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Joseph F. Vance, 16485 F.M. 1957, San Antonio, TX 78253, U.S.A. email: jfv37@aol.com

Charles A. Coppin, Department of Mathematics, Lamar University, P.O. Box 10047, Beaumont, TX 77710, U.S.A. email: ccoppin@gt.rr.com

REPRESENTATION OF LINEAR FUNCTIONALS ON QUASI-CONTINUOUS FUNCTIONS

Abstract

We prove a representation theorem for bounded linear functionals with domain the set of all real-valued, quasi-continuous functions defined on a closed interval; thus, giving a characterization of a class of bounded linear functionals.

1 Introduction.

In this paper, we use a modified mean Stieltjes integral defined on a dense subset of a closed interval whose end points belong to the dense subset. Ultimately, we prove a representation theorem for bounded linear functionals with domain the set of quasi-continuous functions with domain this dense subset. Quasi-continuous functions are also known as regulated functions. In 1934, H.S. Kaltenborn [12] characterized all the bounded linear functionals from the set of quasi-continuous on $[a, b]$ into a subset of the numbers in integral form but with remainder terms. In 1960, J.R. Webb [32] did the same using a single Hellinger integral without remainder terms. Baker [1], Priest [20] and Reneke [21] studied representation theorems for linear functionals for modified Stieltjes integrals with Baker and Reneke using quasi-continuous functions as the domain. See Fraňková [10], Pelant [19], Schwabik [22], and Tvrdý [26], [27], [28], [29]. Priest and Reneke use the mean Stieltjes integral, one of the

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subjects of this paper. R. E. Lane [14], [15] did extensive work on the mean Stieltjes integral.

Modified Stieltjes integrals defined on arbitrary number sets have been studied extensively. Coppin and Muth [6] studied an integral defined on subsets of a closed interval that were not necessarily dense in the closed interval. A special case of this integral was first defined by Coppin [3] and Vance [31] where the integral was defined over dense subsets of an interval containing the end points of that interval. Coppin [4], [5] studied additional properties of this particular modified integral. Coppin and Vance [7] showed necessary and sufficient conditions for f to be g-integrable on a dense subset of $[a, b]$ where $f|M$ and $g|M$ do not have common points of discontinuity.

The Riemann-Stieltjes integral remains a topic of significant interest. See, for example, D'yachkov [9], Kats [13], Liu and Zhao [17], and Tseytlin [25]. Modifications of the Stieltjes integral abound. One only has to sample some of the most recent papers. For some interesting results, see B. Bongiorno and L. Di Piazza [2], A.G. Das and Gokul Sahu [8], Ch. S. Hönig [11], Supriya Pal, D.K. Ganguly and Lee Peng Yee [18], Š. Schwabik, M. Tvrdý, and O. Vejvoda [23], Swapan Kumar Ray and A.G. Das [24] and Ju Han Yoon and Byung Moo Kim [33].

2 Preliminary Definitions and Properties.

Throughout this paper, $[a, b]$ will denote a closed number interval and M will denote a dense subset of $[a, b]$ containing a and b. In general, an interval (or an interval of M) is a set $[c, d]_M = [c, d] \cap M$ where c and d belong to M and $c < d$. Two intervals, A and B, are said to be nonoverlapping if and only if $A \cap B$ does not contain an interval. A nonempty collection of intervals is said to be nonoverlapping if and only if each two distinct members of the collection are nonoverlapping.

Definition 2.1. The collection D is said to be a partition of M if and only if D is a finite collection of non-overlapping subintervals of M whose union is M. $E(D)$ denotes the set of end points of members of D.

Definition 2.2. The partition D' of M is said to be a refinement of D if and only if each end point of a member of D is an end point of a member of D', that is, $E(D) \subseteq E(D')$.

Definition 2.3. If D is a partition of M, and f and g are functions with domain including M, then

$$\Sigma_m(f,g,D) = \sum_{[p,q]_M \in D} \frac{f(q)+f(p)}{2} \cdot [g(q)-g(p)] \tag{1}$$

Right sums, $\Sigma_r(f,g,D)$, are easily defined by replacing $(f(q)+f(p))/2$ in (1) with $f(q)$. Similarly, left sums are defined by replacing $(f(q)+f(p))/2$ in (1) with $f(p)$ to create $\Sigma_l(f,g,D)$.

Definition 2.4. Suppose that f and g are functions with domain including M. Then f is said to be mean g-integrable on M if and only if there exists a number W (called "the mean integral of f with respect to g" and denoted by $(m)\int_M f\,dg$) such that for each $\varepsilon > 0$, there is a partition D of M such that

$$|W - \Sigma_m(f,g,D')| < \varepsilon \tag{2}$$

for each refinement D' of D. Right integrals and left integrals are defined by replacing $\Sigma_m(f,g,D')$ in (2) with $\Sigma_r(f,g,D')$ and $\Sigma_l(f,g,D')$ and denoted by $(r)\int_M f\,dg$ and $(l)\int_M f\,dg$, respectively.

Note. All three integrals, $(m)(l)(r)\int_M f\,dg$, are linear. Moreover,

$$\left|(m)(l)(r)\int_M f\,dg\right| \le \|f\| \cdot V_a^b g \tag{3}$$

where the bounded function f is left, right, mean g-integrable on $[a,b]$ and g is of bounded variation on $[a,b]$.

By $\mathcal{QC}$ we mean the set of all real-valued quasi-continuous functions (both left and right hand limits exist) with domain M. Let $\mathcal{G}$ be the set of all characteristic functions $z_t^- = \mathbf{1}_{(t,b]\cap M}$ and $z_t^+ = \mathbf{1}_{[t,b]\cap M}$ where $t \in [a,b]$. We let $\mathcal{S}$ denote the set of all functions f with domain M where $(m)\int_M f\,dg$ exists for each $g \in \mathcal{G}$ and $\|f\| = \sup_{x\in M}|f(x)|$.

We show that each $L : \mathcal{S} \to \mathbb{R}$ (the set of real numbers) is a bounded, linear functional if and only if for each function $f \in \mathcal{S}$, there are functions, α and β, of bounded variation on $[a,b]$ such that

$$L(f) = (l)\int_M f_R\,d\alpha + (r)\int_M f_L\,d\beta$$

where each of f_R and f_L is a quasi-continuous function with domain M such that f_R is continuous on the right at each of its points, $f_R(b) = 0$, f_L is continuous on the left at each of its points, $f_L(a) = 0$, and $f = f_R + f_L$.

3 Properties of $\mathcal{S}$.

Theorem 3.1. *$\mathcal{S}$ is a linear space.*

Theorem 3.2. *Each member of $\mathcal{S}$ is bounded.*

PROOF. Suppose $f \in \mathcal{S}$. Let $t \in [a,b]$ and let $g \in \mathcal{G}$ where $g = z_t^-$ or $g = z_t^+$. Then, by definition of $\mathcal{S}$, $(m)\int_M f\,dg$ exists. By Definition 2.4, for both choices of g, we can infer that there is a partition D of M such that if D' is a refinement of D, then

$$|\Sigma_m(f,g,D) - \Sigma_m(f,g,D')| < 1. \tag{4}$$

Consider $[u,v]_M \in D$ where $u \le t \le v$. With the goal of showing f is bounded on $(u,v)\cap M$, let $x \in (u,v)\cap M$. Define $D' = (D \setminus [u,v]_M) \cup \{[u,x]_M, [x,v]_M\}$. Then, (4) reduces to

$$\left| \frac{f(u)+f(v)}{2}\cdot[g(v)-g(u)] - \frac{f(u)+f(x)}{2}\cdot[g(x)-g(u)] - \frac{f(x)+f(v)}{2}\cdot[g(v)-g(x)] \right| < 1$$

which, in turn, reduces to

$$|f(x)|\cdot|g(v)-g(u)| \le 2 + |f(u)|\cdot|g(v)-g(x)| + |f(v)|\cdot|g(x)-g(u)|. \tag{5}$$

In case $u < t < v$, because of the definition of $\mathcal{G}$ ($g = \mathbf{1}_{(t,b]}$, $g = \mathbf{1}_{[t,b]}$) and that $u < t < v$, we know that $|g(v)-g(u)| = 1$. Moreover, $|g(v)-g(x)| \le 1$ and $|g(x)-g(u)| \le 1$. As a result, (5) yields $|f(x)| \le 2 + |f(u)| + |f(v)|$. In case $t = u$, since (5) holds for $g = \mathbf{1}_{(t,b]}$, we can still conclude that $|f(x)| \le 2 + |f(u)| + |f(v)|$. For $t = v$, choose $g = \mathbf{1}_{[t,b]}$. From (5), we have $|f(x)| \le 2 + |f(u)| + |f(v)|$.

In summary, for each $t \in [a,b]$, there is an open interval (u,v) containing t such that $|f(x)| \le 2 + |f(u)| + |f(v)|$ for each $x \in (u,v)\cap M$. Therefore, f is bounded on $(u,v)\cap M$. By the Heine-Borel Theorem, there are finitely many of these open intervals H covering $[a,b]$.
$\therefore$ f is bounded.

$\square$

4 Lemmas Concerning Quasi-Continuous Functions.

The following results will be used later. Theorem 4.1, Definition 4.2, Definition 4.1, and Definition 4.3 are repeated here from Coppin and Muth [6] wherein we studied a Stieltjes integral defined over arbitrary subsets of a closed interval not just a dense subset such as M of this paper. The functions in that paper were assumed to be bounded.

Theorem 4.1. *If f is a function with domain $H \subseteq [a,b]$, z is a member of $[a,b]-H$ which is a limit point of the domain of $f|[a,z]$, then there is a number c such that (z,c) is a limit point of the graph of $f|[a,z]$. Similarly, if z is a limit point of the domain of $f|[z,b]$, then there is a number c such that (z,c) is a limit point of the graph of $f|[z,b]$.*

Definition 4.1. In Theorem 4.1, c is said to be a quasi-end value.

Definition 4.2. Suppose $\overline{H} \subseteq [a,b]$. Then a gap G in H (or gap G if no misunderstanding occurs) is a maximal connected subset of (a,b) which contains no points of H.

Definition 4.3. Suppose f is a function with domain $H \subseteq [a,b]$. By f^*, we mean a function such that

(a) $f^*(x) = f(x)$ for each $x \in H$, and

(b) if $x \in [a,b] - H$ and G is a gap containing x, then $f^*(x)$ is equal to a quasi-end value of f with respect to G. It is understood that when there is more than one choice for $f^*(x)$ then only one choice is made and is the same for each value in G. We repeat this process for each gap in H; therefore, f^* has domain $[a,b]$.

Theorem 4.2. *$f \in \mathcal{S}$ if and only if f^* is quasi-continuous.*

Proof of Necessity. Suppose $f \in \mathcal{S}$. Assume that f^* is not quasi-continuous at some $t \in [a,b]$. For the sake of argument, let $a < t$ and f^* is not quasi-continuous on the left at t. This implies that for some $k > 0$ there is an increasing sequence $\{x_n\}$ in M convergent to t such that

$$k < |f(x_n) - f(x_m)| \tag{6}$$

for each positive integer m and n. (Remember that for each $x \in [a,b]$, $x \in M$ or $x \in [a,b] - M$ and $(w, f^*(x))$ is a limit point of the graph of f for some w in some gap.)

Let $g \in \mathcal{G}$ where $g = \mathbf{1}_{[t,b]}$. By definition, $(m)\int_M f\,dg$ exists. As was done in the preceding proof, by Definition 2.4, for k, we know that there is a partition D of M such that if D' and D'' are refinements of D, then

$$|\Sigma_m(f,g,D') - \Sigma_m(f,g,D'')| < k/4. \tag{7}$$

Let $[u,t]_M$ be the member of D for which t is the right hand end point. Let D' be a refinement of D where $D' = (D \setminus [u,t]_M) \cup \{[u,x_n]_M, [x_n,t]_M\}$ for some positive integer n. Similarly, let D'' be a refinement of D where $D'' = (D \setminus [u,t]_M) \cup \{[u,x_m]_M, [x_m,t]_M\}$ for some positive integer m. Because g is 0 or is a constant on all members of D' and D'', except $[x_n,t]_M$ and $[x_m,t]_M$, we conclude that

$$\left| \frac{f(t)+f(x_n)}{2} \cdot [g(t)-g(x_n)] - \frac{f(t)+f(x_m)}{2} \cdot [g(t)-g(x_m)] \right| < k/2. \tag{8}$$

By definition of g, $|g(t)-g(x_n)| = 1$ and $|g(t)-g(x_n)| = 1$. Thus, (8) reduces to

$$|f(x_n) - f(x_m| < k$$

which contradicts (6). Therefore, f^* is quasi-continuous. □

PROOF OF SUFFICIENCY. Suppose f^* is a quasi-continuous function. Clearly, f is also quasi-continuous. To show that $f \in \mathcal{S}$, let $t \in [a,b]$.

Case 1. $t \in M$. For the sake of argument, let $a < t < b$ and $g = \mathbf{1}_{[t,b]}$. Let $\varepsilon > 0$. Since, f is quasi-continuous at t, there is a positive number δ such that $|f(x)-f(y)| < \varepsilon/2$ for each $x,y \in (t-\delta/2,t)\cap M$ and for each $x,y \in (t,t+\delta/2)\cap M$. Let D be a partition of M such that for some $[u,t]_M, [t,v]_M \in D$, $|v-u| < \delta$. Let D' be any refinement of D where $[r,t]_M, [t,s]_M \in D'$, and, of course, $|s-r| < \delta$. Then

$$|\Sigma_m(f,g,D) - \Sigma_m(f,g,D')| =$$
$$\left| \frac{f(t)+f(u)}{2} \cdot [g(t)-g(u)] - \frac{f(t)+f(v)}{2} \cdot [g(t)-g(v)] \right| =$$
$$|f(u)-f(v)| < \varepsilon.$$

Summarizing, we have

$$|\Sigma_m(f,g,D) - \Sigma_m(f,g,D')| < \varepsilon.$$

The proof of case $g_l = \mathbf{1}_{(t,b]}$ would develop in a similar manner as would $t = a$ and $t = b$. Thus, we conclude that f is mean g-integrable on M.

Case 2. $t \notin M$. With minor changes, this case can be argued very much like Case 1. In the interest of space, we omit the proof that f is mean g-integrable on M.

Therefore, if f^* is quasi-continuous, then $f \in \mathcal{S}$. □

Lemma 4.2.1. *If $f \in \mathcal{S}$, then f^* is unique and is quasi-continuous.*

Lemma 4.2.2. *If $g \in \mathcal{S}$, then*

$$g = g_R + g_L$$

where g_R is continuous on the right, $g_R(b) = 0$, g_L is continuous on the left and $g_L(a) = 0$.

PROOF. Suppose $g \in \mathcal{S}$. By Theorem 4.2, g^* is quasi-continuous. From Lane [16], page 380, we know that the quasi-continuous function g^* with domain $[a, b]$ can be written

$$g^* = f_R + f_L$$

where f_R is continuous on the right and f_L is continuous on the left.

Define $h_R(x) = f_R(x)$ for each $x \in [a, b)$, $h_R(b) = 0$, $h_L(x) = f_L(x)$ for each $x \in (a, b]$, and $h_L(a) = 0$. Because our modifications to f_R and f_L to create h_R and h_L, respectively do not influence right and left continuity, h_R remains continuous on the right and h_L remains continuous on the left. Now, since $g = g^*|M$, we define $g_R = h_R|M$ and $g_L = h_L|M$ to yield

$$g = g_R + g_L$$

where g_R is continuous on the right, $g_R(b) = 0$, g_L is continuous on the left and $g_L(a) = 0$ □

Notation. When we say $\mathcal{P}(g, g_R, g_L)$ we mean the proposition "g_R is continuous on the right, $g_R(b) = 0$, g_L is continuous on the left, $g_L(a) = 0$ and $g = g_R + g_L$."

5 A Representation Theorem.

Theorem 5.1. *A function $L : \mathcal{S} \to \mathbb{R}$ is a bounded, linear functional if and only if there are functions α and β of bounded variation on $[a, b]$ such that*

$$L(f) = (l) \int_M f_R \, d\alpha + (r) \int_M f_L \, d\beta$$

for each $f \in \mathcal{S}$ where $\mathcal{P}(f, f_R, f_L)$.

PROOF OF SUFFICIENCY. Suppose $L : \mathcal{S} \to \mathbb{R}$ is defined for functions α and β of bounded variation on $[a, b]$ such that

$$L(f) = (l) \int_M f_R \, d\alpha + (r) \int_M f_L \, d\beta$$

for each $f \in \mathcal{S}$ where $\mathcal{P}(f, f_R, f_L)$.

Let $f \in \mathcal{S}$ and $k \in \mathbb{R}$. We know that

$$k \cdot L(f) = (l) \int_M k \cdot f_R \, d\alpha + (r) \int_M k \cdot f_L \, d\beta$$

where $\mathcal{P}(f, f_R, f_L)$. Clearly, $\mathcal{P}(k \cdot f, k \cdot f_R, k \cdot f_L)$. Therefore,

$$L(k \cdot f) = (l) \int_M k \cdot f_R \, d\alpha + (r) \int_M k \cdot f_L \, d\beta.$$

$\therefore L(k \cdot f) = k \cdot L(f)$ for each $f \in \mathcal{S}$ and each $k \in \mathbb{R}$.

Let $f, g \in \mathcal{S}$. We know that

$$L(f) = (l) \int_M f_R \, d\alpha + (r) \int_M f_L \, d\beta \tag{9}$$

where $\mathcal{P}(f, f_R, f_L)$. Moreover,

$$L(f + g) = (l) \int_M h_R \, d\alpha + (r) \int_M h_L \, d\beta \tag{10}$$

where $\mathcal{P}(f + g, h_R, h_L)$.

Since $f + g = h_R + h_L$, we have $g = (h_R - f_R) + (h_L - f_L)$. Clearly, $\mathcal{P}(g, h_R - f_R, h_L - f_L)$ and

$$L(g) = (l) \int_M (h_R - f_R) \, d\alpha + (r) \int_M (h_L - f_L) \, d\beta. \tag{11}$$

Using the linearity of the left and right integrals and combining (9), (10), and (11), we obtain

$$L(f + g) = L(f) + L(g)$$

for each $f, g \in \mathcal{S}$.

Therefore, $L : \mathcal{S} \to \mathbb{R}$ is a linear functional.

In preparation for the remainder of the proof, we refer to a result on page 380 of R.E. Lane [16], which when applied here states that $||f_R^*|| \le (1.5) \cdot ||f^*||$ and $||f_L^*|| \le (1.5) \cdot ||f^*||$ which, in turn, gives us

$$||f_R|| \le (1.5) \cdot ||f|| \text{ and } ||f_L|| \le (1.5) \cdot ||f|| \tag{12}$$

where $\mathcal{P}(f, f_R, f_L)$.

Now, consider

$$L(f) = (l)\int_M f_R \, d\alpha + (r)\int_M f_L \, d\beta$$

where $\mathcal{P}(f, f_R, f_L)$. Applying the triangle inequality, from (3) and (12), we have

$$\begin{aligned} |L(f)| &\leq \left|(l)\int_M f_R \, d\alpha\right| + \left|(r)\int_M f_L \, d\beta\right| \\ &\leq (1.5)\cdot \|f\| \cdot V_a^b \alpha + (1.5)\cdot \|f\| \cdot V_a^b \beta \\ &= (1.5)\cdot \|f\| \cdot (V_a^b \alpha + V_a^b \beta). \end{aligned}$$

We conclude that L is bounded. □

Proof of Necessity. Suppose that $L : \mathcal{S} \to \mathbb{R}$ is a bounded, linear functional. Define

$$\alpha_t = \mathbf{1}_{[a,t)}, \beta_t = \mathbf{1}_{[a,t]}, \alpha_0 = \mathbf{1}_{\emptyset}|M, \alpha_1 = \mathbf{1}_{\mathbb{R}\setminus\{b\}}|M, \beta_0 = \mathbf{1}_{\{a\}}|M, \beta_1 = \mathbf{1}_M|M.$$

Moreover, remembering that $\mathbf{1}_{[t,b]}|M \in \mathcal{S}, t \in [a,b]$ and $\mathbf{1}_{(t,b]}|M \in \mathcal{S}, t \in [a,b]$, define the functions $\alpha \in \mathcal{S}$ and $\beta \in \mathcal{S}$ as follows:

$$\alpha(t) = L(\mathbf{1}_{[t,b]}|M), t \in M; \beta(t) = L(\mathbf{1}_{(t,b]}|M), t \in M.$$

Note that α and β are real-valued functions with domain M.

For the purpose of showing that α and β are of bounded variation on $[a,b]$, let D be any partition of M. Since L is a bounded linear functional, there is a $k \geq 0$ such that $\|L(f)\| \leq k \cdot \|f\|$ for each $f \in \mathcal{S}$. Define $\delta_{[p,q]}$ to be 1, if

$\alpha(q)-\alpha(p)>0$ and to be -1, otherwise, for each $[p,q]_M \in D$.

$$\begin{aligned}\sum_{[p,q]\in D} |\alpha(q)-\alpha(p)| &= \sum_{[p,q]\in D} |L(\mathbf{1}_{[q,b]}|M) - L(\mathbf{1}_{[p,b]}|M)| \\ &= \sum_{[p,q]\in D} |L(\mathbf{1}_{[q,b]}|M - \mathbf{1}_{[p,b]}|M)| \\ &= \sum_{[p,q]\in D} L(\delta_{[p,q]} \cdot \mathbf{1}_{[p,q)}|M) \\ &= L\left(\sum_{[p,q]\in D} \delta_{[p,q]} \cdot \mathbf{1}_{[p,q)}|M\right) \\ &\le k \cdot \left\| \sum_{[p,q]\in D} \delta_{[p,q]} \cdot \mathbf{1}_{[p,q)}|M \right\| \le k.\end{aligned}$$

Therefore, α is of bounded variation and, by similar argument, β can be shown to be of bounded variation.

Suppose $f \in \mathcal{S}$. By Lemma 4.2.2, there are functions f_R and f_L such that

$$f = f_R + f_L$$

where $\mathcal{P}(f, f_R, f_L)$.

Since L is a bounded, linear function functional, L is continuous. Suppose $\varepsilon > 0$. Note that f, f_R, f_L are members of $\mathcal{S}$, the domain of L. Since L is continuous, there is a common positive number δ such that

$$\begin{gathered} g \in \mathcal{S} \text{ and } \|f - g\| < \delta \to |L(f) - L(g)| < \varepsilon/16 \\ g \in \mathcal{S} \text{ and } \|f_R - g\| < \delta \to |L(f_R) - L(g)| < \varepsilon/16 \\ g \in \mathcal{S} \text{ and } \|f_L - g\| < \delta \to |L(f_L) - L(g)| < \varepsilon/16. \end{gathered} \tag{13}$$

Since f_R is continuous on the right and f_L is quasi-continuous on the left; thus, each is quasi-continuous on M, and each of α and β is of bounded variation on M, we know that each of $(l)\int_M f_R\,d\alpha$ and $(r)\int_M f_L\,d\beta$ exists. Then, there exists a partition D_l of M such that

$$\left|(l)\int_M f_R\,d\alpha - \Sigma_l(f_R, \alpha, D')\right| < \varepsilon/16 \tag{14}$$

for each refinement D' of D_l and there exists a partition D_r of M such that

$$\left|(r)\int_M f_L\,d\beta - \Sigma_r(f_L, \beta, D')\right| < \varepsilon/16 \tag{15}$$

for each refinement D' of D_r. Since each of f_R and f_L is quasi-continuous and continuous on the right and left, respectively, there exists a partition E_r of M and a partition E_l of M such that for each $[p,q]_M \in E_l$,

$$x, y \in [p,q) \cap M \to |f_R(x) - f_R(y)| < \delta/16 \tag{16}$$

and for each $[p,q]_M \in E_r$,

$$x, y \in (p,q] \cap M \to |f_L(x) - f_L(y)| < \delta/16. \tag{17}$$

Now, let D' be a partition of M where $E(D') = E(D_r) \cup E(D_l) \cup E(E_r) \cup E(E_l)$. Define two functions in $\mathcal{S}$ as follows:

$$g_R(x) = \sum_{[p,q]_M \in D'} f_R(p)[\alpha_q(x) - \alpha_p(x)], x \in M \tag{18}$$

$$g_L(x) = \sum_{[p,q]_M \in D'} f_L(q)[\beta_q(x) - \beta_p(x)], x \in M. \tag{19}$$

Consider $x \in M$. Let $[u,v]_M$ be the member of D' that contains x. Keeping in mind that $\alpha_q - \alpha_p = \mathbf{1}_{[p,q)_M}$ and, thus, $\alpha_q(t) - \alpha_p(t) \neq 0$ when $[p,q]_M = [u,v]_M$ and $u = p \leq t < q = v$, we see that

$$|f_R(x) - g_R(x)| \leq |f_R(x) - f_R(u)|.$$

From (16), we see that $|f_R(x) - f_R(u)| < \delta/16$. We can conclude at this point that $|f_R(x) - g_R(x)| < \delta/16$ for each $x \in M$, implying that

$$\|f_R - g_R\| \leq \delta/16 \tag{20}$$

and, using a similar argument,

$$\|f_L - g_L\| \leq \delta/16. \tag{21}$$

Combining (20) and (21), we have

$$\|f - (g_R + g_L)\| = \|(f_R + f_L) - (g_R + g_L)\| < \delta.$$

As a result, from (13), we obtain

$$|L(f) - L(g_R + g_L)| < \varepsilon/16.$$

Using the fact that L is linear, we are allowed to perform the following operations where each sum is taken over all $[p, q]_M \in D'$:

$$\begin{aligned}|L(f) - L(g_R + g_L)| &= |L(f) - L(g_R) - L(g_L)| \\ &= |L(f) - L\left(\sum f(p)[\alpha_q(x) - \alpha_p(x)]\right) \\ &\quad - L\left(\sum f(q)[\beta_q(x) - \beta_p(x)]\right)| \\ &= |L(f) - \sum f(p)[L(\alpha_q(x)) - L(\alpha_p(x))]) \\ &\quad - \sum f(q)[L(\beta_q(x)) - L(\beta_p(x))]| \\ &= |L(f) - \sum f(p)[\alpha(q) - \alpha(p)] \\ &\quad - \sum f(q)[\beta(q) - \beta(p)]|.\end{aligned}$$

Therefore,

$$|L(f) - \Sigma_l(f_R, \alpha, D') - \Sigma_r(f_L, \beta, D')| < \varepsilon/16. \tag{22}$$

Combining (14) and (15) with the preceding, we have

$$|L(f) - (l)\int_M f_R\, d\alpha - (r)\int_M f_L\, d\beta| < \varepsilon.$$

Therefore, giving us the desired conclusion

$$L(f) = (l)\int_M f_R\, d\alpha + (r)\int_M f_L\, d\beta.$$

□

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