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Zbigniew Duszyński,* Pedagogical University, Institute of Mathematics, Chodkiewicza 30, 85-064 Bydgoszcz, Poland.
email: `imath@wsp.bydgoszcz.pl`

THE OSCILLATION FUNCTION ON METRIC SPACES

Abstract

For each metric space (X, ξ) and each bounded function $f\colon X \to \mathbb{R}$ the family of the sets $\Omega_f(y) = \{x \in X\colon \omega_f(x) \geq y\}$ ($\omega_f(x)$ is the oscillation of f) has some well known properties. In this paper it is constructively shown that for each family $\{\Omega(y)\}_{y\in[0,1]}$ of subsets of X (separable and $\mathfrak{c}$-dense in itself) having similar properties there exists a function $f\colon X \to [0,1]$ such that $\Omega_f(y) = \Omega(y)$ for each $y \in [0,1]$.

Let X be any separable metric space of one's choice and $f\colon X \to \mathbb{R}$ an arbitrary function ($\mathbb{R}$ denotes the set of all real numbers). We will use the following description of the set of *limit points of f at x_0* denoted by $L(f, x_0)$: $L(f, x_0)$ is the set of all points $y \in \mathbb{R}$ such that there exists a sequence $\{x_n\} \subset X$ with $\lim x_n = x_0, \quad x_n \neq x_0 \quad (n = 1, 2, \ldots)$ and $\lim f(x_n) = y$. It is well known that $L(f, x)$ is closed in $\mathbb{R}$ for each $x \in X$. Let us recall the natural definition of the function of oscillation for $f\colon X \to [0,1]$: $\omega_f(x)\colon = \max[L(f,x) \cup \{f(x)\}] - \min[L(f,x) \cup \{f(x)\}]$. For each $y \in [0,1]$ let

$$\Omega_f(y)\colon = \{x \in X\colon \omega_f(x) \geq y\}.$$

It is not difficult to see that the following facts hold for every function $f\colon X \to [0,1]$.

1) The set $\Omega_f(y)$ is closed in (X, τ_X), where the distance function on X defines the topology τ_X in a standard way,

2) If $y_1 < y_2$ then $\Omega_f(y_2) \subset \Omega_f(y_1)$,

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3) The set $\bigcup_{y\in[0,1]} \big(\Omega_f(y)\times\{y\}\big)$ is $\tau_X\times\tau$–closed (τ — natural topology on $\mathbb{R}$).

Now let us take an arbitrary family $\{\Omega(y)\}_{y\in[0,1]}$ of nonempty subsets of X such that:

1°) The set $\Omega(y)$ is τ_X–closed for each $y\in[0,1]$,

2°) If $y_1<y_2$ then $\Omega(y_2)\subset\Omega(y_1)$,

3°) The set $\bigcup_{y\in[0,1]} \big(\Omega(y)\times\{y\}\big)$ is $\tau_X\times\tau$–closed,

4°) $\Omega(0)=X$

The main result of this paper is the following,

Theorem 1. *Let X be an arbitrary $\mathfrak{c}$-dense ($\mathfrak{c}=card(\mathbb{R})$) in itself and separable metric space. For each family $\{\Omega(y)\}_{y\in[0,1]}$ that fulfills conditions 1°) – 4°) above, there exists a function $f\colon X\to[0,1]$ such that for every $y\in[0,1]$ we have*

0°) $\Omega(y)=\Omega_f(y)$.

PROOF. In this proof we use the well known Cantor-Bendixson Theorem: each τ_X–closed subset A of separable metric space X can be represented as a sum of two disjoint subsets A_1 and A_2, the first of which is a set of all points of condensation of A, and the second is denumerable. For each $y\in[0,1]$, let $\Omega(y)=A(y)\cup B(y)$ (note that$A(y)\cap B(y)=\varnothing$) where, from the Cantor-Bendixson Theorem, $A(y)$ is τ_X–perfect and $B(y)$ is denumerable. Let $B_a\equiv\{y\in[0,1]\colon\ a\in B(y)\}$ for each $a\in X$. The set F of all points a of X for which $B_a\neq\varnothing$ is at most denumerable. Proof of this fact is analogous to the proof of the lemma in [1], where the notions of condensation and closure are taken in the sense of τ_X.

For each $y\in[0,1]$, let $A'(y)$ denote a denumerable and dense set in $\Omega(y)$ and $U:=\mathbb{Q}\cap[0,1]$ where $\mathbb{Q}$ is the set of all rational numbers. Our function f we define as follows:

$$f(x)=\begin{cases}\sup\{y\in U\colon\ x\in A'(y)\} & \text{for } x\in\bigcup_{y\in U}A'(y)\\ 0 & \text{otherwise}\end{cases}$$

First we shall prove the inclusion
[A] $\Omega_f(y)\subset\Omega(y)$.

A.1 Let us take an arbitrary point $y_0 \in [0,1]$ and $x \in \Omega_f(y_0)$.

A.1.1 If $f(x) \geq y_0$ then the definition of f and condition 3°) imply that there exists $y_1 \in U$ such that $x \in A'(y_1)$ and $y_1 \geq y_0$. Since $\Omega(y_1) \subset \Omega(y_0)$, then $x \in \Omega(y_0)$.

A.1.2 Let $y' = \max L(f,x) \geq y_0$. Therefore, there exists a sequence $\{x_n\} \subset X$ such that $x_n \in \bigcup\limits_{y \in U} A'(y)$ $(x_n \neq x,\, n \in \mathbb{N})$, $\lim x_n = x$, $\lim f(x_n) = y'$ and $y' \geq y_0$.

From the definition of f we infer that $f(x_n) = \sup\{y \in U\colon x_n \in A'(y)\}$. Consider two cases:

(a) Presume that $y' > y_0$. Then there exists the subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $f(x_{n_m}) = y_{n_m} > y_0$. Then $\Omega(y_{n_m}) \subset \Omega(y_0)$ and $x_{n_m} \in \Omega(y_0)$. Since $x_{n_m} \xrightarrow{m \to \infty} x$ and $\Omega(y_0)$ is τ_X–closed we obtain that $x \in \Omega(y_0)$.

(b) Suppose that $y' = y_0$, i.e. max $L(f,x) = y_0$ and for the sequence $\{x_n\}$ we have $\lim f(x_n) = y_0$. There are two cases:

(b') If for infinite number of points of $\{x_n\}$ the value of f is greater than y_0, then there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(x_{n_k}) = y_{n_k} > y_0$. Therefore, as in (a), we have $x_{n_k} \xrightarrow{k \to \infty} x$, $x_{n_k} \in \Omega(y_0)$ and since $\Omega(y_0)$ is closed $x \in \Omega(y_0)$.

(b") If there are only a finite number of elements of the subsequence $\{x_n\}$ whose values of f are greater than y_0, then without loss of generality, we can assume that for each $n \in \mathbb{N}$ $f(x_n) = \sup\{y \in U\colon x \in A'(y)\} \leq y_0$. If $x \notin \Omega(y_0)$, then for a certain n_0, points x_n with $n \geq n_0$ also do not belong to $\Omega(y_0)$. This means that for a sufficiently small number $\delta > 0$

$$x \in \bigcup_{\substack{y \in U \\ y \leq y_0 - \delta}} A'(y) \qquad \text{for each } n > n_0.$$

Hence $\lim f(x_n) \leq y_0 - \delta < y_0$ i.e. $\lim f(x_n) < y_0$, a contradiction.

[B] $\Omega(y) \subset \Omega_f(y)$.

B.1 Let y_0 be an arbitrary point point from $U \setminus \{0\}$.

B.1.1 Let $x \in A'(y_0)$. From the definition of f we have $\omega_f(x) \geq y_0$ and hence $x \in \Omega_f(y_0)$.

B.1.2 Let $x \in A(y_0) \backslash A'(y_0)$. Then there exists a sequence $\{x_n\}$ such that $x_n \in A'(y_0)$ ($x_n \neq x$, $n \in \mathbb{N}$) and $\lim x_n = x$. From the formula for f we infer that $f(x_n) \geq y_0$ ($n \in \mathbb{N}$). Since $\max[L(f,x) \cup \{f(x)\}] \geq \overline{\lim} f(x_n) \geq y_0$ then again $\omega_f(x) \geq y_0$. Therefore $x \in \Omega_f(y_0)$.

B.2 Let $y_0 \in [0,1] \backslash U$ be an arbitrary fixed point. Let us take the sequence $\{y_m\} \subset U$ such that $y_m < y_0$ ($m \in \mathbb{N}$) and $\lim y_m = y_0$.

B.2.1 Let $x \in A(y_0)$, then for every $m \in \mathbb{N}$ $x \in A(y_m)$. Simultaneously for each $m \in \mathbb{N}$ there exists a sequence $\{x_n^{(m)}\}$ such that $x_n^{(m)} \neq x$, $x_n^{(m)} \to x$ and $x_n^{(m)} \in A'(y_m)$. Let us take an open ball V_1 with center at x and choose $x_{n_1}^{(1)} \neq x$ belonging to V_1. Next, for $m = 2$ let us take the ball V_2 with center at x such that $x_{n_1}^{(1)} \notin V_2$ and choose $x_{n_2}^{(2)} \in V_2$, $x_{n_2}^{(2)} \neq x$. Proceeding by induction and having chosen for some k a point $x_{n_k}^{(k)} \in V_k$, let us take the ball V_{k+1} (with center x) such that $x_{n_k}^{(k)} \notin V_{k+1}$ and choose a point $x_{n_{k+1}}^{(k+1)} \neq x$. Denote $x_{n_m}^{(m)} = t_m$, $m \in \mathbb{N}$. Finally, we obtain the inductively defined sequence $\{t_m\} \subset X$ such that $t_m \neq t_{m'}$ for $m \neq m'$, $t_m \neq x$, $t_m \in A'(y_m)$ and $\lim t_m = x$. From the definition of f we obtain $f(t_m) \geq y_m$. Since $\max[L(f,x) \cup \{f(x)\}] \geq \overline{\lim} f(t_m) \geq y_0$, then $x \in \Omega_f(y_0)$.

B.2.2 For $x \in B(y_0)$ the proof is obvious.

B.3 For $y_0 = 0$ the proof is also obvious. $\square$

Corollary 1. *The function f defined above is of the second class of Baire.*

References

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