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THE DOUBLE DIFFERENCE PROPERTY FOR SOME CLASSES OF FUNCTIONS

Abstract

Using some results of M. Laczkovich and T. Keleti we show that for several classes of functions the double difference property holds.

The notions of the difference property and the double difference property for various classes of real functions were investigated by several authors. (See e.g. [1], [5], and [6].) Some new ideas connected with the difference property are contained in the recent results of Keleti ([3], [4]). We develop those studies and obtain new theorems showing that certain classes of functions have the double difference property.

We will consider the additive group $\mathbb{G}$ equal to $\mathbb{R}$ or $\mathbb{T}$ where $\mathbb{T}$ is the circle group $\mathbb{R}/\mathbb{Z}$ (and $\mathbb{Z}$ stands for the set of all integers). Functions defined on $\mathbb{T}$ can be treated as functions defined on $\mathbb{R}$ and being periodic with period 1. We will study pairs $(\mathcal{F}, \mathcal{F}^{(2)})$ where $\mathcal{F}$ and $\mathcal{F}^{(2)}$ will always be classes of functions on $\mathbb{G}$ and $\mathbb{G}^2$, respectively. Usually $\mathcal{F}$ and $\mathcal{F}^{(2)}$ will be classes of one and two variable functions having a given property. For a fixed function $f : \mathbb{G} \to \mathbb{R}$ and any $h \in \mathbb{G}$ we define *the difference function* $\Delta_h f : \mathbb{G} \to \mathbb{R}$ by

$$\Delta_h f(x) = f(x + h) - f(x),$$

and *the double difference function* $Df : \mathbb{G}^2 \to \mathbb{R}$ by

$$Df(x, y) = f(x + y) - f(x) - f(y).$$

Recall that a class $\mathcal{F}$ (respectively, a pair of classes $(\mathcal{F}, \mathcal{F}^{(2)})$) is said to have *the difference property* (respectively, *the double difference property*), if every function $f : \mathbb{G} \to \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ (respectively, $Df \in \mathcal{F}^{(2)}$) for each $h \in \mathbb{G}$, is of the form $f = g + H$ where $g \in \mathcal{F}$ and H is an additive

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function, that is, $H(x+y)=H(x)+H(y)$ for every $x,y\in\mathbb{R}$. Our Proposition 1 shows some connection between these two properties.

A class $\mathcal{F}$ is called *translation invariant* if, for any $f\in\mathcal{F}$ and $a,b\in\mathbb{G}$, the function $g(x):=f(x+a)+b$, $x\in\mathbb{G}$, belongs to $\mathcal{F}$.

Let $\mathcal{I}$ be a translation invariant, proper σ-ideal on $\mathbb{G}$. We are most interested in the special cases of null and the first category sets. Denote these ideals by $\mathcal{N}$ and $\mathcal{M}$. We say that a property $P(x)$ holds $\mathcal{I}$-*almost everywhere* or for $\mathcal{I}$-*almost all* x, if it holds for all points $x\in\mathbb{G}$ except for some of them which form a set in $\mathcal{I}$. A pair $(\mathcal{F},\mathcal{F}^{(2)})$ is called $\mathcal{I}$-*hereditary* if $\mathcal{I}$-almost all sections f^y are in $\mathcal{F}$ for every $f\in\mathcal{F}^{(2)}$ (where $f^y(x)=f(x,y)$, $x\in\mathbb{G}$). We say that a pair $(\mathcal{F},\mathcal{F}^{(2)})$ is *hereditary* if there exists an $\mathcal{I}$ such that $(\mathcal{F},\mathcal{F}^{(2)})$ is $\mathcal{I}$-hereditary.

Lemma 1. *Suppose that $\mathcal{F}$ is a translation invariant additive group of functions $f:\mathbb{G}\to\mathbb{R}$ and $(\mathcal{F},\mathcal{F}^{(2)})$ is hereditary. Then the condition $Df\in\mathcal{F}^{(2)}$ implies that all difference functions $\Delta_h f$ are in $\mathcal{F}$.*

Proof. (Cf. [5] Theorem 5.) Suppose that $Df\in\mathcal{F}^{(2)}$. By the assumption of hereditarity of $(\mathcal{F},\mathcal{F}^{(2)})$ there exists a translation invariant, proper σ-ideal $\mathcal{I}$ and a set $Y\in\mathcal{I}$, such that $(Df)^y\in\mathcal{F}$ for any $y\in Y^c$ (where Y^c stands for the complement of Y). Since $Y^c\cap[(-Y^c)+h]\neq\emptyset$ for any $h\in\mathbb{G}$, there are $y_1,y_2\in Y$ such that $h=y_1+y_2$. Thus
$\Delta_h f(x)=f(x+h)-f(x)=f(x+y_1+y_2)-f(x)$
$=(f(x+y_1+y_2)-f(x+y_2)-f(y_1))+(f(x+y_2)-f(y_2)-f(x))+f(y_1)+f(y_2)$
$=(Df)^{y_1}(x+y_2)+(Df)^{y_2}(x)+f(y_1)+f(y_2)$,
so $\Delta_h f\in\mathcal{F}$ for each $h\in\mathbb{G}$. □

The following proposition is an immediate consequence of Lemma 1.

Proposition 1. *Assume that $\mathcal{F}$ is a translation invariant additive group of functions $f:\mathbb{G}\to\mathbb{R}$ and $(\mathcal{F},\mathcal{F}^{(2)})$ is hereditary. If $\mathcal{F}$ has the difference property then the pair $(\mathcal{F},\mathcal{F}^{(2)})$ has the double difference property.*

Remark. Notice that the inverse implication is false. For example, the family $\mathcal{B}$ of all bounded functions satisfies the assumptions of Proposition 1, the pair $(\mathcal{B},\mathcal{B}^{(2)})$ has the double difference property, but $\mathcal{B}$ does not have the difference property [1].

We consider the following classes of real-valued functions on $\mathbb{G}$:

- $\mathcal{C}$ – the continuous functions,
- $\mathcal{A}p\mathcal{C}$ – the approximately continuous functions (see [8], pp. 131-132),
- $\mathcal{PD}$ – the pointwise discontinuous functions (i.e., those having a dense set of continuity points).

Theorem 1. *The pairs* $(\mathcal{C},\mathcal{C}^{(2)})$, $(\mathcal{A}p\mathcal{C},\mathcal{A}p\mathcal{C}^{(2)})$ *and* $(\mathcal{PD},\mathcal{PD}^{(2)})$ *have the double difference property.*

PROOF. All considered classes of functions (of one variable) are translation invariant additive groups and have the difference property (see respectively, [1], [5] and [6]), so it is enough to check that the assumption about hereditarity is fulfilled. In the case of $(\mathcal{C},\mathcal{C}^{(2)})$ this is obvious and in the case of $(\mathcal{A}p\mathcal{C},\mathcal{A}p\mathcal{C}^{(2)})$ it directly follows from Theorem 4 in [2] (we consider $\mathcal{I}=\{\emptyset\}$). We will show that the pair $(\mathcal{PD},\mathcal{PD}^{(2)})$ is $\mathcal{M}$-hereditary. Let $f\in\mathcal{PD}^{(2)}$. Denoting by C_f the set of continuity points of f, we have that C_f is a dense G_δ subset of $\mathbb{R}^2$. By the Kuratowski-Ulam theorem [7], for $\mathcal{M}$-almost all $y\in\mathbb{R}$, the sections $(C_f)^y\subset C_{(f^y)}$ are dense in $\mathbb{R}$, which means that $\mathcal{M}$-almost all sections f^y are in $\mathcal{PD}$. $\square$

Let $\mathcal{F}$ and $\mathcal{G}$ be classes of real-valued functions on $\mathbb{G}$ with $\mathcal{G}\subset\mathcal{F}$. Tamas Keleti in his doctoral dissertation [3] considered the family $\mathcal{H}(\mathcal{F},\mathcal{G})=\{A\subset\mathbb{G}:\ \exists f\in\mathcal{F}\setminus\mathcal{G}\ \ \forall h\in A\ \ \Delta_h f\in\mathcal{G}\}$ consisting of sets A which do not satisfy the condition:

$$(*)\quad \forall f\in\mathcal{F}\,(\,(\forall h\in A\quad \Delta_h f\in\mathcal{G})\;\Rightarrow\; f\in\mathcal{G}\,).$$

Keleti established $\mathcal{H}(\mathcal{F},\mathcal{G})$ for several classes of measurable functions and gave a characterization of the difference property in the language of property $(*)$ ([4], Lemma 1.1). In a similar way we shall characterize the double difference property, replacing $(*)$ by the following condition:

$$(**)\quad \forall f\in\mathcal{F}\,(\,Df\in\mathcal{G}^{(2)}\;\Rightarrow\; f\in\mathcal{G}\,).$$

Theorem 2. *Let* $\mathcal{F}$ *be an additive group of functions, let* $\mathcal{G}$ *be a subgroup of* $\mathcal{F}$ *containing all linear functions and* $\mathcal{G}^{(2)}\subset\mathcal{F}^{(2)}$. *Assume that*

(i) $(\mathcal{F},\mathcal{F}^{(2)})$ *has the double difference property,*

(ii) every additive function from $\mathcal{F}$ *is linear.*

Then the following conditions are equivalent:

(a) $(\mathcal{F},\mathcal{G})$ *satisfies condition* $(**)$,

(b) $(\mathcal{G},\mathcal{G}^{(2)})$ *has the double difference property.*

PROOF. (a)$\Rightarrow$(b): Let f be a function from $\mathbb{G}$ to $\mathbb{R}$ such that $Df\in\mathcal{G}^{(2)}$. Since $\mathcal{G}^{(2)}\subset\mathcal{F}^{(2)}$ and the pair $(\mathcal{F},\mathcal{F}^{(2)})$ has the double difference property, there exist a $g\in\mathcal{F}$ and an additive function H such that $f=g+H$. Thus $Dg=Df$ belongs to $\mathcal{G}^{(2)}$, so by assumption (a) we have that $g\in\mathcal{G}$.

(b)⇒(a): Let $f \in \mathcal{F}$ and $Df \in \mathcal{G}^{(2)}$. The double difference property of the pair $(\mathcal{G}, \mathcal{G}^{(2)})$ implies that $f = g + H$ where $g \in \mathcal{G} =$ and H is additive. Since $H = f - g$ belongs to $\mathcal{F}$, so by (ii) H is a linear function. Thus f, as a sum of two functions from $\mathcal{G}$, has to be in $\mathcal{G}$. □

Note that under assumptions of Lemma 1 the condition $\mathbb{G} \notin \mathcal{H}(\mathcal{F}, \mathcal{G})$ implies that the pair $(\mathcal{F}, \mathcal{G})$ fulfills condition $(**)$. Omitting assumptions which were not used in the proof of the implication "(a)⇒(b)" we have:

Corollary 1. *Assume that $\mathcal{G} \subset \mathcal{F}$ is a translation invariant additive group of functions, $\mathcal{G}^{(2)} \subset \mathcal{F}^{(2)}$ and $(\mathcal{G}, \mathcal{G}^{(2)})$ is hereditary. If $\mathbb{G} \notin \mathcal{H}(\mathcal{F}, \mathcal{G})$ and the pair $(\mathcal{F}, \mathcal{F}^{(2)})$ has the double difference property then $(\mathcal{G}, \mathcal{G}^{(2)})$ has this property, too.*

In the sequel by L_0 we denote the class of all Lebesgue measurable functions on $\mathbb{G}$ and by M_0 – the class of functions with the Baire property on $\mathbb{G}$. For a fixed family $\mathcal{F}$ we define

$$\mathcal{F}_{\mathcal{I}} = \{f \in \mathbb{R}^{\mathbb{G}} : \exists g \in \mathcal{F} \quad f = g \quad \mathcal{I} - \textit{almost everywhere}\},$$

and we denote by $\mathcal{C}_{\mathcal{I}}$ the family of $\mathcal{I}$-essentially continuous functions and by $\mathcal{B}_{\mathcal{I}}$ the family of $\mathcal{I}$-essentially bounded functions. Let also $L_\infty = L_0 \cap \mathcal{B}_{\mathcal{N}}$.

Theorem 3. *The pairs $(\mathcal{C}_{\mathcal{N}}, \mathcal{C}_{\mathcal{N}}^{(2)})$ and $(L_p(\mathbb{T}), L_p^{(2)}(\mathbb{T}))$ (for $0 < p \leq \infty$) have the double difference property.*

Proof. It suffices to apply Corollary 1 in the case when $\mathcal{F} = L_0$ and $\mathcal{G} = \mathcal{C}_{\mathcal{N}}$ (or respectively, $\mathcal{G} = L_p(\mathbb{T})$). The Fubini theorem implies that respective pairs are $\mathcal{N}$-hereditary. Using the known facts:

- the pair $(L_0, L_0^{(2)})$ has the double difference property (proved by Laczkovich in [5]),
- $\mathbb{R} \notin \mathcal{H}(L_0, \mathcal{C}_{\mathcal{N}})$ and $\mathbb{T} \notin \mathcal{H}(L_0, L_p)$ for $0 < p \leq \infty$ (results of Keleti [4]),

we have the assertion. □

Remark. One can consider the category analogs of the pairs $(\mathcal{C}_{\mathcal{N}}, \mathcal{C}_{\mathcal{N}}^{(2)})$ and $(L_\infty(\mathbb{T}), L_\infty^{(2)}(\mathbb{T}))$ and can ask whether they have the double difference property. That would be true if the following problem had the positive answer:

Problem. Does the pair $(M_0, M_0^{(2)})$ have the double difference property?

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