ for } x \in E,$$

$$|E| > 1 - \epsilon \ ,$$

$0 \leq |a_k| < \delta$ and the non-zero coefficients in $\{|a_k|\}_{k=m_0}^{N}$ are in decreasing order. Taking into account (17), (21)–(23) we have

$$\int_0^1 \mid Q(x) \mid^p \ dx = \sum_{i=1}^{\nu_0} \int_{\Delta_i} \mid \sum_{n=1}^{\nu_0} Q_n(x) \mid^p \ dx$$

$$= \sum_{i=1}^{\nu_0} \int_{\Delta_i} \mid Q_i(x) \mid^p \ dx \leq \sum_{i=1}^{\nu_0} \frac{3^p \mid \gamma_i \mid^p \mid \Delta_i \mid}{\epsilon^{p-1}} \leq 3^p \frac{\int_0^1 \mid f(x) \mid^p \ dx}{\epsilon^{p-1}}.$$

□

## 3 Proof of Theorem 2.

Proof. Let $p \geq 1,\ f(x)$ be an arbitrary element of $L^p[0,1]$, and let $\varepsilon \in (0,1)$. It is easy to see that one can choose a sequence $\{f_n(x)\}_{n=1}^{\infty}$ of polynomials in the Walsh systems such that

$$\lim_{N\to\infty} \left|\left| \sum_{n=1}^{N} f_n(x) - f(x) \right|\right|_p dx = 0,$$

$$||f_n(x)||_p \, dx \leq \varepsilon^{\frac{1}{q}} \cdot 2^{-2(n+1)},\ n \geq 2 \ \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Applying repeatedly Lemma 2, we obtain sequences of sets $\{E_n\}_{n=1}^{\infty}$ and polynomials in the Walsh systems $\{\varphi_n(x)\}$

$$Q_n(x) = \sum_{k=m_{n-1}}^{m_n - 1} a_{s_k} \varphi_{s_k}(x),\ n \geq 1 \ , m_n \ \nearrow,$$

which for all $n \geq 1$ satisfy the following conditions:

$$Q_n(x) = f_n(x),\ \text{for } x \in E_n,$$
$$|E_n| > 1 - \varepsilon 2^{-n},$$
$$||Q_n||_p \leq 3\varepsilon^{-\frac{1}{q}} 2^{\frac{n}{q}} \cdot ||f_n||_p,$$
$$|a_{s_{m_n}}| < |a_{s_{k+1}}| < |a_{s_k}| < 2^{-n},\ \text{for all } k \in [m_{n-1}; m_n).$$

We put

$$g(x) = \sum_{n=1}^{\infty} Q_n(x).$$

Obviously $g(x) \in L^p[0,1]$ , $\{|c_k(g)|,\ k \in spec(g)\}$ is monotonically decreasing,

$$g(x) = f(x),\ \text{for } x \in \bigcap_{n=1}^{\infty} E_n,\ |\bigcap_{n=1}^{\infty} E_n| > 1 - \varepsilon.$$

□

**Remark 2.** Note that the following more general result is true: let $\{\beta_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $\beta_k \to 0$. There exists a sequence $\{A_k\}_{k=1}^{\infty}$ of real numbers with $|A_k| \searrow_0$, $\sum_{n=1}^{\infty} |A_k|\ \beta_k < \infty$, with the following property: for any $0 < \epsilon < 1$, $p \geq 1$ and each function $f \in L^p[0,1]$ one can find a function $g(x) \in \cap_{p\geq 1} L^p$, $mes\{x \in [0,1]\ ;\ g \neq f\} < \epsilon$, such that the sequence $\{|c_k(g)|,\ k \in spec(g)\} \subset \{A_k\}_{k=1}^{\infty}$, and for all $n \geq 0$

$$\left\|\sum_{k=0}^{n} c_k(g)\varphi_k(x)\right\|_p \leq \frac{5}{\epsilon^{1-\frac{1}{p}}}||f||_p,$$

where $\{c_k(g)\}$ is the sequence of Fourier-Walsh coefficients of the function $g(x)$.

From this we have the following corollary:

**Corollary 1.** For any $0 < \epsilon < 1,\ p > 2$ and each function $f \in L^p[0,1]$ one can find a function $g \in L^p[0,1]$, $mes\{x \in [0,1];\ g \neq f\} < \epsilon$, whose greedy algorithm $\{G_m(g)\}$ with respect to the Walsh system converges to $g$ in $L^p[0,1]$ and

$$\|G_m(g)\|_p \leq \frac{5}{\epsilon^{1-\frac{1}{p}}}||f||_p.$$

Greedy algorithms in Banach spaces with respect to normalized bases have been considered in [13]–[20].

Note that in [17] it is proved that for any $p \neq 2$ there exists a function from $L^p[0,1]$, whose greedy algorithm diverges in $L^p[0,1]$.

Note also that in [20] it is proved that there exist a complete orthonormal system $\{\varphi_k(x)\}$ and a function $f(x) \in L^p$, $p > 2$, such that if $g(x)$ is any function from $L^p[0,1]$ with

$$|\{x \in [0,2\pi];\ f(x) = g(x)\}| > 0,$$

then its greedy algorithm with respect to the system $\{\varphi_k(x)\}$ diverges in $L^p[0,1]$.

A question rises concerning Corollary 1:

**Question 3.** Is Corollary 1 true for the trigonometric system?

Note that for $p = 1$ the answer to Question 1 is positive. Note also that, in Theorem 2 the modified function $g$ can be chosen such that $spec(g) \equiv Z_+$.

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## References

[1] N. N. Luzin, *On the fundamental theorem of the integral calculus*, Mat. Sbornik, **28** (1912), 266–294 (in Russian).

[2] D. E. Men'shov, *Sur la representation des fonctions measurables des series trigonometriques*, Mat. Sbornik, **9** (1941), 667–692.

[3] D. E. Men'shov, *On Fourier series of integrable functions*, Trudy Moskov. Mat. Obshch., **1** (1952), 5–38.

[4] F. G. Arutjunjan, *Series with respect to the Haar system*, (Russian. Armenian summary), Akad. Nauk Armjan. SSR Dokl., **42(3)** (1966), 134–140.

[5] J. J. Price, *Walsh series and adjustment of functions on small sets*, Illinois J. Math., **13** (1969), 131–136.

[6] A. M. Olevskii, *Modifications of functions and Fourier series.* (Russian) Uspekhi Mat. Nauk **40** (1985), 157–193.

[7] K. I. Oskolkov, *The uniform modulus of continuity of summable functions on sets of positive measure*, (Russian), Dokl. Akad. Nauk SSSR, **229(2)** (1976), 304–306.

[8] M. G. Grigoryan, *Convergence in the $L^1$-metric and almost everywhere convergence of Fourier series in complete orthonormal systems*, (Russian), Mat. Sb., **181(8)** (1990), 1011–1030; translation in Math. USSR-Sb., **70(2)** (1991), 445–466.

[9] M. G. Grigoryan, *On the representation of functions by orthogonal series in weighted $L^p$ spaces*, Studia. Math., **134(3)** (1999), 207–216.

[10] M. G. Grigoryan, *On the strong $L^p_\mu$-property of orthonormal systems*, (Russian. Russian summary), Mat. Sb., **194(10)** (2003), 77–106; translation in Sb. Math., **194(9-10)** (2003), 1503–1532.

[11] B. I. Golubov, A. F. Efimov, V. A. Skvortsov, (Russian), *Walsh Series and Transforms*, Theory and Applications, "Nauka", Moscow, 1987.

[12] R. E. A. C. Paley, *A remarkable set of orthogonal functions*, Proc. London Math. Soc., **34** (1932), 241–279.

[13] R. A. DeVore, V. N. Temlyakov, *Some remarks on greedy algorithms*, Advances in Computational Math., **5** (1996), 173–187.

[14] S. V. Konyagin and V. N. Temlyakov, *A remark on Greedy approximation in Banach spaces*, East Journal on Approx., **5(1)** (1999), 1–15.

[15] V. N. Temlyakov, *Nonlinear methods of approximation*, Found. Comput. Math., **3** (2003), 33–107.

[16] P. Wojtaszczyk, *Greedy algorithm for general biorthogonal systems*, Journal of Approx. Theory, **107** (2000), 293–314.

[17] R. Gribonval, M. Nielsen, *On the quasi-greedy property and uniformly bounded orthonormal systems*, `http://www.math.auc.dk/research/reports/R-2003-09.pdf`.

[18] T. W. Körner, *Divergence of decreasing rearranged Fourier series*, Ann. of Math., **144** (1996), 167–180.

[19] M. G. Grigoryan and R. E. Zink, *Greedy approximation with respect to certain subsystems of the Walsh orthonormal system*, Proc. of the Amer. Mat. Soc., **134(12)** (2006), 3495–3505.

[20] M. G. Grigoryan, K. S. Kazarian, F. Soria, *Mean convergence of orthogonal Fourier series of mod. func.*, Trans. Amer. Math. Soc., **352(8)** (2000), 3777–3798.