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Real Analysis Exchange
Vol. 35(1), 2009/2010, pp. 65–78

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NON-CONTINUOUS FUNCTIONS ASSOCIATED WITH A COVERING PROPERTY DEFINED BY β-OPEN SETS

Abstract

In this paper, new classes of non-continuous functions and multifunctions stronger than lower (upper) β-continuous functions and multifunctions due to Popa and Noiri [28] and [30] are introduced and investigated for characterizing the covering property β-closedness [7] from different angles.

1 Introduction.

The notions of β-open [1] (=semi-preopen [4]) sets and β-closure of a set in general topology have received a great deal of study in recent years; some of which are found in papers [1-8, 12, 16-18, 19-20, 27-30]. In the paper [1], using β-open sets, Abd El-Monsef et al. introduced the notion of β-continuity as a generalization of semi-continuity [22]. Borsík and Doboš [13] have initiated the notion of almost quasi-continuity. The equivalence of β-continuity and almost quasi-continuity is shown by Borsík [12] and Ewert [17]. The notion of β-continuity has been generalized to multifunction by Popa and Noiri [30]. In this paper, we initiate new classes of non-continuous functions stronger

Mathematical Reviews subject classification: Primary: 54C10, 54C60, 54D20, 54D99
Key words: β-open set, β-closure, β-closed space, β-regular sets, β-θ-closed sets, β-continuous, lower (resp. upper) β-θ-continuous
Received by the editors August 17, 2008
Communicated by: Udayan Darji

than lower (resp. upper) β-continuous functions and multifunctions [28, 30] for studying the covering property β-closedness which has been investigated recently by Basu and Ghosh [7]. Although β-closedness is independent of compactness and stronger than quasi H-closed (QHC, in short) but employing these newly introduced non-continuous functions as well as multifunctions and certain kind of partial orders, we produce analogues for β-closed spaces of the following well known theorems: first of G. Birkhoff [10] that each lower (resp. upper) semi-continuous function from a compact space X to a poset assumes a minimal (resp. maximal) value, second of the corresponding theorem in terms of multifunctions of J. Ceder [15] and in the end, of a theorem of A. D. Wallace [33] that a compact space X has a minimal (resp. maximal) element with respect to each lower (resp. upper) semi-continuous quasi-order on X.

Throughout the paper, X and Y denote topological spaces and $int(S)$ and $cl(S)$ denote the interior and the closure of a subset $S \subset X$ respectively. A subset S is said to be β-open [1] or semi-preopen [4] (resp. preopen [24], semi-open [22], α-open [26]) if $S \subseteq cl(int(cl(S)))$ (resp. $S \subseteq int(cl(S))$, $S \subseteq cl(int(S))$, $S \subseteq int(cl(int(S)))$. β-closed, preclosed etc. are defined in a manner analogous to the corresponding concept of closed sets. The intersection of all β-closed (resp. preclosed, semi-closed, α-closed) sets containing S is called the β-closure [4] (resp. preclosure, semiclosure, α-closure) of S and is denoted by $\beta cl(S)$ (resp. $pcl(S)$, $scl(S)$, $\alpha cl(S)$). A set S is called β-regular [7] (=sp-regular [27]) (resp. semi-regular [23]) if it is both β-open (resp. semi-open) as well as β-closed (resp. semi-closed). A point $x \in X$ is said to be a β-θ-adherent point of a subset S of X if $S \cap \beta cl(U) \neq \emptyset$ for every β-open set U containing x. The set of all β-θ-adherent points of S is called β-θ-closure [7] (=sp-θ-closure [27]) of S and is denoted by β-θ-$cl(S)$ (= sp-θ-$cl(S)$). A subset S is called β-θ-closed [7] (= sp-θ-closed) if β-θ-$cl(S) = S$. The complement of a β-θ-closed set is called a β-θ-open (=sp-θ-open [27]) set. The family of all β-open (resp. pre-open, semi-open, α-open, β-regular, β-θ-open) sets of X is denoted by $\beta O(X)$ (resp. $PO(X)$, $SO(X)$, $\tau_\alpha(X)$, $\beta R(X)$, β-θ-$O(X)$) and that containing a point x of X is denoted by $\beta O(X,x)$ (resp. $PO(X,x)$, $SO(X,x)$, $\tau_\alpha(X,x)$, $\beta R(X,x)$, β-θ-$O(X,x)$). It is well known that for a topological space (X,τ), $\tau \subseteq \tau^\alpha(X) = PO(X) \cap SO(X) \subseteq PO(X) \cup SO(X) \subseteq \beta O(X)$, whereas the reverse inclusions are false. One can check easily that a set S is β-θ-open if for each $x \in S$, there exists a $V \in \beta R(X,x)$ such that $x \in V \subset S$. A net (x_μ) is said to β-θ-converge [7] to a point x of X if it is eventually in $\beta cl(V)$ for every $V \in \beta O(X,x)$. A multifunction $\alpha : X \to Y$ is a correspondence from X to Y with $\alpha(x)$, a non-empty subset of Y, for each $x \in X$. For a multifunction $\alpha : X \to Y$, the upper and lower inverse of a set B of Y will be denoted by $\alpha^+(B)$ and $\alpha^-(B)$ respectively; i.e. $\alpha^+(B) = \{x \in X : \alpha(x) \subseteq B\}$ and $\alpha^-(B) = \{x \in X : \alpha(x) \cap B \neq \emptyset\}$. For lower (resp. upper) semi-continuous multifunctions we refer to the book of C. Berge [9].

2 β-Closed Spaces, Lower (Resp. Upper) β-θ-Continuous Functions and Partial Orders.

Definition 2.1. *A function $\psi : X \to Y$ is said to be β-θ-continuous (resp. β-continuous [1]) if the inverse image of each open set is β-θ-open (resp. β-open).*

Definition 2.2. *A function $\psi : X \to \mathbb{R}$, where $\mathbb{R}$ is the real line is said to be upper (resp. lower) β-θ-continuous if for each $r \in \mathbb{R}$, the set $\{x \in X : \psi(x) < r\}$ (resp. $\{x \in X : \psi(x) > r\}$) is β-θ-open in X.*

If for the function $\psi : X \to \mathbb{R}$, the set $\{x \in X : \psi(x) < r\}$ (resp. $\{x \in X : \psi(x) > r\}) \in \beta O(X)$ for each $r \in \mathbb{R}$, then ψ is called upper (resp. lower) β-continuous [30].

Theorem 2.3. *A function $\psi : X \to \mathbb{R}$ is lower (resp. upper) β-θ-continuous if and only if for each $\mu \in \mathbb{R}$, $\{x \in X : \psi(x) \leq \mu\}$ (resp. $\{x \in X : \psi(x) \geq \mu\}$) is β-θ-closed in X.*

Corollary 2.4. *A subset S of X is β-θ-open (resp. β-θ-closed) if and only if the characteristic function $\chi_S : X \to \mathbb{R}$ is lower (resp. upper) β-θ-continuous.*

Remark 2.5. *The lower (resp. upper) β-θ-continuity is independent respectively of lower (resp. upper) semi-continuity. The following examples establish these facts.*

Example 2.6. *Let $X = \mathbb{R}$, the set of reals and $\tau = \{\emptyset, X, \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rationals. We define a function $\psi : X \to \mathbb{R}$ by $\psi(2) = 3$ and $\psi(\mathbb{Q} \setminus \{2\}) = \{4\}$ and $\psi(\mathbb{R} \setminus \mathbb{Q}) = \{3.5\}$. Then ψ is neither lower nor upper semi-continuous. Since $\beta O(X) = \beta R(X) = P(X)$, ψ is obviously lower as well as upper β-θ-continuous.*

Example 2.7. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Then obviously $\beta O(X) = SO(X) = \tau$ and $\beta R(X) = \beta$-θ-$O(X) = \{\emptyset, X, \{a\}, \{b, c\}\}$. Now we define a function ψ_1 as follows $\psi_1(a) = \psi_1(c) = 2$ and $\psi_1(b) = 3$ and put $\psi_2 = -\psi_1$. Then ψ_1 is lower semi-continuous but not lower β-θ-continuous and ψ_2 is upper semi-continuous but is not upper β-θ-continuous.*

Theorem 2.8. *Let* $\{\psi_\alpha : X \to \mathbb{R} : \alpha \in I\}$ *be any family of* β-θ*-continuous maps. Then*

(a) $M(x) = \sup\{\psi_\alpha(x) : \alpha \in I\}$ *(if exists) is lower* β-θ*-continuous.*

(b) $m(x) = \inf\{\psi_\alpha(x) : \alpha \in I\}$ *(if exists) is upper* β-θ*-continuous.*

Proof. (a) The proof is immediate from the fact that for each $r \in \mathbb{R}$, $\{x \in X : M(x) > r\} = \cup_{\alpha\in I}\{x \in X : f_\alpha(x) > r\}$.
(b) The proof is similar to (a). □

Remark 2.9. *The above theorem can be generalized if* β-θ*-continuity of* ψ_α *in (a) is replaced by lower* β-θ*-continuity and in (b) is replaced by upper* β-θ*-continuity.*

Remark 2.10. *If* ψ_1 *and* ψ_2 *are two lower* β-θ*-continuous functions from* X *into* $\mathbb{R}$*, then the function* $\psi = \min\{\psi_1, \psi_2\}$ *may not be lower* β-θ*-continuous.*

Lemma 2.11. *[27] (a) If* $V \in \beta O(X)$*, then* $\beta cl(V) \in \beta O(X)$*.*
(b) If $V \in \beta R(X)$*, then* V *is* β-θ*-open in* X*, but the reverse is not true.*

Example 2.12. *Let* $X = \{x_1, x_2, x_3\}$, $\tau = \{\emptyset, X, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$.
$\beta O(X) = \{\emptyset, X, \{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}\}$.
β-θ-$O(X) = \{\emptyset, X, \{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}\}$.

Define $\psi_1 : X \to \mathbb{R}$ *and* $\psi_2 : X \to \mathbb{R}$ *as follows:*
$\psi_1(x_1) = 3$, $\psi_1(x_2) = 2$, $\psi_1(x_3) = 4$, $\psi_2(x_1) = 6$, $\psi_2(x_2) = 8$, $\psi_2(x_3) = 2$.

Clearly ψ_1 *and* ψ_2 *are lower* β-θ*-continuous; since* $\{x_2, x_3\}$ *is not* β-θ*-closed, the function* $\psi = \min\{\psi_1, \psi_2\}$ *is not lower* β-θ*-continuous.*

Definition 2.13. *A non-empty subset* A *of a topological space* X *is said to be* β*-closed relative to* X *(*β*-set, for short) if for every cover* $\{U_\alpha : \alpha \in I\}$ *of* A *by* β*-open sets of* X*, there exists a finite subset* I_0 *of* I *such that* $A \subseteq \cup\{\beta cl(U_\alpha) : \alpha \in I_0\}$*. If, in addition* $A = X$*, then* X *is called a* β*-closed space [7].*

Theorem 2.14. *[7] For a space* X*, the following are equivalent:*
(a) X *is* β*-closed.*
(b) Every family of β-θ*-closed sets having finite intersection property has a non-void intersection.*

(c) Every cover by β-regular sets has a finite subcover.
(d) Every net with a well order index set as domain has a β-θ-adherent point.

(e) Every net in X β-θ-adheres at some point in X.

Theorem 2.15. *Let $\psi : X \to \mathbb{R}$ be lower β-θ-continuous. If for some $\mu \in \mathbb{R}$, the set $\Delta_\mu = \{x \in X : \psi(x) \leq \mu\}$ is non-empty and β-closed relative to X (i.e. β-set) then ψ has a minimum value.*

Proof. By the definition of lower β-θ-continuity, for each $x \in X$, the set $\{y \in X : \psi(y) > \psi(x) - m\}$ for $m(> 0) \in \mathbb{R}$ is a β-θ-open set in X containing x. So, there exists $U_x \in \beta R(X, x)$ such that $\psi(x) - m < \psi(y)$ for each $y \in U_x$. Since Δ_μ is a β-set and $\{U_x : x \in \Delta_\mu\}$ is cover of Δ_μ by β-regular sets, there exist $x_1, x_2, \ldots, x_k \in \Delta_\mu$ such that $\{U_{x_1}, \ldots, U_{x_k}\}$ covers Δ_μ. Therefore, $\inf_{j\leq k} \psi(x_j) - m \leq \inf_{y\in\Delta_\mu} \psi(y) = \inf_{y\in X} \psi(y)$. The last equality follows from the fact that whenever $y \notin \Delta_\mu$ then $\psi(y) > \mu$. Therefore ψ is bounded below and hence $\inf_{y\in X} \psi(y)$ exists. Let $\beta = \inf_{y\in X} \psi(y)$. So, for each $k \in \mathbb{N}$, the set of naturals, $\psi(y_k) \leq \min\{\beta + \frac{1}{k}, \mu\}$ for some $y_k \in \Delta_\mu$. Since Δ_μ is β-closed (i.e. β-set), the net $\{y_k : k \in \mathbb{N}\}$ in Δ_μ has a β-θ-adherent point say y in Δ_μ. Clearly, for each $k \in \mathbb{N}$, $y_\lambda \in \Delta_{\beta+\frac{1}{k}}$ for all $\lambda \geq k$ where $\Delta_{\beta+\frac{1}{k}} = \{x \in X : \psi(x) \leq \beta + \frac{1}{k}\}$. Since ψ is lower β-θ-continuous then by Theorem 2.3, $\Delta_{\beta+\frac{1}{k}}$ is β-θ-closed in X and hence $y \in \Delta_{\beta+\frac{1}{k}}$, for each $k \in \mathbb{N}$. Therefore $\psi(y) \leq \beta$. So $\psi(y) = \beta$. □

Theorem 2.16. *Let a function $\psi : X \to \mathbb{R}$ be upper β-θ-continuous. If for some $\mu \in \mathbb{R}$, the set $D_\mu = \{x \in X : \psi(x) \geq \mu\}$ is non-empty and β-closed relative to X (i.e. β-set), then ψ has a maximum value.*

Proof. The proof is similar to the proof of the above theorem. □

Definition 2.17. *A partial order relation '$\leq$' on a topological space X is said to be upper (resp. lower) β-θ-continuous if for each $x^\star \in X$, the set $\{x \in X : x^\star \leq x\}$ (resp. $\{x \in X : x \leq x^\star\}$) is β-θ-closed in X.*

Theorem 2.18. *A topological space X is β-closed if and only if X has a maximal element with respect to each upper β-θ-continuous partial order on X.*

Proof. Let L be a chain in X. Since the partial order '$\leq$' on X is upper β-θ-continuous, for each $y \in X$ the set $L(y) = \{x \in X : y \leq x\}$ is β-θ-closed in X. Therefore the family $\mathcal{F} = \{L(y) : y \in L\}$ has the finite intersection property (since L is a chain). Since X is β-closed, then by Theorem 2.14, $\cap\mathcal{F} = \cap\{L(y) : y \in L\} \neq \emptyset$. If $y_0 \in \cap\mathcal{F}$, then $y \leq y_0$ for all $y \in L$. Hence by Zorn's lemma, X has a maximal element.

Conversely, suppose X be not β-closed. So by Theorem 2.14, there exists a net $\{x_\mu : \mu \in D\}$, where D is a well ordered directed set, which has no β-θ-adherent point in X. So, for each $x \in X$, there exists a $V \in \beta R(X, x)$ and a $\mu_0 \in D$ such that $x_\mu \notin V$ for all $\mu \geq \mu_0$. Consider $\mathcal{B} = \{V \in \beta R(X, x) : \exists\ \mu_0 \in D$ such that $x_\mu \notin V$ for $\mu \geq \mu_0\}$. Now, for each $V \in \mathcal{B}$, let $\mu(V)$ be the smallest element of D satisfying $x_\mu \notin V$ for all $\mu \geq \mu(V)$. Let $\mu(x)$ be the smallest element of $D^\star(x) = \{\mu(V) : V \in \mathcal{B}\}$. Clearly for each $x \in X$, $\mu(x)$ is the first element of $D^\star(x)$ such that $x \notin \beta$-θ-$cl\{x_\mu : \mu \geq \mu(x)\}$. So corresponding to $\mu(x)$ there exists a $V \in \mathcal{B}$, say $V_{\mu(x)}$ such that $\{x_\mu : \mu \geq \mu(x)\} \cap V_{\mu(x)} = \emptyset$ and hence for $\beta < \mu(x)$, $V_{\mu(x)} \cap \{x_\mu : \mu \geq \beta\} \neq \emptyset$. We define the relation on X as follows: $x_1 \leq x_2$ in X if and only if $\mu(x_1) \leq \mu(x_2)$. Clearly $\leq$ is a partial order on X.

In order to show '$\leq$' is upper β-θ-continuous it is sufficient to show that for each $x \in X$, $\{y \in X : x \leq y\}$ is β-θ-closed. Suppose there exists $x^\star \in X$ such that $y^\star \in \beta$-θ-$cl\{y \in X : x^\star \leq y\}$ but $y^\star < x^\star$. Now as discussed above $V_{\mu(y^\star)}$ is a β-regular set containing $y^\star$ such that $y \in V_{\mu(y^\star)}$ with $y^\star < y$; i.e. $\mu(y^\star) < \mu(y)$ and $V_{\mu(y^\star)} \cap \{x_\mu : \mu \geq \mu(y^\star)\} = \emptyset$; i.e. $y \notin \beta$-θ-$cl\{x_\mu : \mu \geq \mu(y^\star)\}$. But $\mu(y)$ is the smallest element such that $y \notin \beta$-θ-$cl\{x_\mu : \mu \geq \mu(y)\}$ — a contradiction. Therefore '$\leq$' is upper β-θ-continuous.

Next we shall show that $(X, \leq)$ has no maximal element. Indeed, if $y^\star \in X$ is a maximal element of X, then for some fixed $\mu_0 = \mu(y^\star) \in D$, β-θ-$cl\{x_\mu : \mu \geq \mu_0\} \subseteq \beta$-$\theta$-$cl\{x_\mu : \mu \geq \lambda\}$ for each $\lambda \in M = \{\mu(x) : x \in X\}$, where $\mu(x)$ is as discussed above. Then $x_{\mu_0} \in \beta$-θ-$cl\{x_\mu : \mu \geq \lambda\}$ for each $\lambda \in M$. But as $x_{\mu_0} \in X$, there must exist $\mu(x_{\mu_0}) \in M$ such that $x_{\mu_0} \notin \beta$-θ-$cl\{x_\mu : \mu \geq \mu(x_{\mu_0})\}$ which is a contradiction. □

Theorem 2.19. *A space X is β-closed if and only if X has a minimal element with respect to each lower β-θ-continuous partial order on X.*

Definition 2.20. *Let $\psi : X \to Y$ be a function where X is a topological space and $(Y, \leq)$ is a poset. Then ψ is said to be upper (resp. lower) β-θ-continuous if for each $y_0 \in Y$, the set $\psi^{-1}\{y \in Y : y_0 \leq y\}$ (resp. $\psi^{-1}\{y \in Y : y \leq y_0\}$) is β-θ-closed in X.*

Theorem 2.21. *A space X is β-closed if and only if each upper β-θ-continuous function from X into a poset assumes a maximal value.*

Proof. Let $\phi : X \to Z$ be an upper β-θ-continuous function, where X is β-closed and $(Z, \leq_Z)$ is a poset. On X, the relation '$\leq_X$' defined by $x_1 \leq_X x_2$ if and only if $\phi(x_1) \leq_Z \phi(x_2)$, is clearly a partial order relation. Since ϕ is upper β-θ-continuous, for each $x_0 \in X$, the set $\{x \in X : x_0 \leq_X x\} = \phi^{-1}\{z \in Z : \phi(x_0) \leq_Z z\}$ is β-θ-closed in X. Hence '$\leq_X$' is an upper β-θ-continuous partial order on X and since X is β-closed, by above Theorem 2.18, X has a maximal element, say $x^{'} \in X$. $\phi(x^{'})$ is therefore a maximal element of $\phi(X)$.

Conversely, let X be not β-closed. Then by Theorem 2.14, there exists a net $(x_\mu)_{\mu \in D}$ with a well ordered directed set $(D, \leq)$, with no β-θ-adherent point. We define a function $\phi : X \to D$ by $\phi(x) = \mu(x)$ for each $x \in X$, where $\mu(x)$ is the first element of the set $\{\mu_0 \in D : x \notin \beta\text{-}\theta\text{-}cl\{x_\mu : \mu_0 \leq \mu\}\}$. Since $(D, \leq)$ is well ordered, the function ϕ is well defined. It is clear that $\phi(X)$ has no maximal element. We shall show that ϕ is upper β-θ-continuous. For this we define a relation $\leq_X$ on X as follows: $x_1 \leq_X x_2$ if and only if $\mu(x_1) \leq \mu(x_2)$. Let $\mu(x^\star) \in D$ for some $x^\star \in X$. Then as $\{x \in X : x^\star \leq x\} = \phi^{-1}\{\mu \in D : \mu(x^\star) \leq \mu\}$ and $\{x \in X : x_0 \leq x\}$ is β-θ-closed (as discussed in the proof of Theorem 2.18), ϕ is upper β-θ-continuous. So we arrive at a contradiction. Therefore X is β-closed. □

Theorem 2.22. *A space X is β-closed if and only if each lower β-θ-continuous function from X into a poset assumes a minimal value.*

Remark 2.23. *It is clear that every lower (resp. upper) β-θ-continuous function is lower (resp. upper) β-continuous [28, 30] but the converses are not true, in general.*

Example 2.24. *Let $\mathbb{R}$ be the set of reals with the co-countable topology τ. Since a subset S is β-open if $S \subseteq cl(int(cl(S)))$, $\beta O(\mathbb{R}) = \{A \subseteq \mathbb{R} : A$ is uncountable or $A = \emptyset\}$. Also it is clear that β-θ-$O(R, \tau) = \{\emptyset, \mathbb{R}\}$. We define a function $\psi : (\mathbb{R}, \tau) \to (\mathbb{R}, \mathcal{U})$, where $(\mathbb{R}, \mathcal{U})$ is the set of reals with the usual topology $\mathcal{U}$, as follows:*

$$\psi(x) = \begin{cases} 1 & \text{if } x \in (-\infty, 0], \\ 2 & \text{if } x \in (0, \infty). \end{cases}$$

Then ψ is lower (resp. upper) β-continuous but is not lower (resp. upper) β-θ-continuous.

3 β-Closed Spaces and β-θ-Continuous Multifunctions.

In this section, we introduce lower (resp. upper) β-θ-continuous multifunctions to characterize β-closed spaces and investigate some of their properties also.

Definition 3.1. *[25] Let (X, τ) be a topological space and $CL(X)$ denotes the class of all non-empty closed subsets of X. The 'upper Vietories topology' denoted by τ_{UV} is the topology generated by the base $\{U^+ : U \in \tau\}$, where $U^+ = \{A \in CL(X) : A \subseteq U\}$.*

The 'lower Vietories topology' denoted by τ_{LV} is the topology generated by the subbase $\{U^- : U \in \tau\}$, where $U^- = \{A \subseteq CL(X) : A \cap U \neq \emptyset\}$.

The 'Vietories topology' denoted by τ_V is the topology generated by the subbase $\{U^+ : U \in \tau\} \cup \{U^- : U \in \tau\}$.

Definition 3.2. *A multifunction $\alpha : X \to Y$ is said to be lower (resp. upper) β-θ-continuous if (when viewed as a function) $\alpha : X \to (CL(Y), \tau_{LV})$ (resp. $\alpha : X \to (CL(Y), \tau_{UV})$) is β-θ-continuous.*

We deduce the following characterizations using the definitions of lower and upper Vietoris topologies τ_{LV} and τ_{UV}.

Theorem 3.3. *For a multifunction $\alpha : X \to Y$, the following are equivalent:*

(a) α is lower (resp. upper) β-θ-continuous.

(b) For each $x \in X$ and each open set U in Y with $x \in \alpha^-(U)$ (resp. $x \in \alpha^+(U)$), there exists a $V \in \beta R(X, x)$ such that $V \subseteq \alpha^-(U)$ (resp. $V \subseteq \alpha^+(U)$).

(c) For each open set U in Y, $\alpha^-(U)$ (resp. $\alpha^+(U)$) is β-θ-open in X.

(d) For each closed set F in Y, $\alpha^+(F)$ (resp. $\alpha^-(F)$) is β-θ-closed in X.

Theorem 3.4. *For a function $f : X \to \mathbb{R}$, let us define a multifunction as $F(x) = \{\mu \in R : f(x) \geq \mu\}$ for each $x \in X$. Then*

(a) f is lower β-θ-continuous if and only if F is lower β-θ-continuous.

(b) f is upper β-θ-continuous if and only if F is upper β-θ-continuous.

Proof. (a) Let f be a lower β-θ-continuous function and U be a non-empty open set in $\mathbb{R}$ such that $x \in F^-(U)$. Let $r \in F(x) \cap U$. Since U is open in $\mathbb{R}$, there is $r^{'} \in U$, $r' < r$. Since f is lower β-θ-continuous, $\{y \in X : f(y) > r^{'}\}$

is a β-θ-open set containing x. Hence there is a $V \in \beta R(X,x)$ such that $V \subseteq \{y \in X : f(y) > r^{'}\}$. Now let $z \in V$. Since $f(z) > r^{'}$, $r^{'} \in F(z) \cap U$. So $V \subseteq F^{-}(U)$. Therefore, by Theorem 3.3, F is lower β-θ-continuous.

Conversely, for each $\mu \in \mathbb{R}$, since $\{x \in X : \mu < f(x)\} = F^{-}(\mu, \infty)$ is β-θ-open then f is lower β-θ-continuous.
(b) The proof is similar to (a). □

Theorem 3.5. *A multifunction $\psi : X \to Y$ is lower β-θ-continuous if and only if $cl(\psi) : X \to Y$ is lower β-θ-continuous (where $cl(\psi)$ is defined as $(cl(\psi))(x) = cl_Y(\psi(x))$ for each $x \in X$).*

PROOF. Since for any open set U in Y, $\{x \in X : \psi(x) \cap U \neq \emptyset\} = \{x \in X : (cl(\psi))(x) \cap U \neq \emptyset\}$. Then the proof follows easily. □

Theorem 3.6. *Let $\alpha : X \to Y$ be a upper β-θ-continuous multifunction such that for each $x \in X$, $\alpha(x)$ is compact. Then for each β-set S of X, $\alpha(S)$ is compact.*

PROOF. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of $\alpha(S)$. For each $x \in X$, since $\alpha(x)$ is compact, there exists $U_{\alpha_1}, ..., U_{\alpha_k} \in \mathcal{U}$ such that $\alpha(x) \subseteq \cup_{i=1}^{k} U_{\alpha_i} = U_x$ (say). Since α is upper β-θ-continuous, then for each $x \in S$, there exists $V_x \in \beta R(X,x)$ such that $\alpha(V_x) \subseteq U_x$. But as S is a β-set, there exists $x_1, ..., x_n \in S$ such that $S \subseteq \cup_{i=1}^{n} V_{x_i}$. Hence $\alpha(S) \subseteq \cup_{i=1}^{n} U_{x_i}$ and therefore $\alpha(S)$ is compact. □

A topological space X is said to be β-connected [3] if X can not be expressed as the union of two non-empty disjoint β-open sets.

Lemma 3.7. *[1] Let A and Y be subsets of a space X. If $Y \in \tau^{\alpha}(X)$ and $A \in \beta O(X)$, then $A \cap Y \in \beta O(Y)$.*

Theorem 3.8. *Let $\alpha : X \to Y$ be a multifunction which is either lower β-θ-continuous or upper β-θ-continuous with connected values. If $S \subseteq X$ is α-open and β-connected subset of X, $\alpha(S)$ is connected.*

PROOF. Let $\alpha : X \to Y$ be a lower β-θ-continuous multifunction. Suppose there exist non-empty disjoint open sets V_1 and V_2 in the subspace $\alpha(S)$ such that $\alpha(S) = V_1 \cup V_2$. Then $V_i = U_i \cap \alpha(S)$ for open sets U_i in Y, $i = 1, 2$. Now, let $W_i = \{x \in X : \alpha(x) \cap U_i \neq \emptyset\}$, $i = 1, 2$. Since α is lower β-θ-continuous, each W_i for $i = 1, 2$, is β-θ-open and hence is β-open in X. Let

$A_i = W_i \cap S$. Since S is an α-open set, by Lemma 3.7, A_i is β-open in S. Clearly $S = A_1 \cup A_2$. Let $x \in S$. Since $\alpha(x) \subseteq V_1 \cup V_2$ and $\alpha(x)$ is connected, either $\alpha(x) \cap V_1 = \emptyset$ or $\alpha(x) \cap V_2 = \emptyset$. Hence either $x \notin W_1$ or $x \notin W_2$. Thus either $x \notin A_1$ or $x \notin A_2$. Therefore $S = A_1 \cup A_2$ where A_1, A_2 are non-empty disjoint β-open sets in the subspace S of X. So S is not β-connected — a contradiction. So $\alpha(S)$ is connected.

For the upper β-θ-continuous case, the proof is quite similar. □

Corollary 3.9. *Let $\alpha : X \to Y$, where Y is Hausdorff, be an upper β-θ-continuous multifunction with, for each $x \in X$, $\alpha(x)$ is connected as well as compact. If X is β-closed and $\alpha(X) = Y$, then Y is a continuum.*

Theorem 3.10. *A space X is β-closed if and only if every lower β-θ-continuous multifunction from X into the closed subsets of a space assumes a minimal value with respect to set inclusion relation.*

Proof. Let $\alpha : X \to Y$ be a lower β-θ-continuous multifunction from the β-closed space X into Y. Also let $CL(Y)$ be the set of all non-empty closed subsets of Y together with the set inclusion relation '$\subseteq$' as a poset. Let $F \in CL(Y)$ and let $x \notin \alpha^{-1}\{K \in CL(Y) : K \subseteq F\}$. Then for each $K \in CL(Y)$ for which $K \subseteq F$, $\alpha(x) \neq K$ and hence $\alpha(x) \cap (Y - F) \neq \emptyset$. Therefore, $x \in \alpha^{-}(Y - F)$. Hence by Theorem 3.3, there exists a $V \in \beta R(X, x)$ such that $V \subset \alpha^{-}(Y - F)$. So for each $v \in V$, $\alpha(v) - F \neq \emptyset$. Hence $V \cap \alpha^{-1}\{K \in CL(Y) : K \subseteq F\} = \emptyset$. So, $\alpha^{-1}\{K \in CL(Y) : K \subseteq F\}$ is β-θ-closed in X for each $F \in CL(Y)$. Therefore $\alpha : X \to (CL(Y), \subseteq)$ is lower β-θ-continuous function and hence by Theorem 2.22, α assumes a minimal value.

Conversely, let X be not β-closed. Then by Theorem 2.14, there is a net $S = \{x_\mu\}_{\mu \in D}$, where D is a well ordered directed set, such that S has no β-θ-adherent point in X. Let D have the order topology. We define a multifunction $\alpha : X \to D$ by $\alpha(x) = \{\mu \in D : \mu \geq \mu(x)\}$, where $\mu(x)$ is as in the proof of Theorem 2.21. Clearly $\alpha(x) \in CL(D)$ and as the set $\{\mu(x) : x \in X\}$ has no greatest element, α does not assume any minimal value with respect to set inclusion relation. In order to show that α is lower β-θ-continuous multifunction, it is enough to show that by Theorem 3.3, that $\alpha^{-}(V)$ is β-θ-open for each open set V of D. Suppose $x \in \alpha^{-}(V)$. Then $\alpha(x) \cap V \neq \emptyset$. Let $\mu_0 \in \alpha(x) \cap V$. Then by definition of α and $\mu(x)$ we have $x \notin \beta\text{-}\theta\text{-}cl\{x_\mu : \mu_0 \leq \mu\}$ with $\mu(x) \leq \mu_0$. So there is a non-empty $W \in \beta R(X, x)$ such that $W \cap \{x_\mu : \mu_0 \leq \mu\} = \emptyset$. Let $x_0 \in W$ be an arbitrary point. Then $x_0 \notin \beta\text{-}\theta\text{-}cl\{x_\mu : \mu_0 \leq \mu\}$. So $\mu(x_0) \leq \mu_0$ and hence $\mu_0 \in \{\mu \in D : \mu(x_0) \leq \mu\} = \alpha(x_0) \cap V$. So $x \in W \subseteq \alpha^{-}(V)$. Therefore $\alpha^{-}(V)$ is β-θ-open. This contradicts the hypothesis of the theorem. □

Theorem 3.11. *A space X is β-closed if and only if each upper β-θ-continuous multifunction from X into a T_1 space assumes a maximal value with respect to set inclusion relation.*

Proof. The proof of the sufficiency is quite similar to that of Theorem 3.10.

For the necessary part, let for a T_1 space Y, $\alpha : X \to Y$ be an upper β-θ-continuous multifunction. If we can show that $\alpha : X \to (P(Y), \subseteq)$ is upper β-θ-continuous, then the proof will be completed by Theorem 2.21. Let $F \in P(Y)$ and let $x \notin \alpha^{-1}\{K \subseteq Y : F \subseteq K\}$. Then obviously $F - \alpha(x) \neq \emptyset$ and let $z \in F - \alpha(x)$. So, $\alpha(x) \subseteq Y - \{z\}$, which is open since Y is T_1. Then by Theorem 3.3, there exists a $V \in \beta R(X, x)$ such that $\alpha(V) \subseteq Y - \{z\}$ and hence $z \in F - \alpha(V)$. Therefore $V \cap \alpha^{-1}\{K \subseteq Y : F \subseteq K\} = \emptyset$. So, $\alpha^{-1}\{K \subseteq Y : F \subseteq K\}$ is β-θ-closed. Therefore, α is an upper β-θ-continuous function. □

Theorem 3.12. *If $\alpha : X \to X$ is a multifunction on a β-closed space X which satisfies $\alpha(K)$ is β-θ-closed whenever K is β-θ-closed, then there exists a non-empty β-set S of X such that $\alpha(S) = S$.*

Proof. Let $\mathcal{G} = \{K \subseteq X : K$ is β-θ-closed and $\alpha(K) \subseteq K\}$. Clearly $\mathcal{G} \neq \emptyset$ as $X \in \mathcal{G}$. Let $\{K_\lambda : \lambda \in I\}$ be a linearly ordered subset of the poset $(\mathcal{G}, \subseteq)$. As X is β-closed, $K = \cap_{\lambda \in I} K_\lambda$ is a non-empty β-θ-closed set. Since $\alpha(K) \subseteq K_\lambda$ for each λ, $\alpha(K) \subseteq K$; i.e. $K \in \mathcal{G}$. Therefore, K is the g.l.b. of $\{K_\lambda : \lambda \in I\}$. Hence by Zorn's lemma, a minimal element of $\mathcal{G}$ is the required fixed set of α. □

Remark 3.13. *Clearly every lower (resp. upper) β-θ-continuous multifunction is lower (resp. upper) β-continuous [30], but the converses are not true. In the Example 2.24, when we define $\psi(x) = \{1\}$ for $x \in (-\infty, 0]$ and $\psi(x) = \{2\}$ for $x \in (0, \infty)$, the multifunction justifies our claim.*

Acknowledgement: The authors are thankful to the learned referee for some constructive comments and suggestions towards some improvements of the earlier version of the paper.

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