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Stanisław Wroński, Institute of Mathematics, Łódź Technical University, al. Politechniki 11, 90-924 Łódź, Poland. e-mail: wronskis@ck-sg.p.lodz.pl

ON A FAMILY OF FUNCTIONS DEFINED BY THE BOUNDARY OPERATOR

Abstract

For a topological space X, let $M(X,R)$ denote the family of all functions $f \in R^X$ such that $f(Fr(A)) \subseteq Fr(f(A))$. Let $N(X,R)$ denote the family of all continuous functions $f \in R^X$ such that $card(f^{-1}(c)) = 1$ for each $c \in \left(\inf\limits_{x\in X} f(x), \sup\limits_{x\in X} f(x)\right)$. We show that $M(X,R) = N(X,R)$ if X is a connected and locally connected Hausdorff space.

We adopt the following notation:

- X - a topological space with the family O of open sets,
- R - the set of real numbers with the natural topology,
- $C(X,R)$ - the family of continuous functions from X into R.

For $f \in R^X$ let i_f and s_f abbreviate $\inf\limits_{x\in X} f(x)$ and $\sup\limits_{x\in X} f(x)$ respectively. By $int(A)$, $cl(A)$ and $Fr(A)$ we denote the interior, the closure and the boundary of the set $A \subseteq X$.

Let us define two classes of functions: $M(X,R)$ and $N(X,R)$ in the following manner:

$$M(X,R) = \left\{ f \in R^X : \forall_{A\subseteq X} f(Fr(A)) \subseteq Fr(f(A)) \right\}$$

$$N(X,R) = \left\{ f \in C(X,R) : \forall_{c\in(i_f,s_f)} card(f^{-1}(c)) = 1 \right\}^1$$

Then we have the following:

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[1]Professor Ryszard Pawlak observed that instead of the family of continuous functions one can take the family of functions having the Darboux property.

Theorem 1. $N(X,R) \subseteq M(X,R) \subseteq C(X,R)$

PROOF. For an indirect proof of the first inclusion suppose that $N(X,R) \setminus M(X,R) \neq \emptyset$. Thus, there exist $f \in N(X,R)$ and $A \subseteq X$ such that $f(Fr(A)) \setminus Fr(f(A)) \neq \emptyset$. Let $y \in f(Fr(A)) \setminus Fr(f(A))$, then there exists an $x \in cl(A) \setminus int(A)$ such that $y = f(x)$ and therefore $y \in cl(f(A))$ because f is continuous. Since $y \notin Fr(f(A))$ then necessarily $y \in int(f(A))$. Since $x \notin int(A)$ and f is continuous then $y \in cl(f(X \setminus A))$. This means that $int(f(A))$, being an open neighborhood of y, intersects the set $f(X \setminus A)$ i.e. $int(f(A)) \cap f(X \setminus A) \neq \emptyset$. Let $c \in int(f(A)) \cap f(X \setminus A)$. Then there exist distinct elements $x' \in A$ and $x'' \in X \setminus A$ such that$f(x') = f(x'') = c$. Since $c \in int(f(A))$ we have $c \in (i_f, s_f)$ which contradicts the assumption that $f^{-1}(c)$ is a singleton.

To prove the second inclusion assume that $f \in M(X,R)$ and $A \subseteq X$. Then $f(cl(A)) = f(A \cup Fr(A)) = f(A) \cup f(Fr(A)) \subseteq f(A) \cup Fr(f(A)) = cl(f(A))$ which means that $f \in C(X,R)$. □

It is worth noticing that the family $C(X,R)$ can be larger than $M(X,R)$. Indeed, the function $f : R \to R$ such that $f(x) = |x|$ belongs to the set $C(R,R) \setminus M(R,R)$. If we take as X the set of the real numbers with the discrete topology then the characteristic function of the interval $[0,\infty)$ belongs to $M(X,R) \setminus N(X,R)$. However, if X is assumed to be a connected and locally connected Hausdorff space then one gets:

Theorem 2. $N(X,R) = M(X,R)$.[2]

To prove the above theorem we shall need several lemmas.

Lemma 1. *If the sets M and N are disjoint, nonempty and closed in a connected and locally connected space then the complement of $M \cup N$ has a component whose closure intersects each of M and N.*

PROOF. See [1, p. 183]. □

Lemma 2. *For every open and connected $V \subseteq X$ and for any $a, b \in V$, if $a \neq b$ then there exists an open and connected set I such that $I \subseteq V$ and $a \in I, b \in Fr(I)$.*

PROOF. Given $V \subseteq X$ and $a, b \in V$ are such as required in the above lemma. Since the subspace V must be locally connected (see [2]) and $\{a\}$, $\{b\}$ are nonempty, disjoint and closed in V then, by lemma 1, the set $V \setminus \{a, b\}$ in the subspace V must have a component U such that $\{a, b\} \subseteq cl(U)$. Since the

[2]The referee has remarked that it would be interesting to know whether Theorem 2 holds for functions $f \in R^X$ where X is only assumed to be arcwise connected.

component U is open and connected in V, it is also open and connected in X. Since U is open and $\{a, b\} \subseteq cl(U)$ then $\{a, b\} \subseteq Fr(U)$. Applying again the assumption that X is a locally connected Hausdorff space we get that there exists a connected open set $W \subseteq X$ such that $a \in W \subseteq V$ and $b \notin W$. Since $a \in Fr(U)$ then $W \cap U \neq \emptyset$. Now, putting $I = W \cup U$ we obtain a connected open set such that $a \in I \subseteq V$. We will show that $b \in Fr(I)$. Indeed, on one hand $b \in Fr(U) \subseteq cl(U) \subseteq cl(I)$. On the other hand, $b \notin U$ because $b \in Fr(U)$. Since at the same time $b \notin W$, we have $b \notin W \cup U = I$.

This together with the fact proved before allows us to conclude that $b \in Fr(I)$. □

In the lemmas that follow we shall assume that $f \in M(X, R)$.

Lemma 3. *For every real number c, the set $Fr(f^{-1}(c))$ has at most one element.*

PROOF. For an indirect argument, let us assume that for some real number c there exist two distinct $x_1, x_2 \in X$ such that $\{x_1, x_2\} \subseteq Fr(f^{-1}(c))$. By the continuity of f (see Theorem 1) it follows that $f^{-1}(c)$ is a closed set and therefore $f(x_1) = f(x_2) = c$. Let K, L be open sets such that $x_1 \in K$, $x_2 \in L$, and $K \cap L = \emptyset$. From assumptions about the space X it follows that there exist connected and open sets A, B such that $x_1 \in A \subseteq K$, $x_2 \in B \subseteq L$. Note that there must exist an element $x_1' \in A$ such that $f(x_1') \neq c$. Indeed, in the opposite case one gets that $A \subseteq f^{-1}(c)$ and consequently $x_1 \notin Fr(f^{-1}(c))$. Analogously one can show the existence of an element $x_2' \in B$ such that $f(x_2') \neq c$. Now we have to consider the following four cases:

1. $f(x_1') < c$ and $f(x_2') > c$
2. $f(x_1') > c$ and $f(x_2') < c$
3. $f(x_1') > c$ and $f(x_2') > c$
4. $f(x_1') < c$ and $f(x_2') < c$.

All the cases above can be dealt with in a similar manner and for this reason only the case 1 will be considered in detail. First, by lemma 2, we pick a connected open set U such that $x_1 \in U$, $x_1' \in Fr(U)$ and $U \subseteq A$. By the same lemma, we get a connected open set V such that $x_2' \in V$, $x_2 \in Fr(V)$ and $V \subseteq B$. Now our assumptions yield that $f(U)$, $f(V)$ are connected subsets of the straight line, $x_1' \in cl(f(U))$, $x_2 \in cl(f(V))$. Let us put $W = U \cup V$, then $f(W) = f(U) \cup f(V) \supseteq (f(x_1'), f(x_2')]$ which implies that $c \notin Fr(f(W))$. Note that $Fr(V) \subseteq Fr(W)$ because U and V are separated. Since $x_2 \in Fr(V)$ then $x_2 \in Fr(W)$ and consequently $c = f(x_2) \in f(Fr(W)) \subseteq Fr(f(W))$, a contradiction. □

Lemma 4. $(\forall c \in (i_f, s_f))\ (\forall s \in Fr(f^{-1}(c)))\ (\forall U \in O,\ s \in U)\ (\exists a \in U)$ $(\exists b \in U)\ \ (f(a) < c < f(b))$

PROOF. For an indirect argument let us suppose that for some $c \in (i_f, s_f)$, for some $s \in Fr(f^{-1}(c))$ and for some neighborhood V_s of s, c is a lower bound of $f(V_s)$ i.e. for every $x \in V_s$, $f(x) \geq c$ (assuming here that c is an upper bound of $f(V_s)$ one can argue further in a similar manner). By the Theorem 1 one gets that the set $\{x \in X : f(x) \geq c\}$ is closed, nonempty and distinct from the whole space. We shall show that it is open too. Indeed, if $f(x_0) = c$ then either $x_0 \neq s$ – in which case, by Lemma 3, x_0 is an interior point of the set $f^{-1}(c) \subseteq \{x \in X : f(x) \geq c\}$ – or $x_0 = s$. But then, by the indirect assumption, the neighborhood V_s of s must be contained in the set $\{x \in X : f(x) \geq c\}$. Next, if $f(x_0) > c$ then by the continuity of f, it follows that the whole f-image of some neighborhood of x_0 lays strictly above c which contradicts the assumption that the space X is connected. □

Lemma 5. *For every $c \in (i_f, s_f)$ the set $f^{-1}(c)$ is nonempty.*

PROOF. It is an immediate consequence of the assumption that the function f is continuous and the space X is connected. □

PROOF OF THEOREM 2. By lemma 5 we need only to prove that for every $c \in (i_f, s_f)$, the set $f^{-1}(c)$ has at most one element. Suppose the contrary, i.e. for some $c \in (i_f, s_f)$ the set $f^{-1}(c)$ has more than one element. From the assumptions it follows that the set $f^{-1}(c)$ is a closed subset of X distinct from the empty set and from X. Since the space X is connected then $f^{-1}(c)$ can not be open and therefore $Fr(f^{-1}(c)) \neq \emptyset$. Let $x' \in Fr(f^{-1}(c)) \subseteq f^{-1}(c)$ and let x'' be an element of $f^{-1}(c)$ which is distinct from x'. Then, by lemma 3, $x'' \in int(f^{-1}(c))$. Let U and V be connected sets such that $x' \in U$, $x'' \in V$, $U \cap V = \emptyset$. By lemma 4, there exist $a, b \in U$ such that $f(a) < c < f(b)$. Next, by lemma 2 one can find a connected set W such that $a \in W$, $b \in Fr(W)$ and $W \subseteq U$. Let us put $A = W \cup \{x''\}$. Then $f(A) \supseteq [f(a), f(b))$ and therefore $c \notin Fr(f(A))$. Since $x'' \in Fr(A)$ then $c = f(x'') \in f(Fr(A)) \subseteq Fr(f(A))$, a contradiction. □

References

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[2] K. Kuratowski, *Topology*, vol. 2, Academic Press, New York, 1968.