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COMPOSITE CONTINUOUS PATH SYSTEMS AND DIFFERENTIATION

Abstract

The concept of composite differentiation was introduced by O'Malley and Weil to generalize approximate differentiation. The concept of continuous path systems was introduced by us. This paper combines these concepts to introduce the notion of composite continuous path systems into differentiation theory. It is shown that a number of results that hold for composite differentiation and for continuous path differentiation also hold for composite continuous path differentiation. In particular, a composite continuous path derivative of a continuous function is a Baire class one function on some dense open set, and extreme composite continuous path derivatives of a continuous function are Baire class two functions. It is also shown that extreme composite continuous path derivatives of a Borel measurable function are Lebesgue measurable. Finally, for each composite continuous path system E, continuous functions typically do not have $E-$derived numbers with $E-$index less than one.

1 Introduction.

The derivative f' of a differentiable real valued function f defined on the real line $\mathbb{R}$ has been generalized in many ways. Generalizations have been achieved by restricting the path of the limit of the difference quotient of f at a fixed point x to a subset E_x of $\mathbb{R}$ — clearly, x must be a member of and a limit point of E_x. Bruckner, O'Malley and Thomson in [5] introduced their concept of path derivative and showed that many known generalized derivatives fall

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into this framework. They showed that most of the nice properties of these derivatives are due to the thickness of the paths as well as the way the path E_y intersects E_x whenever y is close to x for a collection $E = \{E_x : x \in \mathbb{R}\}$ — E is called a path system. (See Section 2 for the definition of a path system.) It is easily seen that the approximate derivative function is associated with a path system. It is shown in [5] that many nice properties possessed by generalized derivative functions are also possessed by path derivative functions. In [7] O'Malley shows that for a real valued function f possessing a finite approximate derivative f'_{ap} everywhere in $[0,1]$, there is a sequence of perfect sets X_n whose union is $[0,1]$ and a sequence of differentiable functions f_n such that $f_n = f$ over X_n and $f'_n = f'_{\mathrm{ap}}$ over X_n. It is also clear that a derivative of a real valued function whose domain is a subset X of $\mathbb{R}$ can be defined if X is a nonempty dense-in-itself set. Using this fact and the decomposition property of approximate derivatives, O'Malley and Weil [9] introduced the concept of composite differentiation of a real valued function defined on $\mathbb{R}$. (See Section 2 for the definition of composite derivative of a function.) Roughly speaking, $\mathbb{R}$ is written as a countable union of closed sets X_i, $i = 1, 2, \ldots$, such that the function restricted to X_i is differentiable for each i. Clearly each piecewise linear, continuous function has a composite derivative which need not be a Darboux function. It is known that composite derivative functions are Baire class one functions, [9] and that every approximate derivative function is a composite derivative, [7].

The notion of a continuous system of paths was introduced in [1], where the path system $E = \{\, E_x : x \in \mathbb{R} \,\}$, which consists of compact sets, is required to be continuous as a function from $\mathbb{R}$ into the metric space of compact subsets of $\mathbb{R}$ endowed with the Hausdorff metric. This notion leads to continuous path derivative of a real valued function defined on $\mathbb{R}$. Several nice properties of continuous path derivatives were shown in [1, 2], and [3]. The present paper extends the notion of continuous path system by using the idea behind composite differentiation. That is, we define the notion of composite continuous path system. (See Definition 3.1. below.) The main results concern various properties of composite continuous path derivatives, extreme composite continuous path derivatives and composite continuous path derived numbers. In Section 4, we generalize the main results given in [1], in particular it is shown that the composite continuous path derivative of a continuous real valued function is an element of Baire class one on a dense open set, the extreme composite continuous path derivatives of Borel measurable functions and of continuous functions are respectively Lebesgue measurable and members of Baire class two. In Section 5, some results concerning the typical properties of composite continuous path derived numbers of continuous real valued functions are presented that generalizes some of the results given in [3]. The definitions and results given here could be stated on the real line, however for simplicity we consider the interval $[a, b]$ as the ambient space.

2 Preliminaries.

In this section we give the notation and terminology. For subsets A of $[a, b]$, A^c is the complement of A in the interval $[a, b]$, and $d(x, A)$ is the usual distance from x to A. Let $f : [a, b] \to \mathbb{R}$ be a function and $x_0 \in [a, b]$. An extended real number α is called a *derived number* (*bilateral derived number*) of f at x_0 if there is a sequence $\{s_n\}_{n=1}^{\infty} \subset [a, b]$ with $\lim_{n\to\infty} s_n = x_0$ and $\lim_{n\to\infty} \frac{f(s_n)-f(x_0)}{s_n-x_0} = \alpha$ (resp., there are sequences $\{s_n\} \subset [a, b]$ and $\{t_n\} \subset [a, b]$ such that $s_n < x_0 < t_n$ for each n and $\lim_{n\to\infty} \frac{f(s_n)-f(x_0)}{s_n-x_0} = \lim_{n\to\infty} \frac{f(t_n)-f(x_0)}{t_n-x_0} = \alpha$.) A function $g : [a, b] \to \mathbb{R}$ is called a *derived function* (*bilateral derived function*) of f if $g(t)$ is a derived number (bilateral derived number) of f at t for each $t \in [a, b]$. A point p is a *point of accumulation* or *limit point (*respectively, *bilateral point of accumulation* or *bilateral limit point)* of $S \subset [a, b]$ if there is a sequence $\{s_n\}_{n=1}^{\infty}$ in S such that $s_n \neq p$ for $n \geq 1$ and $\lim_{n\to\infty} s_n = p$ (respectively, if there are sequences $\{s_n\}$ and $\{t_n\}$ in S such that for each $n \geq 1$, $s_n < p < t_n$ and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = p$.) Let $x \in [a, b]$. A *path leading to* x is a set $E_x \subseteq [a, b]$ containing x and having x as a point of accumulation. A *path system* is a collection $E = \{E_x : x \in [a, b]\}$ such that each E_x is a path leading to x. The restriction of a path system E on a set $A \subseteq [a, b]$ is $E|_A = \{E_x \in E : x \in A\}$. If for each x, E_x has x as a bilateral point of accumulation, E is called a *bilateral path system*. We should point out that, a bilateral path system or bilateral derived number on a closed interval $[a, b]$, is interpreted as unilateral at both endpoints a and b. Let $f : [a, b] \to \mathbb{R}$ and let $E = \{E_x : x \in [a, b]\}$ be a system of paths. If $\lim_{y\to x,\ y\in E_x} \frac{f(y)-f(x)}{y-x} = g(x)$ is finite, then f is $E-$*differentiable* at x and $f'_E(x) = g(x)$. The *extreme E-derivatives* of f at a point x are $\bar{f}'_E(x) = \limsup_{y\to x,\ y\in E_x} \frac{f(y)-f(x)}{y-x}$ and $\underline{f}'_E(x) = \liminf_{y\to x,\ y\in E_x} \frac{f(y)-f(x)}{y-x}$. If f is $E-$differentiable at every point x and $f'_E(x) = g(x)$, then f is said to be $E-$*differentiable* and g is called the $E-$*derivative* of f. The *extreme E-derivatives* of f are $\bar{f}'_E$ and $\underline{f}'_E$. Note that, if f is continuous and $E-$differentiable at x, the path E_x can be replaced with its closure $\overline{E}_x$. By a *decomposition of the interval* $[a, b]$ we mean closed sets X_n for $n = 1, 2, 3, \cdots$ such that $\cup_{n=1}^{\infty} X_n = [a, b]$. A function $f : [a, b] \to \mathbb{R}$ is said to be *compositely differentiable* to a function g: $[a, b] \to \mathbb{R}$ relative to the decomposition $\{X_n\}_{n=1}^{\infty}$ of the interval $[a,b]$ if for each n and each $x \in X_n$

$$\lim_{t\to x,\ t\in X_n} \frac{f(t)-f(x)}{t-x} = g(x).$$

In this case the above limit need not be unique unless x is a limit point of X_n.

3 Composite Continuous Path Systems.

In this section we introduce the composite continuous path systems by considering the compact subsets of the interval $[a,b]$ endowed with the Hausdorff metric as the underlying metric space.

Definition 3.1. *Let $A \subseteq [a,b]$ be closed and let $E = \{E_x : x \in [a,b]\}$ be a system of paths so that for each $x \in [a,b]$, E_x is a compact subset of the interval $[a,b]$. If the function $E : x \to E_x$ is a continuous function on A, then we say E is a continuous system of paths* on A. Let $\{A_i\}_{i=1}^{\infty}$ be a decomposition of $[a,b]$ such that the function $E : x \to E_x$ is a continuous function on A_i for each $i \geq 1$; then we say E is a *composite continuous system of paths* on $[a,b]$. A composite continuous path derivative is a path derivative with respect to a composite continuous path system. Similarly, extreme composite continuous path derivatives are defined.

Example 3.2. *There exists a path system $E = \{E_x : x \in [0,1]\}$ which is composite continuous but not continuous.*

Proof. Let $A \subset [0,1]$ be the Cantor set, $\{(c_n,d_n)\}_{n=1}^{\infty}$ be the complementary intervals of A in $[0,1]$, and let $\delta(x) = \frac{1}{2}d(x,A)$ for each $x \in [0,1] \setminus A$. Define the path system $E = \{E_x : x \in [0,1]\}$ as $E_x = A$ for $x \in A$. If $c_n < x < d_n$ for some $n = 1, 2, \cdots$, define $E_x = [x-\delta(x), x]$ for $c_n < x \leq \frac{c_n+d_n}{2}$, and $E_x = [x, x+\delta(x)]$ for each $\frac{c_n+d_n}{2} < x < d_n$. It is easy to see that E is a composite continuous path system, but not a continuous path system. $\square$

In general composite derivatives are not path derivatives unless some extra conditions are imposed, see [8]. The following theorem shows that each composite derivative that can be expressed as path derivative is also a composite continuous path derivative.

Theorem 3.3. *Let f and g be real valued functions on $[0,1]$. If g is a derived function of f and f has g as a composite derivative, then there exists a composite continuous path system $E = \{E_x : x \in [0,1]\}$ such that $f'_E = g$.*

PROOF. Let $A_i,\ i = 1, 2, 3, \cdots$, be the decomposition of $[0, 1]$ associated with the composite derivative g and $\lim_{t\to x,\ t\in A_i} \frac{f(t)-f(x)}{t-x} = g(x)$ whenever x is a limit point of A_i. Let P_i be the perfect part of A_i. Then $C_i = A_i \setminus P_i$ is countable. There is no loss in assuming that the F_σ set $P_n \setminus \cup_{k<n} P_k$ is not empty. Let K_{n_j} be closed sets such that $\cup_{j=1}^{\infty} K_{n_j} = P_n \setminus \cup_{k<n} P_k$, and for each x in K_{n_j} let $E_x = A_n$. Then $\lim_{t\to x, t\in E_x} \frac{f(t)-f(x)}{t-x} = g(x)$, and $\{E_x : x \in K_{n_j}\}$ is continuous. Let $C = [0, 1] \setminus \cup_{n=1}^{\infty} P_n$. Then C is a countable set contained in $\cup_{n=1}^{\infty} C_n$. For each singleton x in C let $E_x = \{x_n : n = 1, 2, 3, \cdots\} \cup \{x\}$, where $\{x_n\}$ is a sequence in $[0, 1]$ that converges to x and $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} = g(x)$. Clearly $\{E_x : x \in \{x\}\}$ is continuous for each $x \in C$. The collection $E = \{E_x : x \in [0, 1]\}$ so defined is a composite continuous path system such that $f'_E(x) = g(x)$ for each x. □

In Theorem 3.3, We may replace the derived function g of f with a bilateral one and use the following theorem to obtain a bilateral system of paths with each path being a perfect set.

Theorem 3.4. [[6], Lemma 1, Page 86]. *Let $f : [0, 1] \to \mathbb{R}$ be compositely differentiable to $g : [0, 1] \to \mathbb{R}$. Suppose, for each $x \in [0, 1]$, that $g(x)$ is a bilateral derived number of f at x. Then there exists a nondecreasing sequence P_n of perfect sets such that*

(i) every point of P_n is a bilateral limit point of P_{n+1},

(ii) $\cup_{n=1}^{\infty} P_n = [0, 1]$, and

(iii) $f|_{P_n}$ is differentiable to $g|_{P_n}$ for each $n \geq 1$.

Theorem 3.5. *Let f and g be real valued functions on $[0, 1]$. If g is a bilateral derived function of f and f has g as a composite derivative, then there exists a bilateral composite continuous path system $E = \{E_x : x \in [0, 1]\}$ such that for each x, E_x is a perfect set and $f'_E = g$.*

PROOF. Since f is compositely differentiable to g, using Theorem 3.4., we may obtain an increasing sequence of perfect sets P_n such that, (i) every point of P_n is a bilateral limit point of P_{n+1}, (ii) $\cup_{n=1}^{\infty} P_n = [0, 1]$, and (iii) $f|_{P_n}$ is differentiable to $g|_{P_n}$ for each $n \geq 1$. We have $[0, 1] = \cup_{n=1}^{\infty} P_n = \cup_{n=2}^{\infty} (P_n \setminus P_{n-1}) \cup P_1$. For each $x \in P_1$ let $E_x = P_2$. For $n \geq 2$ let K_{n_j} be closed sets such that $\cup_{j=1}^{\infty} K_{n_j} = P_n \setminus P_{n-1}$, and for $x \in K_{n_j}$ let $E_x = P_{n+1}$. Then for each $x \in K_{n_j}$, where $j = 1, 2, 3, \cdots$ (resp. $x \in P_1$) x is a bilateral limit point of $E_x = P_{n+1}$ (resp. $E_x = P_2$) and $\lim_{t\to x, t\in E_x} \frac{f(t)-f(x)}{t-x} = g(x)$. The collection $E = \{E_x : x \in [0, 1]\}$ so defined is a bilateral composite continuous path system with each path being a perfect set and $f'_E(x) = g(x)$ for each x. □

4 Baire Classification and Measurability.

Let us turn to the Baire classes of composite continuous path derivative functions and extreme composite continuous path derivatives. Here B_i denotes the Baire functions of class i for $i = 1,\ 2$. Recall that a function f is in B_1 if and only if each nonempty perfect subset P of the domain of f contains a point of continuity of $f|_P$, and a function f is in B_2 if and only if the sets $\{x : f(x) < r\}$ and $\{x : f(x) > r\}$ are $G_{\delta\sigma}$ sets for each $r \in \mathbb{R}$.

An approximate derivative is a derived function and a composite derivative, see [7] and thus it is a composite continuous path derivative. We know that all composite derivatives are members of B_1, see [9]. In [1] we gave an example of a B_2 function f and a continuous path system E with $\bar{f}'_E$ not being Borel measurable. Thus, in general an extreme composite continuous path derivative cannot be Borel measurable. It is also known that continuous path derivatives of arbitrary functions need not be elements of B_1, but continuous path derivatives of continuous functions are members of B_1, see [1]. In this section we give a generalization of Theorem 4.1. in the setting of composite continuous path systems.

Theorem 4.1. [[1], Theorem 5 and Theorem 16] *Let $E = \{E_x : x \in [a, b]\}$ be a continuous system of paths on $[a, b]$.*

a) For continuous functions $f : [a, b] \to \mathbb{R}$,

(i) $f'_E \in B_1$ whenever f'_E exits,

(ii) $\bar{f}'_E$ and $\underline{f}'_E$ are members of B_2.

b) For Borel measurable functions $f : [a, b] \to \mathbb{R}$, $\bar{f}'_E$ and $\underline{f}'_E$ are Lebesgue measurable.

As an immediate consequence of this theorem and the Baire Category Theorem we have the following result.

Theorem 4.2. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and let $E = \{E_x : x \in [0, 1]\}$ be a bilateral composite continuous system of paths. If $f'_E(x)$ exists for all $x \in [0, 1]$, then there is a dense open subset U of $[0, 1]$ such that f'_E restricted to U is a Baire class one function.*

Proof. Let $[0, 1] = \cup_{i=1}^{\infty} A_i$, where the path system E is continuous on each closed set A_i. Let I be a closed subinterval of [0,1] with positive length. Then by Baire Category Theorem for some $i \geq 1$, there exists a closed interval J of positive length with $J \subset (A_i \cap I)$. Since $E = \{E_x : x \in A_i\}$ is a continuous and bilateral system of paths on A_i, the path system $R = \{R_x = E_x \cap J : x \in J\}$

is a continuous path system on $J \subseteq A_i$. On the other hand the function $f : J \to \mathbb{R}$ is continuous and for each $x \in J$, $f'_R(x) = f'_E(x)$. Thus by Theorem 4.1., $f'_E = f'_R$ is of Baire class one on the closed interval J. Now let $\mathcal{I} = \{(a_i, b_i)\}_{i=1}^{\infty}$ be an enumeration of the open subintervals of [0,1] with rational endpoints and let $\mathcal{A} = \{K_i = (a_i, b_i) \in \mathcal{I} : f'_E\big|_{[a_i, bi]} \in B_1 \text{ on } [a_i, b_i]\}$. Let $U = \cup_{K_i \in \mathcal{A}} K_i$. It is clear that U is an open dense subset of $[0,1]$ and f'_E restricted to U is a Baire class one function. □

The next lemma is a Urysohn extension theorem for continuous path systems. We use the following theorem, which is proved in [4].

Theorem 4.3. *Let (X, d) be a metric space, let $\mathcal{Z}$ be a normed space, and let $\mathcal{F}(\mathcal{Z})$ be the metric space of non-empty bounded closed subsets of $\mathcal{Z}$ with Hausdorff metric d_H. Given any non-empty closed subset $A \subset X$ and any continuous mapping $F : A \to \mathcal{F}(\mathcal{Z})$, there exists a continuous mapping $G : X \to \mathcal{F}(\mathcal{Z})$ such that $G(x) = F(x)$ for each $x \in A$ and $G(x)$ lies in the closure of the convex hull of $\bigcup_{a \in A} F(a)$ for every $x \in X$.*

Lemma 4.4. *Let R be a continuous mapping from the closed set $A \subseteq [0,1]$ into $\mathcal{F}(\mathbb{R})$ (endowed with the Hausdorff metric d_H) such that, for each $x \in A$, $R(x)$ is a path at x and $R(x) \subseteq [0,1]$. Then there exists a continuous system of paths $E = \{E_x : x \in [0,1]\}$ such that $E_x = R(x)$ for each $x \in A$.*

Proof. Take $\mathcal{Z} = \mathbb{R}$ and $X = [0,1]$. Since R is a continuous mapping from the closed set $A \subseteq [0,1]$ into $\mathcal{F}(\mathbb{R})$, from Theorem 4.3., it follows that there exists a continuous mapping $G : [0,1] \to \mathcal{F}(\mathbb{R})$ such that $G(x) = R(x)$ for each $x \in A$ and $G(x)$ lies in the closure of the convex hull of $\bigcup_{a \in A} F(a) \subseteq [0,1]$. Let $\delta(x) = \frac{1}{2}\inf\{x, 1-x, d(x, G(x) \setminus \{x\})\}$ for $x \in (0,1)$, $\delta(0) = \frac{1}{2}d(0, G(0) \setminus \{0\})$, and let $\delta(1) = \frac{1}{2}d(1, G(1) \setminus \{1\})$. Let $E_0 = G(0) \cup [0, \delta(0)]$, $E_1 = G(1) \cup [1 - \delta(1), 1]$, and $E_x = G(x) \cup [x - \delta(x), x + \delta(x)]$ for each $x \in (0,1)$. It is easy to see that for each x, $E_x \subseteq [0,1]$ is a path at x, hence $E = \{E_x : x \in [0,1]\}$ is a path system and we have $d_H(E_x, E_y) \leq \mid x - y \mid + \mid \delta(x) - \delta(y) \mid + d_H(G(x), G(y))$ for each x and y in $X = [0,1]$. Thus E is a continuous system of paths on [0,1] with $E_x = R(x)$ for each $x \in A$. □

Theorem 4.5. *Let $E = \{E_x : x \in [0,1]\}$ be a composite continuous system of paths.*

(i) If $f : [0,1] \to \mathbb{R}$ is a continuous function, then $\underline{f}'_E$ and $\bar{f}'_E$ are B_2 functions.

(ii) If $f[0,1] \to \mathbb{R}$ is a Borel measurable function, then $\underline{f}'_E$ and $\bar{f}'_E$ are Lebesgue measurable.

PROOF. Let $[0,1] = \cup_{m=1}^{\infty} A_m$ be the associated decomposition where the path system E is continuous on each closed set A_m. Utilizing Lemma 4.4. for each $m \geq 1$, there exists a continuous path system $R_m = \{R_m(x) : x \in [0,1]\}$ such that $R_m(x) = E_x$ for each $x \in A_m$. In case (i), the function f is continuous and R_m is a continuous system of paths. Thus by Theorem 4.1., $\bar{f}'_{R_m} \in B_2$. Let $r \in \mathbb{R}$ be arbitrary. We then have

$$\begin{aligned}\{x \in [0,1] : \bar{f}'_E(x) > r\} &= \cup_{m=1}^{\infty}\{x \in A_m : \bar{f}'_E(x) > r\}\\ &= \cup_{m=1}^{\infty}(A_m \cap \{x \in [0,1] : \bar{f}'_{R_m}(x) > r\}),\end{aligned}$$

which is a $G_{\delta\sigma}$ set. Similarly the set

$$\begin{aligned}\{x \in [0,1] : \bar{f}'_E(x) < r\} &= \cup_{m=1}^{\infty}\{x \in A_m : \bar{f}'_E(x) < r\}\\ &= \cup_{m=1}^{\infty}(A_m \cap \{x \in [0,1] : \bar{f}'_{R_m}(x) < r\}),\end{aligned}$$

which is also a $G_{\delta\sigma}$ set. Thus $\bar{f}'_E \in B_2$. In a similar way we can show that $\underline{f}'_E \in B_2$.

In case (ii), the function f is Borel measurable, and for each $m \geq 1$, R_m is a continuous system of paths. Thus by Theorem 4.1., $\bar{f}'_{R_m}$ and $\underline{f}'_{R_m}$ are Lebesgue measurable. By the same argument as in case (i), we see that $\bar{f}'_E$ and $\underline{f}'_E$ are Lebesgue measurable. □

Of interest is the following question whose positive resolution will generalize Theorem 5.5.2. on page 209 of [10].

Question 4.6. *Let $A \subseteq [0,1]$ be closed and $R = \{R_x : x \in A\}$ be a continuous system of paths, such that for each $x \in A$, $R_x \subseteq [0,1]$. If the function $f : [0,1] \to \mathbb{R}$ is continuous and $R-$differentiable on A, is it possible to extend the path system R to a continuous path system $E = \{E_x : x \in [0,1]\}$ and the function $f|_A$ to a continuous function $g : [0,1] \to \mathbb{R}$ so that g is $E-$differentiable and $f'_R(x) = g'_E(x)$ for each $x \in A$?*

5 Typical Behavior.

As usual, in the normed space $C[0,1]$ of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the maximum norm, a property is said to hold typically if it is satisfied by members of a residual subset. In [3], the index of a path derived number of a function was defined, and it was shown that typically a continuous function has no finite $E-$derived number with index less than one at each x in $[0,1]$ when E is a continuous system of paths. Let us define the terms used in the property. We shall restrict the definitions to continuous functions.

We first define $E-$derived number at a point of $[0,1]$. Note that derived numbers were defined earlier. Here we introduce a path system $E = \{E_x : x \in [0,1]\}$, an extended real number α is called an $E-$derived number of a continuous function f at x if there exists a sequence $\{x_n\}$ in E_x such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} = \alpha$. The collection of $E-$derived numbers of f at x will be denoted by $D(E,f,x)$. The union $\cup\{D(E,f,x) : x \in [0,1]\}$, denoted by $D(E,f)$, is called the set of $E-$derived numbers of f.

For the definition of the index at x of an $E-$derived number in $D(E,f,x)$, we used the notation

$$\overline{\gamma}(\{x_n\},x) = \limsup_{n\to\infty} \frac{x_n - x_{n+1}}{x_n - x},$$

where $\{x_n\}$ is a monotone sequence that converges to x. Obviously, $0 \leq \overline{\gamma}(\{x_n\},x) \leq 1$. For α in $D(E,f,x)$, let $S(\alpha,E,f,x)$ be the collection of all monotone sequences in E_x that converge to x and satisfies $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} = \alpha$. The number

$$C(\alpha,E,f,x) = \inf\{\overline{\gamma}(\{x_n\},x) : \{x_n\} \in S(\alpha,E,f,x)\}$$

is called the $E-$index at x of α, where $\alpha \in D(E,f,x)$. The $E-$index of α in $[0,1]$ is defined to be

$$C(\alpha,E,f) = \inf\{C(\alpha,E,f,x) : x \in [0,1]\}.$$

With the aid of Lemma 4.4, the proof of Theorem 5.2 is reduced to the following theorem on continuous path systems which is Theorem 4 on page 361 of [3].

Theorem 5.1. *Let $E = \{E_x : x \in [0,1]\}$ be a continuous system of paths on $[0,1]$. Typically, a continuous function f has no finite $E-$derived number α with $C(\alpha,E,f) < 1$.*

We are now ready to state and prove our result on path derived numbers of continuous functions with respect to composite continuous path systems.

Theorem 5.2. *Let $E = \{E_x : x \in [0,1]\}$ be a composite continuous system of paths on [0,1]. Typically, a continuous function f has no finite $E-$derived number α with $C(\alpha,E,f) < 1$.*

Proof. Let $[0,1] = \cup_{m=1}^{\infty} A_m$ be the associated decomposition where the path system E is continuous on each closed set A_m. For each $m \geq 1$, Lemma 4.4. gives a continuous path system $R_m = \{R_m(x) : x \in [0,1]\}$ such that

$R_m(x) = E_x$ whenever $x \in A_m$. By Theorem 5.1. for each $m \geq 1$, the collection H_m of all continuous functions f such that f has a finite R_m-derived number α with $C(\alpha, R_m, f) < 1$ is a set of first category. Hence $H = \cup_{m=1}^{\infty} H_m$ is a set of first category. Let f be such that for some $x \in [0, 1]$ there is a finite α in $D(E, f, x)$ with $C(\alpha, E, f, x) < 1$. There is an m such that $x \in A_m$, as $R_m(x) = E_x$, it follows that α is an R_m-derived number with $C(\alpha, R_m, f) < 1$. That is $f \in H$. The theorem is proved. $\square$

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