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A STRUCTURAL THEOREM FOR METRIC SPACE VALUED MAPPINGS OF Φ-BOUNDED VARIATION

Abstract

In this paper we introduce the notion of Φ-bounded variation for metric space valued mappings defined on a subset of the real line. Such a notion generalizes the one for real functions introduced by M. Schramm, and many previous generalized variations. We prove a structural theorem for mappings of Φ-bounded variation. As an application we show that each mapping of Φ-bounded variation defined on a subset of $\mathbb{R}$ possesses a Φ-variation preserving extension to the whole real line.

1 Introduction.

Let f be a map defined on a non-empty subset E of the real line $\mathbb{R}$ and taking values in a metric space (X, d) (i.e. $f \in X^E$). The classical Jordan decomposition criterion ([15]) asserts that, in the case $E = [a, b]$ is a compact interval, and $X = \mathbb{R}$ is the real line endowed with the usual Euclidean metric, a function $f : [a, b] \to \mathbb{R}$ is of bounded variation (BV, for short) if and only if f can be represented as a difference of two non-decreasing bounded functions on $[a, b]$. This simple and elegant criterion is a useful tool in the classical theory of BV-functions. The notion of bounded variation works as well if (X, d) is a metric space, (see [13, Sec. 2.5.16], [21, Ch. 4, Sec. 9] and [1, Ch.1, Sec.2]), but, in this case, the Jordan decomposition loses meaning. The following is a

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decomposition criterion applicable both to the classical case and to the general case proved by Chistyakov ([2], [3]):

Theorem A. *A mapping $f : E \to X$ is of bounded variation if and only if it can be represented as a composite mapping $f = g \circ \varphi$, where $\varphi : E \to \mathbb{R}$ is a bounded, nondecreasing function and $g : \varphi(E) \to X$ is a natural mapping.*

A mapping $g : E \to X$ is said to be *natural* (see [2, Sec. 3], [3, Sec. 3], [6, Sec. 3]) if $V(g, E_x^y) = y - x$ for all $x, y \in E, \quad x < y$, where $V(g, E_x^y)$ denotes the Jordan variation of g in the set $E_x^y = E \cap [x, y]$. Since $d(g(x), g(y)) \leq V(g, E_x^y) \;\; \forall x, y \in E, x < y$, we have that each natural mapping g is Lipshitzian, with constant $Lip(g) \leq 1$. We remember that a particular case of Theorem A was outlined by Federer ([13]).

During the last years there has been a considerable interest in the study of variation, due to its applications in various fields of analysis, as the convergence of Fourier series ([12], [15], [19], [20], [22], [24], [25]), and the study of the multivalued functions ([2]- [11]). Many mathematicians have generalized and deepened the notion of variation. We recall some of such notions:

(a) the p-variation in the Wiener sense ([24], [25]);

(b) the Φ-variation, introduced by L. C. Young ([25], [26]);

(c) the Λ-variation, introduced by Waterman ([22]);

(d) the Φ-bounded variation, introduced by Schramm and Waterman ([19], [20]), where $\Phi = \{\phi_n\}$ is a sequence of functions;

(e) the nonlinear q-variation in the sense of Riesz ([16], [17]);

(f) the (Φ, σ)-variation, introduced by Chistyakov ([4], [5]);

(g) the Φ- variation in the Jordan-Riesz-Orlicz sense ([6]).

The above listed notions of variation have been obtained proceeding along different ways and give place to different classes of mappings of generalized bounded variation. In the following we will speak of mappings of Φ-*bounded variation* for variations in cases (a)-(d) and, on the other hand, of mappings of *bounded Φ-variation* for the ones in cases (e)-(g).

The definitions (a)-(d) will be given in Section 2. Now we recall the definitions (e)-(g).

Let $\Phi : \mathbb{R}_0^+ = [0, +\infty[\to \mathbb{R}_0^+$ be a convex, continuous function with $\Phi(0) = 0$ and $\Phi(t) > 0$ for positive t, and let $\sigma : [a, b] \to \mathbb{R}$ be an increasing function. In

[4] Chistyakov introduced the (Φ, σ)-*variation* of $f \in X^{[a,b]}$ as the supremum of the sums

$$\sum_{i=1}^{n} \Phi\left(\frac{d(f(t_i), f(t_{i-1}))}{\sigma(t_i) - \sigma(t_{i-1})}\right)(\sigma(t_i) - \sigma(t_{i-1}))$$

over all subdivisions, $t_0 = a < t_1 < \cdots < t_n = b$, of $[a, b]$.

The notion of $(\Phi, \sigma)-$variation is a generalization of the classical concepts introduced by Jordan ([15]) and Riesz ([16] and [17]) for real valued functions, in the case $\sigma(t) = t$ and $\Phi(t) = t^r$. Precisely, if $r = 1$ we obtain the Jordan variation, if $r > 1$ we obtain the Riesz variation. We denote (as in [6]) by $BV_\Phi(E, X)$ the set of all mappings from E into X of bounded (Φ, σ)-variation in the case $\sigma(t) = t$.

A wide study of such classes of mappings, under suitable hypotheses on the function Φ, has been made by Chistyakov in [5] and [6]: he proved new structural theorems, and, using such theorems, many properties of the mappings in $BV_\Phi(E, X)$.

In Section 2 we introduce the notion of Φ-bounded variation for metric space valued mappings defined on a subset E of $\mathbb{R}$. Such a notion generalizes the variations recalled in (a)-(d). The subset of X^E of all mappings of Φ-bounded variation is denoted by Φ-$BV(E, X)$.

Our main result (Theorem 1) is a structural theorem claiming that any mapping $f \in \Phi$-$BV(E, X)$ can be characterized as a composition $f = g \circ \varphi$ where $\varphi : E \to \mathbb{R}$ is bounded, nondecreasing and $g : T = \varphi(E) \to X$ is uniformly continuous, of Φ-bounded variation, and such that $V_\Phi(g, T) = V_\Phi(f, E)$. As an application of such a result we prove (Theorem 2) that, if X is a complete metric space, each mapping $f \in \Phi$-$BV(E, X)$ possesses a Φ-variation preserving extension to the whole real line $\mathbb{R}$.

2 Notations and Known Facts.

Let $f : E \to X$ be a mapping defined on a non-empty $E \subseteq \mathbb{R}$, with values in a metric space (X, d). Let $\{I_n\}$ denote a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset \mathbb{R}$ whose endpoints $a_n, b_n \in E$. We write $|f(I_n)| = d(f(b_n), f(a_n))$. Throughout this paper, when we consider a collection of intervals, they will be assumed to be non-overlapping and with endpoints in the domain of the mapping without further reference to that fact.

Let $\Phi = \{\phi_n : n \in \mathbb{N}^+\}$ be a sequence of increasing convex functions, defined on $\mathbb{R}_0^+$ and fulfilling the following properties: $\phi_n(0) = 0$ and $\phi_n(x) > 0$, $\forall n \in \mathbb{N}^+$ and $\forall x > 0$; $\phi_{n+1}(x) \leq \phi_n(x)$ $\forall n \in \mathbb{N}^+$ and $\forall x \in \mathbb{R}_0^+$; $\sum_n \phi_n(x)$ diverges for $x > 0$. Let us denote by $\mathcal{S}$ the set of all the sequences Φ fulfilling the above properties. The following definition is due to Schramm ([20]) in case

$E = [a, b]$ is a bounded, closed interval and $X = \mathbb{R}$ endowed with the usual metric.

Definition 1. Given $\Phi = \{\phi_n\} \in \mathcal{S}$, a mapping $f : E \to X$ is said to be of Φ-*bounded variation* in E if there exists a positive real constant M such that, $\sum_n \phi_n(|f(I_n)|) \leq M$, for every choice of $\{I_n\}$.

In such a case we call *total* Φ*-variation of* f in E the number

$$V_\Phi(f, E) = \sup \sum_n \phi_n(|f(I_n)|),$$

where the supremum is taken over all collections of intervals $\{I_n\}$.

In this paper we will denote by Φ-$BV(E, X)$ the class of all the mappings $f : E \to X$ of Φ-bounded variation.

In case $E = [a, b]$ and $X = \mathbb{R}$, it was pointed out by Schramm ([20]) that the notion of total Φ-variation contains, as a special case, the notions of variation recalled in (a)-(d) of the introduction:

i) If ϕ is an N-function in the sense of Young ([25]), and $\phi_n = \phi$ for all n, we obtain the Φ-variation, introduced by Young ([25], [26]). It is well known that if $\phi(x) = x^p, p \geq 1$, we have the classical notions of p-variation due respectively to Jordan in case $p = 1$ ([15]), to Wiener if $p > 1$ ([24], [25]);

ii) let $\Lambda = \{\lambda_n\}$ denotes a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_n$ diverges and $\phi_n(x) = x/\lambda_n$, then $V_\Phi(f, [a, b])$ is the total Λ-variation introduced by Waterman ([22]);

iii) let ϕ and Λ be respectively as in i) and ii); if $\phi_n(x) = \phi(x)/\lambda_n$, then $V_\Phi(f, [a, b])$ is the total variation introduced by Schramm and Waterman ([19]).

An immediate consequence of the definition is the *monotonicity property of the total* Φ*-variation with respect to the set*, in the sense that if $f \in \Phi$-$BV(E, X)$ and $\emptyset \neq E_1 \subset E$, then $f \in \Phi$-$BV(E_1, X)$ and $V_\Phi(f, E_1) \leq V_\Phi(f, E)$.

To each mapping $f \in \Phi$-$BV(E, X)$ we associate its Φ-variation function defined, for $x \in \mathbb{R}$, as follows:

$$v_{\Phi,f}(x) = \begin{cases} \sup\{V_\Phi(f, E_t) : t \in E,\ t \leq x\} & \text{if} \quad E_x \neq \emptyset, \\ \inf\{V_\Phi(f, E_t) : t \in E\} & \text{if} \quad E_x = \emptyset, \end{cases}$$

where $E_t =]-\infty, t] \cap E$ for every $t \in \mathbb{R}$.

Obviously, $v_{\Phi,f}$ is a bounded nondecreasing real function defined on $\mathbb{R}$, and $v_{\Phi,f}(x) = V_\Phi(f, E_x)$ if $x \in E$.

Remark 1. We observe that Definition 1 does not lose generality if we suppose that the sequence $\{I_n\}$ is such that $|f(I_n)| \neq 0 \ \forall\, n$. Indeed, if the sequence $\{I_n\}$ is such that $|f(I_n)| = 0$ for some n, let $\{I_{n_k}\}$ be the subsequence obtained removing the intervals such that $|f(I_n)| = 0$ and leaving unchanged the order. We have: $n_k \geq k \ \forall\, k$, so $\phi_{n_k}(|f(I_{n_k})|) \leq \phi_k(|f(I_{n_k})|)$. Therefore $\sum_n \phi_n(|f(I_n)|) = \sum_k \phi_{n_k}(|f(I_{n_k})|) \leq \sum_k \phi_k(|f(I_{n_k})|)$.

Remark 2. It was proved by Schramm ([20, Lemma 2.6]) that for $y > x \geq 0$ and positive integer n, we have $\phi_n(x) - \phi_n(y) \leq (y-x)\phi_1(y+1)$. Then, fixed $b \in]0, +\infty[$, for each n we have $|\phi_n(x) - \phi_n(y)| \leq |y-x|\phi_1(b+1) \quad \forall x, y \in [0, b]$. Therefore, in $[0, b]$, the ϕ_n are Lipschitz continuous with the same Lipschitz constant.

Remark 3. Let us remember that V. V. Chistyakov and O. E. Galkin ([9], [10]) developed the theory of functions of bounded variation in the sense of N. Wiener and of L. C. Young, for mappings defined on a subset of the real line and taking values in a metric or normed space.

3 Main Results.

Let $\Phi = \{\phi_n\}$ be a sequence in $\mathcal{S}$, M and δ be positive real numbers. Set

$$K(M, \delta) = \min\{n : \sum_{j=1}^{n} \phi_j(\delta/2) > M\},$$

$$H(M, \delta) = \min\{1 - \frac{\phi_j(\delta/2)}{\phi_j(\delta)} : j = 1, 2. \dots, K(M, \delta)\}.$$

Given $E \subseteq \mathbb{R}$, $f \in \Phi\text{-}BV(E, X)$ and $x \in E$, in the next lemma we use the following notation: $k(x, \delta) = K(v_{\Phi,f}(x), \delta)$, $h(x, \delta) = H(v_{\Phi,f}(x), \delta)$ if $v_{\Phi,f}(x) \neq 0$, $h(x, \delta) = k(x, \delta) = 1$ if $v_{\Phi,f}(x) = 0$.

Lemma 1. *Let $f : E \to X$ be such that $V_\Phi(f, E) < +\infty$. If $x, y \in E$, $x < y$ and $d(f(x), f(y)) \geq \delta > 0$, then $v_{\Phi,f}(y) - v_{\Phi,f}(x) \geq h(x, \delta)\phi_{k(x,\delta)}(\delta)$.*

The following proof is analogous to the one of ([20, Lemma 2.5]).

PROOF. If $v_{\Phi,f}(x) = 0$ the proof is immediate. If $v_{\Phi,f}(x) \neq 0$, given $\eta > 0$, there exist intervals $I_n,\ n = 1, 2, \ldots, N$, whose endpoints are in $E \cap (-\infty, x]$ such that $\{|f(I_n)|\}$ is nonincreasing and $v_{\Phi,f}(x) \leq \sum_{n=1}^{N} \phi_n(|f(I_n)|) + \eta$. Put $c_n = |f(I_n)|$ and $T = \sum_{n=1}^{N} \Phi_n(c_n)$. We will construct a second sum S as follows:

i) if $c_n \geq \delta/2$ for $n = 1, 2, \ldots, k$ and $c_{k+1} < \delta/2$, set $S = \phi_1(c_1) + \phi_2(c_2) + \cdots + \phi_k(c_k) + \phi_{k+1}(\delta) + \phi_{k+2}(c_{k+1}) + \cdots + \phi_{N+1}(c_N)$. Then $S - T \geq \phi_{k+1}(\delta) - \phi_{k+1}(\delta/2)$;

ii) if $c_n \geq \delta/2$ for $n = 1, 2, \ldots, N$ we set $S = T + \phi_{N+1}(\delta)$;

iii) if $c_n < \delta/2$ for $n = 1, 2, \ldots, N$ we set $S = \phi_1(\delta) + \phi_2(c_1) + \cdots + \phi_{N+1}(c_N)$. Then $S - T \geq \phi_1(\delta) - \phi_1(\delta/2)$.

Since $v_{\Phi,f}(y) \geq S$ we have $v_{\Phi,f}(y) - v_{\Phi,f}(x) \geq S - T - \eta$. Thus

$$v_{\Phi,f}(y) - v_{\Phi,f}(x) \geq \begin{cases} \phi_{k+1}(\delta) - \phi_{k+1}(\delta/2) - \eta & \text{if } c_{k+1} < \delta/2 \leq c_k, \\ \phi_{N+1}(\delta) - \phi_{N+1}(\delta/2) - \eta & \text{if } c_n \geq \delta/2 \quad \forall n, \\ \phi_1(\delta) - \phi_1(\delta/2) - \eta & \text{if } c_n < \delta/2 \quad \forall n. \end{cases}$$

Hence

$$v_{\Phi,f}(y) - v_{\Phi,f}(x) \geq \left(1 - \frac{\phi_m(\delta/2)}{\phi_m(\delta)}\right) \phi_m(\delta) - \eta,$$

with $m = \min(\{n : |f(I_n)| < \delta/2\} \cup \{N+1\})$.

If $m = 1$, obviously $k(x, \delta) \geq m$. If $m = N + 1$, then

$$v_{\Phi,f}(x) \geq \sum_{n=1}^{N} \phi_n(c_n) \geq \sum_{n=1}^{N} \phi_n(\delta/2)$$

so that $k(x, \delta) \geq N + 1$. If $1 < m < N + 1$, then

$$v_{\Phi,f}(x) \geq \sum_{n=1}^{m-1} \phi_n(c_n) \geq \sum_{n=1}^{m-1} \phi_n(\delta/2),$$

hence $k(x, \delta) > m - 1$. In any case, $k(x, \delta) \geq m$, so that $\left(1 - \frac{\phi_m(\delta/2)}{\phi_m(\delta)}\right) \phi_m(\delta) - \eta \geq h(x, \delta)\phi_{k(x,\delta)}(\delta) - \eta$. By the arbitrariness of η, since $h(x, \delta)$ and $k(x, \delta)$ do not depend on η, we have $v_{\Phi,f}(y) - v_{\Phi,f}(x) \geq h(x, \delta)\phi_{k(x,\delta)}(\delta)$. □

As an immediate consequence we get the following corollary:

Corollary 1. *Let $f \in \Phi\text{-}BV(E,X)$ and $\delta > 0$. For every $x, y \in E$ such that $d(f(x), f(y)) \geq \delta$, then*

$$|v_{\Phi,f}(x) - v_{\Phi,f}(y)| \geq h(\delta)\phi_{k(\delta)}(\delta),$$

where

$$k(\delta) = K(V_\Phi(f,E),\delta) \quad \textit{and} \quad h(\delta) = H(V_\Phi(f,E),\delta). \tag{1}$$

Corollary 2. *Let $x, y \in E$ with $x < y$. Then a mapping $f \in \Phi\text{-}BV(E,X)$ is constant in $E \cap [x,y]$ if and only if $v_{\Phi,f}(x) = v_{\Phi,f}(y)$.*

PROOF. Let us suppose that f is constant in $E \cap [x,y]$. Let $\{I_n\}$ be a collection of intervals, $I_n = [a_n, b_n]$ with $a_n, b_n \in E \cap]-\infty, y]$, such that $|f(I_n)| \neq 0 \ \forall\ n$. We have $I_n \subset]-\infty, x] \ \forall n$ excepting, at most, an index h such that $x \in]a_h, b_h[$. In such a case we put $I'_h = [a_h, x]$, then $|f(I'_h)| = |f(I_h)|$. Therefore

$$\sum_n \phi_n(|f(I_n)|) = \sum_{n \neq h} \phi_n(|f(I_n)|) + \phi_h(|f(I'_h)|) \leq v_{\Phi,f}(x).$$

The arbitrariness of the collection $\{I_n\}$ implies $v_{\Phi,f}(y) \leq v_{\Phi,f}(x)$. By the monotonicity of the function $v_{\Phi,f}$ we have $v_{\Phi,f}(y) = v_{\Phi,f}(x)$. On the other hand, if f is not constant in $[x,y]$ there exists $\xi \in]x,y]$ such that $f(\xi) \neq f(x)$. By Corollary 1 we obtain

$$v_{\Phi,f}(\xi) - v_{\Phi,f}(x) \geq h(\delta)\phi_{k(\delta)}(\delta) > 0$$

where $\delta = d(f(\xi), f(x)) > 0$. Then $v_{\Phi,f}(y) \geq v_{\Phi,f}(\xi) > v_{\Phi,f}(x)$. □

Theorem 1. *A mapping $f \in \Phi\text{-}BV(E,X)$ if and only if $f = g \circ \varphi$ where $\varphi : E \to \mathbb{R}$ is a bounded, nondecreasing function and $g : T = \varphi(E) \to X$, is uniformly continuous and of Φ-bounded variation with $V_\Phi(g,T) = V_\Phi(f,E)$.*

PROOF. In what follows J denotes a generic interval in $\mathbb{R}$. For the sake of clarity the proof is divided into three steps.

Step 1. In this step we suppose that $V_\Phi(f,E) < +\infty$ and that for each J, such that $J \cap E$ contains at least two points, the restriction $f_{/J\cap E}$ is not a constant mapping. Then, by Corollary 2, the function $\varphi = v_{\Phi,f}$ is increasing in E. Let $\varphi^{-1} : T = \varphi(E) \to E$ be its inverse, then $f = g \circ \varphi$ with $g = f \circ \varphi^{-1}$. The mapping $g : T \to X$ is uniformly continuous: $\forall\ \epsilon > 0$ let $\delta_\epsilon = h(\epsilon)\phi_{k(\epsilon)}(\epsilon)$, where h and k are the functions defined in (1). Let $t_1, t_2 \in T$ and $x_i =$

$\varphi^{-1}(t_i),\ i = 1, 2$. By Corollary 1 we get that if $|t_1 - t_2| = |\varphi(x_1) - \varphi(x_2)| < \delta_\epsilon$ then $d(g(t_1), g(t_2)) = d(f \circ \varphi^{-1}(t_1), f \circ \varphi^{-1}(t_2)) = d(f(x_1), (x_2)) < \epsilon$.

Now, to each collection $\{\tilde{I}_n = [\alpha_n, \beta_n]\}$ of intervals whose endpoints are in T we associate the collection $\{I_n\}$ of non-overlapping intervals whose endpoints are in E, $I_n = [a_n, b_n]$, with $a_n = \varphi^{-1}(\alpha_n)$ and $b_n = \varphi^{-1}(\beta_n)$ fulfilling

$$\sum_n \phi_n(|g(\tilde{I}_n)|) = \sum_n \phi_n(d(g(\varphi(a_n)), g(\varphi(b_n))) = \sum_n \phi_n(|f(I_n)|). \tag{2}$$

Vice-versa, with the same technique, to each collection $\{I_n = [a_n, b_n]\}$ of intervals whose endpoints are in E we associate the intervals collection $\{\tilde{I}_n = [\alpha_n, \beta_n]\}$ with $\alpha_n = \varphi(a_n), \beta_n = \varphi(b_n) \in T$, fulfilling (2). So we have proved that $V_\Phi(f, E) = V_\Phi(g, T)$.

Step 2. We suppose, now, that $V_\Phi(f, E) < +\infty$ and that there exists an interval J such that $J \cap E$ contains at least two points and the restriction $f_{/J\cap E}$ is constant. Let $\{J_i,\ \ i \in I\}$ be the collection of the maximal intervals with respect to such property, in the sense that $f_{/J_i \cap E} \equiv c_i$ and, if $J \supset J_i$ is an interval such that $J \cap E \neq J_i \cap E$, there exists at least one point $x \in J \cap E$ such that $f(x) \neq c_i$. Obviously such intervals are pairwise disjoint and the set I is, at most, countable. Let $\tilde{E} = (E \setminus \cup J_i) \cup \{\xi_i : i \in I\}$ with $\xi_i \in E \cap J_i$ arbitrarily fixed, and let $\tilde{f} = f_{/\tilde{E}}$. Then $V_\Phi(\tilde{f}, \tilde{E}) = V_\Phi(f, \tilde{E}) \leq V_\Phi(f, E) < +\infty$. Therefore the mapping $\tilde{f} : \tilde{E} \to X$ is of the type described in step 1, then the real function $\varphi = v_{\Phi,\tilde{f}}$ is nondecreasing in $\mathbb{R}$, and $\varphi_{/\tilde{E}}$ is increasing. Let g be the real function defined in $T = \varphi(\tilde{E})$ by $g(t) = \tilde{f} \circ \varphi^{-1}(t)$. It was proved in step 1 that $V_\Phi(g, T) = V_\Phi(\tilde{f}, \tilde{E})$ and $\tilde{f} = g \circ \varphi$.

Now we are proving that $V_\Phi(\tilde{f}, \tilde{E}) = V_\Phi(f, E)$, and $v_{\Phi,\tilde{f}}(x) = v_{\Phi,f}(x),\ \ \forall x \in \mathbb{R}$.

Let $\{I_n = [a_n, b_n]\}$ be an arbitrary collection of intervals whose endpoints are in E, such that $|f(I_n)| \neq 0\ \forall\ n$, we get the collection of intervals $I'_n = [a'_n, b'_n]$, as follows:

a) if $a_n \in E \setminus \cup J_i$ and $b_n \in J_h \cap E,\ \ h \in I$, (respectively, $b_n \in E \setminus \cup J_i$ and $a_n \in J_h \cap E$) we set $a'_n = a_n$ and $b'_n = \xi_h$ (resp., $b'_n = b_n$ and $a'_n = \xi_h$);

b) if $a_n \in J_h \cap E$ and $b_n \in J_k \cap E$ with $h, k \in I,\ \ h \neq k$, we set $a'_n = \xi_h$ and $b'_n = \xi_k$;

c) if $a_n, b_n \in E \setminus \cup J_i$ we set $a'_n = a_n$ and $b'_n = b_n$;

Therefore

$$\sum_n \phi_n(|f(I_n)|) = \sum_n \phi_n(|f(I'_n)|) = \sum_n \phi_n(|\tilde{f}(I'_n)|) \leq V_\Phi(\tilde{f}, \tilde{E}).$$

Thus $V_\Phi(f,E) \leq V_\Phi(\tilde f,\tilde E) = V_\Phi(f,\tilde E) \leq V_\Phi(f,E)$ and the equalities follow. Analogously, for each $x\in\mathbb{R}$ we have $V_\Phi(\tilde f,\tilde E\cap]-\infty,x]) = V_\Phi(f,E\cap]-\infty,x])$ and then $v_{\Phi,\tilde f}(x) = v_{\Phi,f}(x)$.

Let $x \in E$. If $x\in\tilde E$ we have $f(x) = \tilde f(x) = g(\varphi(x))$. If $x\in E\setminus\tilde E$ there exists $h\in I$ such that $x\in J_h$, therefore, by Corollary 2 and the above passages, $\varphi(x)=\varphi(\xi_h)$. Then $g(\varphi(x)) = g(\varphi(\xi_h)) = f(\xi_h) = f(x)$. Moreover $T=\varphi(\tilde E)=\varphi(E)$.

Step 3. In this step we suppose that $f=g\circ\varphi$ with $\varphi: E\to\mathbb{R}$ bounded, nondecreasing and $g: T=\varphi(E)\to X$ such that $V_\Phi(g,T)<+\infty$.

In case φ is increasing, if $\{I_n=[a_n,b_n]\}$ is a collection of intervals whose endpoints are in E, the intervals $\tilde I_n=[\alpha_n,\beta_n]$, where $\alpha_n=\varphi(a_n)$ and $\beta_n = \varphi(b_n)$, are non-overlapping with endpoints in T. Respectively, if $\tilde I_n=[\alpha_n,\beta_n]$ is a collection of intervals whose endpoints are in T, the intervals $I_n=[a_n,b_n]$, where $a_n=\varphi^{-1}(\alpha_n)$ and $b_n=\varphi^{-1}(\beta_n)$, are non-overlapping with endpoints in E. Then

$$\sum_n \phi_n(|f(I_n)|) = \sum_n \phi_n(|g\circ\varphi(I_n)|) = \sum_n \phi_n(|g(\tilde I_n)|).$$

By the arbitrariness of the $\{I_n\}$ (resp. $\{\tilde I_n\}$) we obtain $V_\Phi(f,E)=V_\Phi(g,T)$.

If there exist intervals J such that $J\cap E$ contain at least two points and the restrictions $\varphi_{/J\cap E}$ are constant mappings, we can proceed as in step 2 in order to find the non-overlapping intervals $J_i\subseteq E$ maximal with respect to such a property. Let $\tilde E=(E\setminus\cup J_i)\cup\{\xi_i\}$ with $\xi_i\in E\cap J_i$ arbitrarily chosen, and let $\tilde\varphi=\varphi_{/\tilde E}$.

Let $I_n=[a_n,b_n]$ be a collection of intervals whose endpoints are in E, such that $\varphi(a_k)\neq\varphi(b_k)$. In the following the symbols have analogous meaning to those in step 2: let $I'_n=[a'_n,b'_n]\subseteq\tilde E$ and $\tilde I_n=[\alpha_n,\beta_n]$, where $\alpha_n=\varphi(a'_n)$ and $\beta_n=\varphi(b'_n)$, are non-overlapping with endpoints in T. We have

$$\sum_n \phi_n(|g\circ\varphi(I_n)|) = \sum_n \phi_n(|g\circ\varphi(I'_n)|) = \sum_n \phi_n(|g(\tilde I_n)|) \leq V_\Phi(g,T).$$

Therefore $V_\Phi(f,E)\leq V_\Phi(g,T)$. Analogously we can obtain the inverse inequality, and then we get $V_\Phi(f,E)=V_\Phi(g,T)$. □

Theorem 2. *Let $\emptyset\neq E\subseteq\mathbb{R}$ and (X,d) be a complete metric space. If $f\in\Phi\text{-}BV(E,X)$, then there exists $\widetilde f\in\Phi\text{-}BV(\mathbb{R},X)$ such that $\widetilde f/E=f$ and $V_\Phi(\widetilde f,\mathbb{R})=V_\Phi(f,E)$.*

PROOF. Let $f \in \Phi\text{-}BV(E,X)$. By Theorem 1, $f = g \circ \varphi$, with $\varphi : E \to [0, V_\Phi(f,E)]$ non-decreasing and $g : T = \varphi(E) \to X$ uniformly continuous and of Φ-bounded variation such that $V_\Phi(g,T) = V_\Phi(f,E)$.

Let $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be the Saks' extension of the function φ ([18, Ch. 7, Sec. 4]); i.e.

$$\widetilde{\varphi}(t) = \begin{cases} \sup\{\varphi(s) : s \in E_t, \} & \text{if} \quad E_t = (-\infty, t] \cap E \neq \emptyset, \\ \\ \inf\{\varphi(s) : s \in E\} & \text{if} \quad E_t = \emptyset. \end{cases}$$

Clearly $\widetilde{\varphi}$ is bounded and nondecreasing. Moreover $\widetilde{\varphi}(\mathbb{R}) \subseteq \overline{\varphi(E)} = \overline{T} \subseteq [0, V_\Phi(f,E)]$. Let $\widetilde{g} : \overline{T} \to X$ be the continuous extension of g. Since g is uniformly continuous, also $\widetilde{g}$ is uniformly continuous. Now we prove that $V_\Phi(g,T) = V_\Phi(\widetilde{g}, \overline{T})$. Let $\{J_n = [\alpha_n, \beta_n]\}$ be a sequence of non-overlapping intervals whose endpoints $\alpha_n, \beta_n \in \overline{T}$. Fixed $\epsilon > 0$ and $N \in \mathbb{N}^+$ there exist $a_n, b_n \in T$, $n = 1, 2, \ldots, N$, such that the intervals $\{I_n = [a_n, b_n] : n = 1, 2, \ldots, N\}$ are non overlapping and

$$d(\widetilde{g}(\alpha_n), \widetilde{g}(a_n)) < \epsilon / M 2^{n+1}, \quad d(\widetilde{g}(\beta_n), \widetilde{g}(b_n)) < \epsilon / M 2^{n+1}$$

where $M = \phi_1(V_\Phi(f,E) + 1)$. Then

$$\begin{aligned} |d(\widetilde{g}(\alpha_n), \widetilde{g}(\beta_n)) - d(\widetilde{g}(a_n), \widetilde{g}(b_n))| &\leq |d(\widetilde{g}(\alpha_n), \widetilde{g}(a_n)) + d(\widetilde{g}(b_n), \widetilde{g}(\beta_n))| \\ &\leq \epsilon / M 2^n. \end{aligned}$$

Therefore, in view of Remark 2, $|\phi_n(\widetilde{g}(J_n)) - \phi_n(\widetilde{g}(I_n))| \leq \epsilon / 2^n$, and

$$\sum_{n=1}^{N} \phi_n(\widetilde{g}(J_n)) \leq V_\Phi(\widetilde{g}, T) + \epsilon.$$

The arbitrariness of ϵ, N and of the sequence $\{J_n\}$ implies $V_\Phi(\widetilde{g}, \overline{T}) \leq V_\Phi(\widetilde{g}, T) = V_\Phi(g,T)$. The equality follows by the property of monotonicity of the total Φ-variation.

The mapping $\widetilde{f} = \widetilde{g} \circ \widetilde{\varphi} : \mathbb{R} \to X$ is the required extension of f. □

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