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POROSITY AND THE DARBOUX PROPERTY OF FRÉCHET DERIVATIVES

Abstract

We study a relation between the porosity of sets in Euclidean spaces and the Darboux property of (relative) Fréchet derivatives.

1 Introduction and Main Result.

A set A in a real Banach space X is said to be porous at $a \in X$ if there are $c > 0$ and $x_n \in X$, $x_n \neq a$, with $x_n \to a$ such that $x \notin A$ whenever $n \in \mathbb{N}$ and $\|x - x_n\| < c\|a - x_n\|$. Let $B \subset X$ be non-empty without isolated points and $f : B \to \mathbb{R}$ be given. We say that $g : B \to X^*$ is a (relative) Fréchet derivative of f on B if

$$\lim_{x \to a,\, x \in B} \frac{f(x) - f(a) - g(a)(x - a)}{\|x - a\|} = 0$$

for each $a \in B$.

The following two results have appeared in [1].

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Lemma 1.1. *Let X be a real Banach space, $G \subset X$ open, $a \in \partial G$ and let $X \setminus G$ be porous at a. Let $M := G \cup \{a\}$ and suppose that $g : M \to X^*$ is a Fréchet derivative of a function $f : M \to \mathbb{R}$ on M. Then $(a, g(a))$ belongs to the closure of the graph of $g|_G$ in $X \times X^*$. In particular, $g(a) \in \overline{g(G)}$.*

Theorem 1.2. *Let X be a real Banach space and $B \subset X$ be non-empty such that the interior of B is connected and $X \setminus B$ is porous at every $a \in B \cap \partial B$. Let $g : B \to X^*$ be a Fréchet derivative of a function $f : B \to \mathbb{R}$ on B. Then the graph of g is a connected subset of $X \times X^*$. In particular, $g(B)$ is connected in X^*.*

In this paper, we prove converses of these results in the case of Euclidean spaces. Proposition 4.2 below corresponds with Lemma 1.1, while the following theorem corresponds with Theorem 1.2.

Theorem 1.3. *Let $B \subset \mathbb{R}^d$ be non-empty without isolated points such that the interior of B is connected. Then the following assertions are equivalent:*

(i) *$\mathbb{R}^d \setminus B$ is porous at every $a \in B \cap \partial B$.*

(ii) *The graph of g is connected whenever g is a Fréchet derivative of a function $f : B \to \mathbb{R}$ on B.*

(iii) *$g(B)$ is connected whenever g is a Fréchet derivative of a function $f : B \to \mathbb{R}$ on B.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 1.2 and (ii) $\Rightarrow$ (iii) is clear. Suppose (i) does not hold. There is $a \in B$ such that $\mathbb{R}^d \setminus B$ is not porous at a. By Proposition 4.2 below, there is $f : \mathbb{R}^d \to \mathbb{R}$, Fréchet differentiable on B, such that $f'(a) = 0$ and $|f'(u)| \geq 1$ for any $u \in B \setminus \{a\}$. Then $g = f'|_B$ is a Fréchet derivative of $f|_B$ on B and 0 is an isolated point of $g(B)$. Thus (iii) does not hold, and the remaining (iii) $\Rightarrow$ (i) is proved. □

2 Preliminaries.

Let $d \in \mathbb{N}$ be fixed throughout the whole paper. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^d$ and by $B(x, r)$ the open ball around x with radius $r > 0$. We fix ψ a mollification kernel; i.e. a function with properties

1) $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$,
2) $\psi > 0$ on $B(0, 1)$ and $\psi = 0$ on $\mathbb{R}^d \setminus B(0, 1)$,
3) $\psi(x) = \psi(y)$ if $|x| = |y|$,
4) $\int_{\mathbb{R}^d} \psi = 1$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be open and $\rho : \Omega \to (0, \infty)$ be a continuous function. Let $c > 0$. Then there is $\delta \in \mathcal{C}^1(\Omega)$ satisfying $0 < \delta < \rho$ on Ω, Lipschitz with the constant c on Ω.*

PROOF. Let $\{B_k\}_{k\in\mathbb{N}}$ be a covering of Ω by open balls such that $\overline{B_k} \subset \Omega$ for each $k \in \mathbb{N}$. Put $m_k = \min_{x\in\overline{B_k}} \rho(x)$. Then the desired function is

$$\sum_{k=1}^{\infty} \frac{m_k}{2^k} \Psi_k,$$

where $\Psi_k : \Omega \to [0,1)$ is a continuously differentiable function such that $\Psi_k > 0$ on B_k, $\Psi_k = 0$ on $\Omega \setminus B_k$ and $|\Psi_k'| \le c/m_k$ on Ω. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be open, $\varphi \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and let $\delta \in \mathcal{C}^1(\Omega)$ be positive on Ω. Then, for the function $F : \Omega \to \mathbb{R}$ defined as*

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x)y)\psi(y)\,dy,$$

we have $F \in \mathcal{C}^1(\Omega)$.

PROOF. We note first that F can be equivalently expressed as

$$F(x) = \frac{G(x)}{\delta(x)^d},$$

where

$$G(x) = \int_{\mathbb{R}^d} \varphi(t)H_t(x)\,dt \quad \text{and} \quad H_t(x) = \psi\Big(\frac{x-t}{\delta(x)}\Big).$$

Fix $x \in \Omega$ and a direction $\nu \in \mathbb{R}^d$. We will prove that

I. $\frac{\partial G}{\partial \nu}(x)$ exists and

$$\frac{\partial G}{\partial \nu}(x) = \int_{\mathbb{R}^d} \varphi(t)\frac{\partial H_t}{\partial \nu}(x)\,dt,$$

II. the mapping

$$s \mapsto \int_{\mathbb{R}^d} \varphi(t)\frac{\partial H_t}{\partial \nu}(s)\,dt$$

is continuous at x.

Choose $\varepsilon > 0$ such that $\overline{B(x,\varepsilon)} \subset \Omega$ and put

$$\Gamma = \overline{\bigcup_{s\in\overline{B(x,\varepsilon)}} B(s,\delta(s))}.$$

Note that, for $s \in \overline{B(x,\varepsilon)}$ and $t \in \mathbb{R}^d \setminus \Gamma$, we have $\frac{\partial H_t}{\partial \nu}(s) = 0$. Moreover, the function $(s,t) \mapsto \frac{\partial H_t}{\partial \nu}(s)$ is continuous on the compact set $\overline{B(x,\varepsilon)} \times \Gamma$, and so there is a constant $C > 0$ with $|\frac{\partial H_t}{\partial \nu}(s)| \le C$ for $(s,t) \in \overline{B(x,\varepsilon)} \times \Gamma$. So

$$\Big|\varphi(t)\frac{\partial H_t}{\partial \nu}(s)\Big| \le C\chi_\Gamma(t)|\varphi(t)|$$

for $s \in \overline{B(x,\varepsilon)}$ and $t \in \mathbb{R}^d$, where χ_Γ is the characteristic function of the set Γ. Now, since $\chi_\Gamma|\varphi| \in \mathcal{L}^1(\mathbb{R}^d)$, I and II are consequences of the standard theorems on integral depending on parameter.

We proved, in particular, that the partial derivatives of G are continuous on Ω, and so $G \in \mathcal{C}^1(\Omega)$. Immediately, $F \in \mathcal{C}^1(\Omega)$ as well. □

Lemma 2.3. *Let $L, K > 0$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a function which is Lipschitz with the constant L, let $\Omega \subset \mathbb{R}^d$ be an open set and let $\delta \in \mathcal{C}^1(\Omega)$ be positive and Lipschitz with the constant K/L. Suppose that, for each $x \in \Omega$, there is $\nu_x \in \mathbb{R}^d$, $|\nu_x| = 1$, such that $\frac{\partial\varphi}{\partial\nu_x}(y) \geq 2K$ for almost every $y \in B(x, \delta(x))$. Then the function*

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x)y)\psi(y)\, dy$$

belongs to $\mathcal{C}^1(\Omega)$ and $|F'(x)| \geq K$ for each $x \in \Omega$. Moreover, F is Lipschitz.

Proof. First, note that $F \in \mathcal{C}^1(\Omega)$ due to Lemma 2.2. Now, choose $x \in \Omega$ and a sequence $\{\lambda_n\}$ of non-zero real numbers with $\lambda_n \to 0$. Since $F \in \mathcal{C}^1(\Omega)$, it is sufficient to write

$$\begin{aligned}
\frac{\partial F}{\partial \nu_x}(x) &= \lim_{n\to\infty} \frac{F(x+\lambda_n\nu_x) - F(x)}{\lambda_n} \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \frac{\varphi(x+\lambda_n\nu_x + \delta(x+\lambda_n\nu_x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\, dy \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{\varphi(x+\lambda_n\nu_x + \delta(x+\lambda_n\nu_x)y) - \varphi(x+\lambda_n\nu_x+\delta(x)y)}{\lambda_n}\psi(y)\, dy \\
&\quad + \liminf_{n\to\infty} \int_{\mathbb{R}^d} \frac{\varphi(x+\lambda_n\nu_x + \delta(x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\, dy \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^d} -L\frac{|\delta(x+\lambda_n\nu_x) - \delta(x)|}{\lambda_n}|y|\psi(y)\, dy \\
&\quad + \int_{\mathbb{R}^d} \liminf_{n\to\infty} \frac{\varphi(x+\lambda_n\nu_x + \delta(x)y) - \varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\, dy \\
&\geq \int_{\mathbb{R}^d} -L\frac{K}{L}|y|\psi(y)\, dy + \int_{B(0,1)\setminus N} \frac{\partial\varphi}{\partial\nu_x}(x+\delta(x)y)\psi(y)\, dy \\
&\geq \int_{B(0,1)} -K|y|\psi(y)\, dy + \int_{B(0,1)\setminus N} 2K\psi(y)\, dy \\
&\geq \int_{B(0,1)} K\psi(y)\, dy = K,
\end{aligned}$$

where N has measure 0. We could use the Fatou lemma because

$$\frac{\varphi(x+\lambda_n\nu_x+\delta(x)y)-\varphi(x+\delta(x)y)}{\lambda_n}\psi(y)\geq -L\psi(y)$$

for $n\in\mathbb{N}$ and $y\in\mathbb{R}^d$.

To prove that F is Lipschitz, we write

$$\begin{aligned}|F(u)-F(v)| &\leq \int_{\mathbb{R}^d}|\varphi(u+\delta(u)y)-\varphi(v+\delta(v)y)|\psi(y)\,dy\\ &\leq \int_{\mathbb{R}^d}L(|u-v|+|\delta(u)-\delta(v)||y|)\psi(y)\,dy\\ &\leq \int_{\mathbb{R}^d}(L|u-v|+K|u-v||y|)\psi(y)\,dy\\ &= \int_{B(0,1)}(L|u-v|+K|u-v||y|)\psi(y)\,dy\\ &\leq \int_{B(0,1)}(L+K)|u-v|\psi(y)\,dy=(L+K)|u-v|.\end{aligned}$$

□

Lemma 2.4. *Let (P,ϱ) be a metric space and functions $s,t:P\to\mathbb{R}$ be bounded by M_s,M_t on P. Then the function st is Lipschitz with the constant $M_sL_t+M_tL_s$ in the case that s,t are Lipschitz with the constants L_s,L_t.*

Proof. We have

$$\begin{aligned}|s(x)t(x)-s(y)t(y)| \leq&|s(x)t(x)-s(x)t(y)|+|s(x)t(y)-s(y)t(y)|\\ =&|s(x)||t(x)-t(y)|+|t(y)||s(x)-s(y)|\\ \leq& M_sL_t\varrho(x,y)+M_tL_s\varrho(x,y)\end{aligned}$$

for $x,y\in P$. □

3 Functions on Special Domains.

Let $r_i,s_i\in\mathbb{R},p_i\in\mathbb{N}$ for $i\in\mathbb{N}$ satisfying

- $r_1>r_2>\dots>0$,
- $p_1\leq p_2\leq\dots$,
- $r_i\to 0$,

- $\frac{r_{i+1}}{r_i} \to 1$,
- $\frac{s_i}{r_i} \to 0$,
- $p_i \to \infty$
- $|\frac{s_i - s_{i+1}}{r_i - r_{i+1}}| = 1$,
- $\frac{r_i}{r_i - r_{i+1}} \frac{1}{p_i} \leq 2$,

be fixed throughout this section. We put

$$D_p = \Big\{(x_1, \dots, x_d) \in \partial([-1,1]^d) : 2px_1, \dots, 2px_d \in \mathbb{Z}\Big\}, \quad p \in \mathbb{N},$$

$$D = \bigcup_{i \in \mathbb{N}} r_i D_{p_i}.$$

In this section, we denote

$$\|x\| = \|x\|_\infty = \max\{|x_1|, \dots, |x_d|\}$$

for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.

Lemma 3.1. *There is a Lipschitz function $F : \mathbb{R}^d \to \mathbb{R}$ with properties*

1. $F'(0) = 0$,
2. $F'(x)$ *exists and* $|F'(x)| \geq 1/(4\sqrt{d})$ *whenever* $x \in \mathbb{R}^d \setminus (D \cup \{0\})$.

The whole section is dedicated to the proof of this lemma.
Define

$$h(x) = \operatorname{dist}(x, \mathbb{Z}), \quad h_0(x) = \operatorname{dist}(x, \{-p_1, \dots, 0, \dots, p_1\}), \quad x \in \mathbb{R},$$

$$g(x_1, \dots, x_d) = \sum_{j=1}^{d} h(x_j), \quad g_0(x_1, \dots, x_d) = \sum_{j=1}^{d} h_0(x_j), \quad (x_1, \dots, x_d) \in \mathbb{R}^d,$$

$$g_t(x) = t^{-1} g(tx), \quad g_{t,0}(x) = t^{-1} g_0(tx), \quad x \in \mathbb{R}^d,\ t > 0.$$

Put $C = 1 + 4d$. For $x \in \mathbb{R}^d$, define

$$\varphi(x) = \begin{cases} 0, & x = 0, \\ \frac{\|x\| - r_{i+1}}{r_i - r_{i+1}} (Cs_i + g_{p_i/r_i}(x)) & \\ \qquad + \frac{r_i - \|x\|}{r_i - r_{i+1}} (Cs_{i+1} + g_{p_{i+1}/r_{i+1}}(x)), & r_{i+1} \leq \|x\| < r_i, \\ Cs_1 + g_{p_1/r_1,0}(x), & r_1 \leq \|x\|. \end{cases}$$

Claim 3.2. *$\varphi(x)/\|x\| \to 0$ as $x \to 0$.*

Proof. For $x \in \mathbb{R}^d$ and $i \in \mathbb{N}$ with $r_{i+1} \le \|x\| < r_i$, we obtain

$$\begin{aligned}
|\varphi(x)| &\le \Big|\frac{\|x\| - r_{i+1}}{r_i - r_{i+1}}\Big| \Big|Cs_i + g_{p_i/r_i}(x)\Big| \\
&\quad + \Big|\frac{r_i - \|x\|}{r_i - r_{i+1}}\Big| \Big|Cs_{i+1} + g_{p_{i+1}/r_{i+1}}(x)\Big| \\
&\le C|s_i| + |g_{p_i/r_i}(x)| + C|s_{i+1}| + |g_{p_{i+1}/r_{i+1}}(x)| \\
&\le C|s_i| + \frac{r_i}{p_i}\frac{d}{2} + C|s_{i+1}| + \frac{r_{i+1}}{p_{i+1}}\frac{d}{2}, \\
\frac{|\varphi(x)|}{\|x\|} \le \frac{|\varphi(x)|}{r_{i+1}} &\le C\Big|\frac{s_i}{r_i}\Big|\frac{r_i}{r_{i+1}} + C\Big|\frac{s_{i+1}}{r_{i+1}}\Big| + \frac{1}{p_i}\frac{r_i}{r_{i+1}}\frac{d}{2} + \frac{1}{p_{i+1}}\frac{d}{2}.
\end{aligned}$$

The properties of the sequences r_i, s_i and p_i guarantee that the right side converges to 0 as i tends to ∞. □

Claim 3.3. *φ is Lipschitz.*

Proof. Obviously, h is Lipschitz with the constant 1 and g, g_t are Lipschitz with the constant d on $\mathbb{R}^d$ (all the Lipschitz constants in the proof are with respect to $\|\cdot\|$). Fix $i \in \mathbb{N}$ and put $U = \{x \in \mathbb{R}^d : r_{i+1} \le \|x\| < r_i\}$. We will investigate separately the functions

$$\begin{aligned}
\varphi_1(x) &= \frac{\|x\| - r_{i+1}}{r_i - r_{i+1}} Cs_i + \frac{r_i - \|x\|}{r_i - r_{i+1}} Cs_{i+1}, \\
\varphi_2(x) &= \frac{\|x\| - r_{i+1}}{r_i - r_{i+1}} g_{p_i/r_i}(x), \\
\varphi_3(x) &= \frac{r_i - \|x\|}{r_i - r_{i+1}} g_{p_{i+1}/r_{i+1}}(x),
\end{aligned}$$

which satisfy that $\varphi_1 + \varphi_2 + \varphi_3 = \varphi$ on U. For $x, y \in U$, we have

$$\varphi_1(x) - \varphi_1(y) = C(\|x\| - \|y\|)\frac{s_i - s_{i+1}}{r_i - r_{i+1}},$$

and thus $|\varphi_1(x) - \varphi_1(y)| \le C\|x - y\|$. It follows from Lemma 2.4 that φ_2, φ_3 are Lipschitz with the constants

$$d + \frac{r_i}{p_i}\frac{d}{2}\frac{1}{r_i - r_{i+1}}, \quad d + \frac{r_{i+1}}{p_{i+1}}\frac{d}{2}\frac{1}{r_i - r_{i+1}}$$

on U. Together, we get that φ is Lipschitz with the constant $C + 4d$ on U. Even, φ is Lipschitz with this constant on $\overline{U} = \{x \in \mathbb{R}^d : r_{i+1} \le \|x\| \le r_i\}$ because $\lim_{x\to z, x\in U} \varphi(x) = Cs_i + g_{p_i/r_i}(z) = \varphi(z)$ whenever $\|z\| = r_i$.

We have proved that φ is Lipschitz with the constant $C + 4d$ on $\{x \in \mathbb{R}^d : r_{i+1} \le \|x\| \le r_i\}$ for every $i \in \mathbb{N}$. It is also Lipschitz with this constant (in fact, Lipschitz with the constant d) on $\{x \in \mathbb{R}^d : r_1 \le \|x\|\}$. Considering the continuity of φ at 0 (Claim 3.2), we see that φ is Lipschitz with the constant $C + 4d$ on $\mathbb{R}^d$. □

Fix $k \in \{1, \dots, d\}$ and $i \in \mathbb{N}$ and differentiate φ on the set $\{x \in \mathbb{R}^d : r_{i+1} < \|x\| < r_i, \|x\| = x_k > |x_j| \text{ for } j \ne k\}$:

$$\varphi(x) = \frac{x_k - r_{i+1}}{r_i - r_{i+1}}\Big(Cs_i + \frac{r_i}{p_i} g\big(\frac{p_i}{r_i}x\big)\Big) + \frac{r_i - x_k}{r_i - r_{i+1}}\Big(Cs_{i+1} + \frac{r_{i+1}}{p_{i+1}} g\big(\frac{p_{i+1}}{r_{i+1}}x\big)\Big),$$

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_j}(x) &= \frac{x_k - r_{i+1}}{r_i - r_{i+1}} h'\big(\frac{p_i}{r_i}x_j\big) + \frac{r_i - x_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}}x_j\big), \quad j \ne k,\\
\frac{\partial\varphi}{\partial x_k}(x) &= \frac{x_k - r_{i+1}}{r_i - r_{i+1}} h'\big(\frac{p_i}{r_i}x_k\big) + \frac{r_i - x_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}}x_k\big)\\
&\quad + C\frac{s_i - s_{i+1}}{r_i - r_{i+1}} + \frac{1}{r_i - r_{i+1}}\frac{r_i}{p_i} g\big(\frac{p_i}{r_i}x\big) - \frac{1}{r_i - r_{i+1}}\frac{r_{i+1}}{p_{i+1}} g\big(\frac{p_{i+1}}{r_{i+1}}x\big)
\end{aligned}$$

(if the derivatives of h exist). For almost every x with $r_{i+1} < \|x\| < r_i$ and $\|x\| = x_k > |x_j|$ for $j \ne k$, we obtain

$$\begin{aligned}
\frac{\partial\varphi}{\partial\nu_x}(x) &\ge \frac{s_i - s_{i+1}}{r_i - r_{i+1}}\frac{\partial\varphi}{\partial x_k}(x) - \sum_{j\ne k}\Big|\frac{\partial\varphi}{\partial x_j}(x)\Big|\\
&\ge C - \Big|\frac{1}{r_i - r_{i+1}}\frac{r_i}{p_i} g\big(\frac{p_i}{r_i}x\big)\Big| - \Big|\frac{1}{r_i - r_{i+1}}\frac{r_{i+1}}{p_{i+1}} g\big(\frac{p_{i+1}}{r_{i+1}}x\big)\Big|\\
&\quad - \sum_{j=1}^{d}\Big|\frac{x_k - r_{i+1}}{r_i - r_{i+1}} h'\big(\frac{p_i}{r_i}x_j\big)\Big| - \sum_{j=1}^{d}\Big|\frac{r_i - x_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}}x_j\big)\Big|\\
&\ge C - 4d = 1,
\end{aligned}$$

where ν_x denotes $(((s_i - s_{i+1})/(r_i - r_{i+1}))/\|x\|)x$.

Claim 3.4. *For every $x \in \mathbb{R}^d \setminus (D \cup \{0\})$, there is a direction $\nu \in \mathbb{R}^d, \|\nu\| = 1$, and a neighborhood U_x of x such that $\frac{\partial\varphi}{\partial\nu}(y) \ge 1/2$ for almost every $y \in U_x$.*

PROOF. Due to the symmetry, we may suppose that $x_j \ge 0, j = 1, \dots, d$.

Consider cases:

(1) Let $\|x\| = r_i$ for some $i \in \mathbb{N}, i \geq 2$. As $x \notin r_i D_{p_i}$, there is $j \in \{1, \dots, d\}$ such that $2p_i x_j / r_i \notin \mathbb{Z}$. Denote $\tau = h'(p_i x_j / r_i) \in \{-1, 1\}$ and choose $\varepsilon > 0$ such that $\varepsilon \leq (1/4) \min\{r_i - r_{i+1}, r_{i-1} - r_i\}, 2\varepsilon < r_i - x_j$ and $h'(p_i a / r_i) = \tau$ whenever $|x_j - a| \leq \varepsilon$. Put $\nu = \tau e_j$ and $U_x = \{y \in \mathbb{R}^d : \|y - x\| \leq \varepsilon\}$. For almost every $y = (y_1, \dots, y_d) \in U_x$, there is $k \in \{1, \dots, d\}$ such that $\|y\| = y_k > |y_{j'}|$ for $j' \neq k$ and the derivatives $h'(\frac{p_{i+1}}{r_{i+1}} y_j)$ and $h'(\frac{p_{i-1}}{r_{i-1}} y_j)$ exist (in such a case, $k \neq j$ because $y_k \geq \|x\| - \varepsilon = r_i - \varepsilon > x_j + \varepsilon \geq y_j$ by the choice of ε). So, for almost every $y = (y_1, \dots, y_d) \in U_x$ with $\|y\| < r_i$, we have (for some k)

$$\begin{aligned}
\frac{\partial \varphi}{\partial \nu}(y) = \tau \frac{\partial \varphi}{\partial x_j}(y) &= \tau \frac{y_k - r_{i+1}}{r_i - r_{i+1}} h'\big(\frac{p_i}{r_i} y_j\big) + \tau \frac{r_i - y_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}} y_j\big) \\
&= \frac{y_k - r_{i+1}}{r_i - r_{i+1}} + \tau \frac{r_i - y_k}{r_i - r_{i+1}} h'\big(\frac{p_{i+1}}{r_{i+1}} y_j\big) \\
&\geq \frac{y_k - r_{i+1}}{r_i - r_{i+1}} - \frac{r_i - y_k}{r_i - r_{i+1}} \\
&= 1 - 2 \frac{\|x\| - \|y\|}{r_i - r_{i+1}} \geq 1 - 2 \frac{\varepsilon}{r_i - r_{i+1}} \geq 1/2,
\end{aligned}$$

while, for almost every $y = (y_1, \dots, y_d) \in U_x$ with $\|y\| > r_i$, we have (for some k)

$$\begin{aligned}
\frac{\partial \varphi}{\partial \nu}(y) = \tau \frac{\partial \varphi}{\partial x_j}(y) &= \tau \frac{y_k - r_i}{r_{i-1} - r_i} h'\big(\frac{p_{i-1}}{r_{i-1}} y_j\big) + \tau \frac{r_{i-1} - y_k}{r_{i-1} - r_i} h'\big(\frac{p_i}{r_i} y_j\big) \\
&= \tau \frac{y_k - r_i}{r_{i-1} - r_i} h'\big(\frac{p_{i-1}}{r_{i-1}} y_j\big) + \frac{r_{i-1} - y_k}{r_{i-1} - r_i} \\
&\geq \frac{r_{i-1} - y_k}{r_{i-1} - r_i} - \frac{y_k - r_i}{r_{i-1} - r_i} \\
&= 1 - 2 \frac{\|y\| - \|x\|}{r_{i-1} - r_i} \geq 1 - 2 \frac{\varepsilon}{r_{i-1} - r_i} \geq 1/2.
\end{aligned}$$

(2) Let $\|x\| = r_1$. In this case, the procedure is similar to the procedure of (1) (choosing $j, \tau, \varepsilon, \nu$ and U_x as in (1), we have $\frac{\partial \varphi}{\partial \nu}(y) \geq 1/2$ for almost every $y = (y_1, \dots, y_d) \in U_x$ with $\|y\| < r_1$ and we can easily check that $\frac{\partial \varphi}{\partial \nu}(y) = 1$ for every $y = (y_1, \dots, y_d) \in U_x$ with $\|y\| > r_1$).

(3) Let $r_{i+1} < \|x\| < r_i$ for some $i \in \mathbb{N}$. We define

$$V = \big\{y \in \mathbb{R}^d : r_{i+1} < \|y\| < r_i, \|y\| = y_k \geq \max_{j \neq k} |y_j| \text{ for some } k\big\}.$$

We supposed that $x_j \geq 0, j = 1, \dots, d$. Therefore, V is a neighborhood of x.

We have

$$\frac{\partial\varphi}{\partial\nu_x}(y) = \frac{\partial\varphi}{\partial\nu_y}(y) + \varphi'(y)(\nu_x - \nu_y) \geq 1 - |\varphi'(y)||\nu_x - \nu_y|$$

for almost every $y \in V$, where ν_x and ν_y denote $(((s_i - s_{i+1})/(r_i - r_{i+1}))/\|x\|)x$ and $(((s_i - s_{i+1})/(r_i - r_{i+1}))/\|y\|)y$, as above. Now, the existence of an appropriate U_x follows from the continuity of $y \mapsto \nu_y$ and from Claim 3.3.

(4) Let $\|x\| > r_1$. We choose a k with $x_k > r_1$ and take $U_x = \{(y_1, \ldots, y_d) \in \mathbb{R}^d : y_k > r_1\}$. If ν denotes e_k, then

$$\frac{\partial\varphi}{\partial\nu}(y) = \frac{\partial g_{p_1/r_1,0}}{\partial x_k}(y) = h_0'\big(\frac{p_1}{r_1}y_k\big) = 1$$

for every $y \in U_x$. □

Now, for every $x \in \mathbb{R}^d \setminus (D \cup \{0\})$, we define $\rho(x)$ as the supremum of numbers $r \leq |x|$ for which there is $\nu \in \mathbb{R}^d, |\nu| = 1$, such that $\frac{\partial\varphi}{\partial\nu}(y) \geq 1/(2\sqrt{d})$ for almost every $y \in B(x,r)$. By Claim 3.4, $\rho > 0$ on $\mathbb{R}^d \setminus (D \cup \{0\})$. Obviously, ρ is Lipschitz (with the constant 1 with respect to $|\cdot|$). By Claim 3.3, we can take $L > 0$ such that φ is Lipschitz with the constant L (with respect to $|\cdot|$). By Lemma 2.1, there is $\delta \in \mathcal{C}^1(\mathbb{R}^d \setminus (D \cup \{0\}))$ satisfying $0 < \delta < \rho$, Lipschitz with the constant $1/(4\sqrt{d}L)$. We define F on $\mathbb{R}^d \setminus (D \cup \{0\})$ first by

$$F(x) = \int_{\mathbb{R}^d} \varphi(x + \delta(x)y)\psi(y)\, dy, \quad x \in \mathbb{R}^d \setminus (D \cup \{0\}).$$

By Lemma 2.3 (applied on $K = 1/(4\sqrt{d})$), F is Lipschitz and differentiable on $\mathbb{R}^d \setminus (D \cup \{0\})$ and Property 2 from Lemma 3.1 is satisfied. We extend F on $\mathbb{R}^d$ to be Lipschitz. Property 1 follows now from Claim 3.2 and from

$$\sup_{x \in B(0,r)} |F(x)| \leq \sup_{x \in B(0,r) \setminus (D \cup \{0\})} \ \sup_{t \in B(x,\delta(x))} |\varphi(t)| \leq \sup_{t \in B(0,2r)} |\varphi(t)|$$

for $r > 0$. This completes the proof of Lemma 3.1.

4 General Case.

Lemma 4.1. *Let $r > 0$ and $x, y \in \mathbb{R}^d$ be such that $|x - y| < r/2$. Then there is a diffeomorphism $\Psi : \mathbb{R}^d \to \mathbb{R}^d$, Lipschitz with the constant 2, such that $\Psi(u) = u$ for $u \in \mathbb{R}^d \setminus B(x,r)$, $\Psi(y) = x$ and $|v \circ \Psi'(u)| \geq \frac{2}{3}|v|$ for any $u \in \mathbb{R}^d$ and $v \in (\mathbb{R}^d)^*$.*

PROOF. Without loss of generality $x = 0$, $y = (|y|, 0, 0, \dots, 0)$ and $r = 1$. Let $\phi : [0, \infty) \to \mathbb{R}$ be a function which is differentiable everywhere in $(0, \infty)$ and right differentiable at 0 such that $\phi(0) = |y|$, $\phi(\xi) = 0$ for $\xi \geq 1$, $\phi'_+(0) = 0$ and $|\phi'(\xi)| \leq 1/2$ for $\xi > 0$. Define $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $\Theta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi(s) = \phi(|s|) \quad \text{and} \quad \Theta(s) = (s_1 + \Phi(s), s_2, \dots, s_d),$$

where $s = (s_1, s_2, \dots, s_d) \in \mathbb{R}^d$. Now, Θ is a diffeomorphism on $\mathbb{R}^d$ which is identity on $\mathbb{R}^d \setminus B(0,1)$ and $\Theta(0) = y$. Put $\Psi = \Theta^{-1}$. For $s \in \mathbb{R}^d$ and $t \in (\mathbb{R}^d)^*$, we have

$$|t \circ \Theta'(s)| = |t + t(e_1)\Phi'(s)| \leq \frac{3}{2}|t|.$$

Moreover, for $s, s' \in \mathbb{R}^d$, we have

$$|\Theta(s) - \Theta(s')| \geq |s - s'| - |\Phi(s) - \Phi(s')| \geq \frac{1}{2}|s - s'|.$$

So $|v \circ \Psi'(u)| \geq \frac{2}{3}|v|$ for $u \in \mathbb{R}^d, v \in (\mathbb{R}^d)^*$, and Ψ is Lipschitz with the constant 2. □

Proposition 4.2. *Let $a \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d \setminus \{a\}$ be a set which is not porous at a. Then there is a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$, Fréchet differentiable on $\mathbb{R}^d \setminus E$, such that $f'(a) = 0$ and $|f'(u)| \geq 1$ for any $u \in \mathbb{R}^d \setminus (E \cup \{a\})$.*

PROOF. Without loss of generality $a = 0$. Put $I = [-1, 1]^d$. Since E is not porous at 0, there is, for any $k \in \mathbb{N}$, some minimal $n_k \in \mathbb{N}$ such that, for any $r \in (0, 2^{-n_k}]$, $rI \subset E + B(0, r/10^{2k})$. Put

$$k(n) = \max_{n_k \leq n} k \quad \text{for} \quad n \geq n_1,$$

$$r_{n,l} = \frac{1}{2^n} - \frac{10l}{2^{n+1} \cdot 10^{2k(n)}} \quad \text{and} \quad p_{n,l} = 10^{2k(n)-1} \quad \text{for } l = 0, \dots, 10^{2k(n)-1} - 1.$$

Rearrange $r_{n,l}$ into the decreasing sequence $\{r_i\}_{i=1}^{\infty}$ and $\{p_i\}_{i=1}^{\infty}$ be the sequence of the corresponding $p_{n,l}$'s. Put

$$s_1 = 0 \quad \text{and} \quad s_{i+1} = s_i + (-1)^{i+1}(r_i - r_{i+1}) \quad \text{for} \quad i \geq 1.$$

Note that $s_i = 0$ and $s_{i+1} = r_i - r_{i+1}$ if i is odd. One can compute that

$$\frac{r_i}{r_i - r_{i+1}} \frac{1}{p_i} = 2 - \frac{10l}{10^{2k(n)}} \quad \text{and} \quad 1 \geq \frac{r_{i+1}}{r_i} \geq 1 - \frac{10}{10^{2k(n)}}$$

for the $n \in \mathbb{N}$ and $l \in \{0, \ldots, 10^{2k(n)-1} - 1\}$ corresponding to i, and so

$$\sup \frac{r_i}{r_i - r_{i+1}} \frac{1}{p_i} \leq 2 \quad \text{and} \quad \lim_{n\to\infty} \frac{r_{i+1}}{r_i} = 1.$$

Moreover,

$$\left| \frac{s_i - s_{i+1}}{r_i - r_{i+1}} \right| = 1 \quad \text{and} \quad \frac{s_i}{r_i} \leq \frac{10}{10^{2k(n)}} \quad \text{for all } \ i \in \mathbb{N},$$

and so $s_i/r_i \to 0$ for $i \to \infty$. Let F be a function which Lemma 3.1 gives for these r_i's and p_i's.

Now, choose $x \in r_i D_{p_i}$. There are some n and l such that $r_i = r_{n,l}$ and $p_i = p_{n,l}$. So there is some $u_x \in E$ with $|x - u_x| < r_{n,l}/10^{2k(n)}$. Put $B_x = B(x, 2r_{n,l}/10^{2k(n)})$ and, by Lemma 4.1, choose a diffeomorphism $\Psi_x : \mathbb{R}^d \to \mathbb{R}^d$, Lipschitz with the constant 2, which is identity on $\mathbb{R}^d \setminus B_x$ and maps u_x onto x such that $|v \circ \Psi'_x(u)| \geq \frac{2}{3}|v|$ for any $u \in \mathbb{R}^d$ and $v \in (\mathbb{R}^d)^*$. Let x_1, x_2 be distinct elements of $D = \bigcup_{i\in\mathbb{N}} r_i D_{p_i}$ with the corresponding r_{n_1,l_1}, p_{n_1,l_1}, r_{n_2,l_2} and p_{n_2,l_2}. We may suppose that $r_{n_1,l_1} \geq r_{n_2,l_2}$. Then

$$|x_1 - x_2| \geq \frac{r_{n_1,l_1}}{2p_{n_1,l_1}} = 5\frac{r_{n_1,l_1}}{10^{2k(n_1)}}$$

if $r_{n_1,l_1} = r_{n_2,l_2}$ and

$$|x_1 - x_2| \geq r_{n_1,l_1} - r_{n_2,l_2} \geq \frac{10}{2^{n_1+1} \cdot 10^{2k(n_1)}} = 5\frac{r_{n_1,0}}{10^{2k(n_1)}} \geq 5\frac{r_{n_1,l_1}}{10^{2k(n_1)}}$$

if $r_{n_1,l_1} > r_{n_2,l_2}$. In both cases,

$$|x_1 - x_2| \geq 5\frac{r_{n_1,l_1}}{10^{2k(n_1)}} > \frac{2r_{n_1,l_1}}{10^{2k(n_1)}} + \frac{2r_{n_2,l_2}}{10^{2k(n_2)}}.$$

So $B_{x_1} \cap B_{x_2} = \emptyset$ and we can define a one-to-one mapping $\Psi : \mathbb{R}^d \to \mathbb{R}^d$, differentiable on $\mathbb{R}^d \setminus \{0\}$ and Lipschitz with the constant 2, by

$$\Psi(u) = \begin{cases} \Psi_x(u) & \text{if } u \in B_x, \\ u & \text{if } u \in \mathbb{R}^d \setminus \bigcup_{x\in D} B_x. \end{cases}$$

Put $f = (6\sqrt{d})F \circ \Psi$. Since f is a composition of Lipschitz mappings, it is Lipschitz. We have $\Psi^{-1}(D) \subset E$, and thus f is differentiable everywhere in $\mathbb{R}^d \setminus E$. For $u \in \mathbb{R}^d \setminus (E \cup \{0\})$, we have

$$|f'(u)| = (6\sqrt{d})|F'(\Psi(u)) \circ \Psi'(u)| \geq \frac{2}{3}(6\sqrt{d})|F'(\Psi(u))| \geq 1$$

by Property 2 of the function F. Finally, $f'(0) = 0$. It follows from Property 1 and from

$$f(B(0,r)) = (6\sqrt{d})F(\Psi(B(0,r))) \subset (6\sqrt{d})F(B(0,2r))$$

for every $r > 0$. □

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References

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