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GREEDY APPROXIMATION IN CERTAIN SUBSYSTEMS OF THE SCHAUDER SYSTEM

Abstract

Although the sequence of greedy approximants associated with the Schauder expansion of a function, f, continuous on $[0,1]$, may fail to converge, there always will be a continuous function, arbitrarily close to f, whose Schauder expansion does have a convergent sequence of greedy approximants. Further examination of this problem shows that the same sort of proposition is valid for a multitude of subsystems of the Schauder system.

1 Introduction.

One part of the rich and interesting theory that has grown around the notion of the greedy algorithm concerns the failure, in certain cases, of the greedy approximants of a given function to converge to the function. Examples of this type connected with the trigonometric system have been provided by Temlyakov [7], who has demonstrated the existence of functions in $L^p[0,1], 1 \leq p < 2$, for which a sequence of corresponding greedy approximants diverges in measure, and by Körner [5], who has constructed a continuous function

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whose Fourier sum, when taken in decreasing order of coefficient magnitudes, diverges unboundedly almost everywhere.

On the other hand, Grigorian [2] has shown that it is always possible to alter the values of an integrable function, on a small set, so that the greedy algorithm, when applied to the function thus altered, will yield a sequence of greedy approximants, that converges to the altered function in the L^1 norm.

Continuing this work, in [4], there were established similar results, for the Walsh system, as well as for a multitude of subsystems of that noble system.

More recently Grigoryan and Sargsyan [3] have shown that a continuous function can be modified, on a small set, in such a manner that the sequence of greedy approximants associated with the Faber–Schauder expansion of the function thus altered converges uniformly to the altered function.

Motivated, perhaps, by the earlier treatment of the Walsh subsystems, these Authors have suggested that it might not be necessary to have the entire Schauder system in order for their result to obtain. That this is indeed the case follows from an appeal to ancient work of Goffman **[1]**. In fact, the techniques developed by that estimable scholar can be employed to show that, for a Schauder subsystem that is total in measure, each real–valued measurable function f can be closely approximated by a continuous function g the subsystem expansion of which has greedy approximants that converge uniformly to g.

2 Preliminary Observations.

A Schauder basis for a Banach space X is a countable set $\Psi = \{\psi_n : n \in \mathbb{N}\}$ of elements of X with respect to which each f in X can be represented by an unique series $\sum_{n=1}^{\infty} \mathcal{C}_n(f)\psi_n$ that converges to f in the norm of the space.

If σ is a permutation of the natural numbers for which

$$|\mathcal{C}_{\sigma(n)}(f)| \geq |\mathcal{C}_{\sigma(n+1)}(f)| \quad (n),$$

then

$$G_m(f) = G_m(f, \Psi, \sigma) =: \sum_{n=1}^{m} \mathcal{C}_{\sigma(n)}\psi_{\sigma(n)}$$

is the m^{th} greedy approximant of f with respect to the basis Ψ and the permutation σ.

In his treatment of the Schauder–series representation of measurable real functions, Goffman has employed a Schauder basis, Ψ, that results from a minor alteration of the basis originally considered by Schauder and Faber. In this system one has

$$x_{-1}^{(1)}(t) = t,\ \forall t \in [0,1];$$
$$x_{-1}^{(2)}(t) = 1-t,\ \forall t \in [0,1];$$

and, for $m = 0, 1, \ldots,$ and $k = 1, \ldots, 2^m$,

$$x_m^{(k)}(t) = \begin{cases} 2^{m+1}t - 2(k-1), & \text{if } \frac{k-1}{2^m} \le t \le \frac{2k-1}{2^{m+1}}; \\ 2k - 2^{m+1}t, & \text{if } \frac{2k-1}{2^{m+1}} \le t \le \frac{k}{2^m}; \\ 0, & \text{otherwise}; \end{cases}$$

and the system is ordered lexicographically.

For the purpose of representation of measurable real functions, Price and Zink [6] have observed that, in the following sense, the Schauder system is exceedingly rich.

A system Φ of measurable functions, defined and a.e. real valued on a measurable set G, is *total (closed) in measure* if for each measurable f defined on G there is a sequence of Φ polynomials that converges in measure to f. Of course, the Schauder system is total in measure on $[0, 1]$, but so also are many of its subsystems [6].

Remark A. *Let $\Phi = \{\varphi_n : n \in \mathbb{N}\}$ be a subsystem of the Schauder system, and, for each n, let E_n be the support of φ_n. Then Φ is total in measure on $[0, 1]$ iff*

$$\mu(\limsup_n E_n) = 1. \tag{*}$$

The proposition promised in the introduction treats those subsystems of Ψ that enjoy this closure property (*).

Theorem 1. *Let Φ be a subsystem of the Schauder system that is total in measure, let f be an a.e. real–valued function defined and measurable on $[0, 1]$, and let ε be an arbitrarily small positive real number. There exists a continuous function g such that*

$$\mu(\{t : g(t) \neq f(t)\}) < \varepsilon,$$

the Schauder expansion of g involves only elements of Φ, and the sequence of greedy approximants of g converges uniformly to g.

Following Goffman, one makes the following observations upon which the proof of the theorem depends.

A point $k/2^m, m = 0, 1, \ldots, k = 0, \ldots, 2^m$, is termed a dyadic point, and a dyadic interval is one whose end points are dyadic. The rank of a dyadic point $(2j+1)/2^m$ is the number m, and the rank of a dyadic interval of length $1/2^m$ is the number m.

Remark B. *If $g \in C[0,1]$ vanishes at 0 and at 1 and at each dyadic point of rank m or smaller, then the Schauder expansion of g*

$$a_{-1}^{(1)} x_{-1}^{(1)} + a_{-1}^{(2)} x_{-1}^{(2)} + a_0 x_0^{(1)} + \cdots$$

has

$$a_{-1}^{(1)} = a_{-1}^{(2)} = a_0^{(1)} = \cdots = a_m^{(2^m)} = 0.$$

Moreover, if $h \in C[0,1]$, then for each $\varepsilon > 0$ and $m \geq 0$, there is $g \in C[0,1]$ such that

$$\|g\| \leq \|h\|, \quad \mu(\{t : g(t) \neq h(t)\}) < \varepsilon,$$

and g vanishes at 0 and 1 and at each dyadic point of rank m or smaller.

For the first assertion one recalls that $a_{-1}^{(1)} = g(1)$, and if the k^{th} point in the lexicographic ordering of the dyadic points be denoted by t_k, and if the corresponding Schauder function be denoted by x_k, then the coefficient of x_k in the Schauder expansion of g is

$$g(t_k) - (a_{-1}^{(1)} x_{-1}^{(1)} + a_{-1}^{(2)} x_{-1}^{(2)} + \cdots + a_{k-1} x_{k-1})(t_k),$$

so that a simple induction argument suffices.

As for the second assertion one may surround each of the interior dyadic points involved by an open interval $(t_k - \delta_k, t_k + \delta_k)$ and take half–open intervals $[0, \delta_0)$ and $(1 - \delta_1, 1]$ for the endpoints, with the sum of the lengths of these intervals smaller than ε; let $g(t) = 0$, for each of the dyadic points t_k; let g coincide with h on the complement of the union of these intervals; and let g be linear on each of the intervals $[0, \delta_0], [1 - \delta_1, 1], [t_k - \delta_k, t_k]$, and $[t_k, t_k + \delta_k]$.

Remark C. *Let $\{I_1, \ldots, I_r; J_1, \ldots, J_s\}$ be a partition of $[0,1]$ into dyadic intervals such that the J_i have the same rank $m-1$ and each I_j has rank less than $m-1$. If f be a continuous function that vanishes on each I_j and at the endpoints of each J_i, and if f be either nonnegative or nonpositive on each J_i, then the Schauder expansion of f has the form*

$$f = a_m^{(1)} x_m^{(1)} + \cdots + a_m^{(2^m)} x_m^{(2^m)} + \cdots .$$

Moreover, every partial sum of each subseries of the expansion has norm not exceeding $\|f\|$, vanishes on each I_j, is nonnegative on each J_i where f is nonnegative and is nonpositive on each J_i where f is nonpositive.

This remark is an immediate consequence of Remark B and the algorithm by means of which the Schauder coefficients are determined.

Remark D. *Corresponding to $f \in C[0,1], n \in \mathbb{N}$, and $\varepsilon > 0$, there are an $m > n$, a partition of $[0,1]$, $\{I_1, \ldots, I_r; J_1, \ldots, J_s\}$, and a continuous function g such that:*

(α.) $\mu(\{t : g(t) \neq f(t)\}) < \varepsilon$;

(β.) *Each J_i has rank m, and each I_j has rank less that m;*

(γ.) *g is either nonnegative or nonpositive on each J_i;*

(δ.) *g vanishes on each I_j and the endpoints of each J_i; and*

(ε.) $\|g\| \leq \|f\|$.

For this Goffman chooses pairwise disjoint intervals $K_1, \ldots, K_k$ contained in $\{t : f(t) \neq 0\}$, such that $\mu(\cup_{i=1}^k K_i) > \mu(\{t : f(t) \neq 0\}) - \frac{\varepsilon}{3}$, shrinks and partitions each K_i so that it is composed of dyadic intervals, partitions the complementary intervals so that they are dyadic and denotes these by $I_1, \ldots, I_r$. The total shrinking involved being an amount less than $\varepsilon/3$. The subintervals of the K_i are further partitioned so that they have the desired rank, and these new intervals are $J_1, \ldots, J_s$. Finally, f is altered so as to be unchanged on each J_i, to be 0 on each I_j, and to be linear on each of the intervals that separate the intervals of $\{I_1, \ldots, I_r\}$ from those of $\{J_1, \ldots, J_s\}$.

Lemma E. *If $f \in C[0,1]$, then for each $\varepsilon > 0, \delta > 0$, and $m \in \mathbb{N}$, there is a Schauder polynomial*

$$a_m^{(1)} x_m^{(1)} + \cdots + a_n^{(2^n)} x_n^{(2^n)}, \quad n > m,$$

such that no coeffieient exceeds $\|f\|$ in absolute value,

$$|f(t) - (a_m^{(1)} x_m^{(1)} + \cdots + a_n^{(2^n)} x_n^{(2^n)})(t)| < \varepsilon,$$

on a set of measure greater than $1 - \delta$, and for each subset B of $\{(i,j) : m \leq i \leq n, 1 \leq j \leq 2^i\}$,

$$\| \sum_{(i,j) \in B} a_i^{(j)} x_i^{(j)} \| \leq \|f\|.$$

Here one takes g to be the approximating function described in Remark D. By virtue of Remark C, the Schauder series for g has a partial sum that provides the specified approximation.

A simple refinement of this result allows one further to control the size of the coefficients involved in the approximating polynomial.

Lemma F. *If $f \in C[0,1]$, then for each $\varepsilon > 0, \delta > 0, \eta > 0$, and $m \in \mathbb{N}$, there is a Schauder polynomial*

$$a_m^{(1)} x_m^{(1)} + \cdots + a_n^{(2^n)} x_n^{(2^n)}, \quad n > m,$$

such that $|a_i^{(j)}| \leq \eta$, for each $(i,j) \in S = \{(i,j) : \ 1 \leq i \leq n, \ 1 \leq j \leq 2^i\}$,

$$|f(t) - (a_m^{(1)} x_m^{(1)} + \cdots + a_n^{(2^n)} x_n^{(2^n)})(t)| < \varepsilon,$$

on a set of measure greater than $1 - \delta$, and for each $B \subset S$,

$$\| \sum_{(i,j) \in B} a_i^{(j)} x_i^{(j)} \| < \eta.$$

One chooses continuous functions $f_1, \dots, f_r$, such that each has norm less than η/r, and $f = f_1 + \cdots + f_r$. Applying Lemma E to f_1, with $\varepsilon/r, \delta/r$, and m, one obtains a Schauder polynomial

$$a_m^{(1)} x_m^{(1)} + \cdots + a_{n_1}^{(2^{n_1})} x_{n_1}^{(2^{n_1})}$$

that approximates f_1 to within ε/r on a set of measure greater than $1 - \delta/r$, with

$$\| \sum_{(i,j) \in B} a_i^{(j)} x_i^{(j)} \| < \eta/r, \ \forall B \subset \{(i,j) : 1 \leq i \leq n_1, 1 \leq j \leq 2^i\}.$$

Again, one applies Lemma E to f_2, with $\varepsilon/r, \delta/r$, and $n_1 + 1$, and continues this process inductively.

Lemma G. *If $\Phi = \{\varphi_1, \varphi_2, \dots\}$ be a subsystem of the Schauder system that is total in measure, then every Schauder function, f can be expressed as a series $\sum_{n=1}^{\infty} a_n \varphi_n$ that converges a.e. to f, wherein $0 \leq a_n \leq 1$, for every n, and $\sum_{j \in B} a_j \varphi_j \leq f$, for every $B \subset \mathbb{N}$.*

The demonstration of this final piece of the Goffman program is effected by the determination of an increasing sequence of natural numbers $\{k(n)\}_{n=1}^{\infty}$ and a sequence of nonnegative Φ polynomials $\{\varphi^{(n)} = \sum_{i=k(n-1)+1}^{k(n)} a_i \varphi_{n_i}\}_{n=1}^{\infty}$ $(k(0) = 1)$ such that the sequence $\{f_n = \sum_{i=1}^{n} \varphi^{(i)}\}_{n=1}^{\infty}$ increases monotonically a.e. to f.

The confluence of Lemmas F and G provides the core of the proof of the theorem.

Lemma H. *Let $\Phi = \{\varphi_1, \varphi_2, \dots\}$ be a subsystem of Ψ that is total in measure. For each $f \in C[0,1], \varepsilon > 0$, $\delta > 0, \eta > 0$, and $m \in \mathbb{N}$, there is a Φ polynomial*

$$Q = \sum_{j=m+1}^{n} \mathcal{C}_j \varphi_j,$$

such that $|\mathcal{C}_j| < \eta$, $j = m+1, \dots, n$, $|f(t) - Q(t)| < \varepsilon$ on a set of measure greater than $1 - \delta$, and, for each $B \subset \{m+1, \dots, n\}$

$$\| \sum_{j \in B} \mathcal{C}_j \varphi_j \| < \eta.$$

It is helpful to relabel the Schauder functions

$$x_{-1}^{(1)} = x_1,\ x_{-1}^{(2)} = x_2,\ x_0^{(1)} = x_3, \dots,$$

in order to simplify the rather complex notation that arises in the demonstration of this proposition.

By virtue of Lemma F, there is a Schauder polynomial,

$$P = a_k x_k + \cdots + a_\ell x_\ell,$$

and a measurable set E_0, such that $|a_i| < \eta$, $i = k, \dots, \ell$,

$$\| \sum_{j \in B} a_j x_j \| < \eta, \quad \forall B \subset \{k, \dots, \ell\},$$

$$|f(t) - P(t)| < \frac{\varepsilon}{2}, \quad \forall t \in E_0, \quad \text{and} \quad \mu(E_0) > 1 - \frac{\delta}{2}.$$

Since Φ is total in measure, so also are each of the subsystems $\Phi_r = \{\varphi_j \in \Phi : j \geq r\}$, $r \in \mathbb{N}$. Thus, for each $i \in \{k, \dots, \ell\}$, one may approximate x_i, as in Lemma G, by a polynomial

$$P_i = b_{m(i)+1}^{(i)} \varphi_{m(i)+1} + \cdots + b_{n(i)}^{(i)} \varphi_{n(i)},$$

where $m(1) = m, m(2) = n(1), \dots, m(\ell) = n(\ell - 1), n(\ell) = n$, such that each coefficient lies in $[0,1]$,

$$|x_i(t) - P_i(t)| < \frac{\varepsilon}{2M(1 + |a_i|)},\ \forall t \in E_i,\ \mu(E_i) > 1 - \frac{\delta}{2M},$$

where $M = \ell - k + 1$, and, for each $B_i \subset \{1, \ldots, n(i) - m(i)\}$,

$$0 \leq \sum_{s \in B_i} b^{(i)}_{m(i)+s}\, \varphi_{m(i)+s}(t) \leq x_i(t), \quad \forall t.$$

Let

$$\begin{aligned} Q = \sum_{i=k}^{\ell} a_i P_i &= \sum_{i=k}^{\ell} \sum_{s=1}^{n(i)-m(i)} a_i b^{(i)}_{m(i)+s}\, \varphi_{m(i)+s} \\ &= \sum_{j=m+1}^{n} \mathcal{C}_j \varphi_j, \end{aligned}$$

where

$$\begin{aligned} &\mathcal{C}_{m+1} = a_k b^{(1)}_{m+1}, \mathcal{C}_{m+2} = a_k b^{(1)}_{m+2}, \cdots, \mathcal{C}_{n(1)} = a_k b^{(1)}_{n(1)}, \\ &\mathcal{C}_{n(1)+1} = a_{k+1} b^{(2)}_{n(1)+1}, \cdots, \mathcal{C}_{n(2)} = a_{k+1} b^{(2)}_{n(2)}, \\ &\qquad\qquad \cdots\cdots \\ &\mathcal{C}_{n(\ell-1)+1} = a_\ell b^{(\ell)}_{n(\ell-1)+1}, \cdots, \mathcal{C}_n = a_\ell b^{(\ell)}_n. \end{aligned}$$

For each $t \in \bigcap_{i=k}^{\ell} E_i \cap E$, one has

$$\begin{aligned} |f(t) - Q(t)| &\leq |f(t) - P(t)| + |P(t) - Q(t)| \\ &< \frac{\varepsilon}{2} + \Big| \sum_{i=k}^{\ell} a_i (x_i(t) - P_i(t) \Big| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=k}^{\ell} |a_i| \frac{\varepsilon}{2M(1+|a_i|)} \\ &< \varepsilon, \end{aligned}$$

and $\mu\left(\bigcap_{i=k}^{\ell} E_i \cap E \right) > 1 - \delta$.

As for the last estimate, if $B \subset \{m+1, \ldots, n\}$, let $B = \bigcup_{i=k}^{\ell} B_i$, where

$$B_i = \{j \in B : m(i) + 1 \leq j \leq n(i)\},\ m(0) = m.$$

Then

$$\sum_{j\in B}\mathcal{C}_j\varphi_j=\sum_{i=k}^{\ell}\sum_{j\in B_i}\mathcal{C}_j\varphi_j.$$

Since

$$0\le\sum_{j\in B_i}\mathcal{C}_j\varphi_j=a_i\sum_{j\in B_i}b_{m(i)+s}\,\varphi_{m(i)+s}\le a_ix_i,\ \text{if } a_i\ge 0,$$

and

$$a_ix_i\le a_i\sum_{j\in B_i}b_{m(i)+s}\varphi_{m(i)+s}=\sum_{j\in B_i}\mathcal{C}_j\varphi_j\le 0,\ \text{if } a_i\le 0,$$

one has

$$\|\sum_{j\in B}\mathcal{C}_j\varphi_j\|<\eta.$$

3 Concluding the Proof.

By virtue of the theorem of Lusin, if f is measurable and real valued on $[0,1]$, and if ε is a positive real number, then there is a function h, continuous on the unit interval, that differs from f on a set of measure less than ε. Thus, for the proof of the theorem, one may assume that f is continuous.

Following Goffman et al, there is an $f_0=\sum_{i=1}^{n_1}a_i\varphi_i\in$ Span Φ such that $\|f_0\|\le\|f\|$, and $|(f-f_0)(t)|<\frac{\varepsilon}{2}$, $\forall t\in E_0$, with $\mu(E_0)>1-\frac{\varepsilon}{4}$. Since $\{t:|(f-f_0)(t)|<\frac{\varepsilon}{2}\}$ is an open set, there is a finite set of pairwise–disjoint open intervals $\{V_{11},\dots,V_{1m_1}\}$ such that

$$\bigcup_{j=1}^{m_1}V_{1j}\subset E_0\quad\text{and}\quad\mu(\bigcup_{j=1}^{m_1}V_{1j})>\mu(E_0)-\frac{\varepsilon}{8}.$$

Let $\{I_{11},\dots,I_{1r_1}\}$ be the set of closed intervals complementary to $\bigcup_{j=1}^{m_1}V_{1j}$, and let $F_1=\bigcup_{j=1}^{r_1}I_{1j}$.

By shrinking the $V_{1j}, j=1,\dots,m$, by a total amount less than $\varepsilon/8$, designating the new (open) intevals by $U_{11},\dots,U_{1m_1}$, and setting $U_1=\bigcup_{j=1}^{m_1}U_{1j}$, one may obtain a continuous function $(f-f_0)^{\sim}$, linear on each of the intervals that separate the U_{1j} from the I_{1i}, with

$$(f-f_0)^{\sim}(t)=\begin{cases}(f-f_0)(t), & \text{if } t\in U_1;\\ 0, & \text{if } t\in F_1;.\end{cases}$$

such that $\|(f-f_0)^{\sim}\|<\dfrac{\varepsilon}{2}$, and

$$f-f_0=(f-f_0)^{\sim}\quad\text{on } U_1,\ \mu(U_1)>1-\frac{\varepsilon}{2}.$$

Next, let $f_1 = \sum_{j=n_1+1}^{n_2} a_j\varphi_j$ satisfy the conditions

$$|\left((f-f_0)^{\sim} - f_1\right)(t)| < \frac{\varepsilon}{4},\ \forall t \in E_1, \mu(E_1) > 1 - \frac{\varepsilon}{8},$$
$$|a_j| \le \min\{|a_i| : 1 \le i \le n_1, a_i \ne 0\}, j = n_1+1, \dots, n_2,$$
$$\|f_1\| \le \|(f-f_0)^{\sim}\|.$$

Then

$$\left((f-f_0)^{\sim} - f_1\right)(t) = (f - f_0 - f_1)(t),\ \forall t \in E_1 \cap U_1, \quad \text{and}$$
$$\mu(E_1 \cap U_1) > 1 - \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{8}\right).$$

As before, there are pairwise–disjoint open intervals $V_{21}, \dots, V_{2m_2}$, such that

$$\bigcup_{j=1}^{m_2} V_{2j} \subset E_1, \quad \text{and} \quad \mu\left(\bigcup_{j=1}^{m_2} V_{2j}\right) > \mu(E_1) - \frac{\varepsilon}{16}.$$

One shrinks each V_{2j}, the total shrinkage being an amount less than $\varepsilon/16$; one designates the new open intervals thus obtained by $U_{21}, \dots, U_{2m_2}$; one sets $U_2 = \bigcup_{j=1}^{m_2} U_{2j}$; and one defines $((f-f_0)^{\sim} - f_1)^{\sim}$, a further modification of f, to be the continuous function that agrees with $(f-f_0)^{\sim} - f_1$ on U_2, that vanishes on each of the closed intervals that make up $[0,1] \setminus \bigcup_{j=1}^{m_2} V_{2j}$ and is linear otherwise.
Then

$$\|((f-f_0)^{\sim} - f_1)^{\sim}\| < \frac{\varepsilon}{4},$$
$$((f-f_0)^{\sim} - f_1)^{\sim}(t) = (f - f_0 - f_1)(t),\ \forall t \in U_1 \cap U_2,$$

and

$$\mu(U_1 \cap U_2) > 1 - \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{4}\right).$$

Proceeding thus, inductively, one constructs a sequence $\{f_k\}_{k=1}^{\infty}$ of Φ–polynomials

$$f_k = \sum_{j=n_k+1}^{n_{k+1}} a_j\varphi_j, \quad k = 0, 1, \dots,$$

and a corresponding sequence of open sets $\{U_k\}_{k=1}^{\infty}$ such that

$$|a_j| \le \min\{|a_i| : a_i \neq 0, 1 \le i \le n_k\}, j > n_k;$$
$$\|f_k\| \le \frac{\varepsilon}{2^k}, \ \forall k;$$
$$\mu\Big(\bigcap_{j=1}^{k} U_j\Big) > 1 - \sum_{j=1}^{k} \frac{\varepsilon}{2^j};$$

and

$$\Big|\Big(f - \sum_{j=0}^{k} f_j\Big)(t)\Big| < \frac{\varepsilon}{2^{k+1}}, \ \forall t \in \bigcap_{j=1}^{k} U_j.$$

Certainly $\sum_{j=0}^{\infty} f_k$ converges uniformly, on $[0,1]$, to a continuous function g which differs from f on a set of measure less than ε. Moreover, $\sum_{i=1}^{\infty} a_i\varphi_i$ is the Schauder series for g, since, for n_k the greatest n_i not exceeding n, one has

$$\|g - \sum_{j=1}^{n} a_j\varphi_j\| \le \|g - \sum_{j=1}^{k} f_j\| + \|\sum_{j=n_k+1}^{n} a_j\varphi_j\| < \|g - \sum_{j=1}^{k} f_j\| + \frac{\varepsilon}{2^{k+1}},$$

and, since $\| \sum_{j=n_k+1}^{n} a_{\sigma(j)}\varphi_{\sigma(j)}\| = \| \sum_{j\in B_k} a_j\varphi_j\|$, for some $B_k \subset \{n_k+1, \ldots, n_{k+1}\}$, one has

$$\begin{aligned}\|g - G_n(f,\Phi,n)\| &\le \|g - \sum_{j=1}^{n_k} a_{\sigma(j)}\varphi_{\sigma(j)}\| + \|\sum_{j=n_k+1}^{n} a_{\sigma(j)}\varphi_{\sigma(j)}\| \\ &\le \|g - \sum_{j=1}^{k} f_j\| + \|\sum_{j\in B_k} a_j\varphi_j\| \\ &< \|g - \sum_{j=1}^{k} f_j\| + \frac{\varepsilon}{2^{k+1}},\end{aligned}$$

so that the sequence of greedy approximants of g also converges uniformly to g.

References

[1] Casper Goffman, *Remark on a problem of Lusin*, Acta Math., **111** (1964), 63–72.

[2] M. G. Grigorian, *On the convergence of the greedy algorithm in the L'norm*, J. of Contemp. Math. Anal., **39** (2004), 89–116.

[3] M. G. Grigoryan and A. A. Sargsyan, *Greedy approximation with respect to the Faber–Schauder system,* J. Approx. Theory, (to appear).

[4] Martin G. Grigorian and Robert E. Zink, *Greedy approximation with respect to certain subsystems of the Walsh orthogonal system,* Proc. Amer. Math. Soc. **134** (2006), 3495–3505.

[5] T. W. Körner, *Decreasing rearranged Fourier series,* J. Fourier Anal. and Appl. **5** (1999), 1–19.

[6] J. J. Price and Robert E. Zink, *On sets of completeness for families of Haar functions,* Trans. Amer. Math. Soc. **119** (1964), 262–269.

[7] V. N. Temlyakov, *Greedy algorithm and m–term approximation,* Constr. Approx. **14** (1998), 569–587.

[8] Robert E. Zink, *On a theorem of Goffman concerning Schauder series,* Proc. Amer. Math. Soc. **21** (1969), 523–529.