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ON A. C. LIMITS OF DECREASING SEQUENCES OF CONTINUOUS OR RIGHT CONTINUOUS FUNCTIONS

Abstract

The a.c. limits (**i.e.** the discrete limits introduced by Császár and Laczkovich) of decreasing sequences of continuous (resp. right continuous) functions are investigated.

Let $\mathbb{R}$ be the set of all reals. (X, τ) or X in this paper always denotes a perfectly normal Hausdorff topological space. A function $f : X \to \mathbb{R}$ is a B_1^* function (belongs to the class B_1^*) if there is a sequence of continuous functions $f_n : X \to \mathbb{R}$ with $f = \mathrm{a.c.}\lim_{n\to\infty} f_n$, **i.e.** for each point $x \in X$ there is a positive integer k such that $f_n(x) = f(x)$ for every $n > k$ (compare [2, 3]).

From the results obtained in [2] it follows that the function $f : X \to \mathbb{R}$ belongs to $\mathcal{B}_1^*$ if and only if there are closed sets A_n, $n = 1, 2, \ldots$, such that the restricted functions $f \restriction A_n$ are continuous and $X = \bigcup_{n=1}^{\infty} A_n$.

1 The Discrete Limits of Decreasing Sequences of Continuous Functions.

In the first part of this article we will investigate B_1^* functions which are upper semicontinuous. Recall that the function $f : X \to \mathbb{R}$ is upper semicontinuous if for every real a the set $\{x \in X; f(x) < a\}$ belongs to τ. Evidently the pointwise limit of each decreasing sequence of upper semicontinuous functions $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$, is upper semicontinuous.

The following theorem can be found on page 51 of [4].

Key Words: upper semicontinuity, decreasing sequences of functions, B_1^* class, right continuity.

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Remark 1. *If the function $f : X \to \mathbb{R}$ is upper semicontinuous, then there is a decreasing sequence of continuous functions $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$, such that $f = \lim_{n\to\infty} f_n$.*

We will prove the following theorem.

Theorem 1. *Let (X, τ) be a perfectly normal σ-compact Hausdorff topological space. Then an upper semicontinuous function $f : X \to \mathbb{R}$ belongs to class B_1^* if and only if there is a decreasing sequence of continuous functions $f_n : X \to \mathbb{R}$ such that $f = \mathrm{a.c.}\lim_{n\to\infty} f_n$.*

We start from the following lemma.

Lemma 1. *Let $f : X \to \mathbb{R}$ be a function. If there are sets A_n and continuous functions $f_n : X \to \mathbb{R}$ such that $A_1 \subset A_2 \subset \cdots$, $X = \bigcup_n A_n$, $f_n \geq f$ and $f_n \restriction A_n = f \restriction A_n$ for $n = 1, 2, \ldots$, then there is a decreasing sequence of continuous functions $g_n : X \to \mathbb{R}$ with $f = \mathrm{a.c.}\lim_{n\to\infty} g_n$.*

PROOF. Of course, the functions $g_n = \min_{k\leq n} f_n$ satisfy all required conditions. □

PROOF OF THEOREM 1. If f is the discrete limit of a decreasing sequence of continuous functions $f_n : X \to \mathbb{R}$, then evidently $f \in \mathcal{B}_1^*$. So, we assume that $f \in \mathcal{B}_1^*$. Since f is upper semicontinuous and X is perfectly normal, by Remark 1 there is a decreasing sequence of continuous functions $f_n : X \to \mathbb{R}$ which converges to f at each point $x \in X$.

On the other hand f is the discrete limit of continuous functions; so there are closed sets A_n, $n = 1, 2, \ldots$, such that every restricted function $f \restriction A_n$ is continuous and $X = \bigcup_{n=1}^{\infty} A_n$. We can assume that A_n is compact for each $n = 1, 2, \ldots$. Fix a positive integer k. On A_k the sequence (f_n) tends uniformly to f due to Dini's lemma. So we can also assume that

$$\max\{(f_n(x) - f_{n+1}(x)); x \in A_k\} \leq 2^{-n}.$$

By Tietze's theorem for $n = 1, 2, \ldots$ there is a continuous extension $g_n : X \to [0, 2^{-n}]$ of the restricted function $(f_n - f_{n+1}) \restriction A_k$. Let

$$h_n = \min(g_n, f_n - f_{n+1}) \text{ for } n = 1, 2, \ldots,$$

and let $l_k = f_1 - \sum_{n=1}^{\infty} h_n$. Since the series $\sum_{n=1}^{\infty} h_n$ converges uniformly, the function l_k is continuous. Moreover, for $k = 1, 2, \ldots$ we have $l_k \geq f$ and $f \restriction A_k = l_k \restriction A_k$. So, by Lemma 1 we obtain our theorem. □

Theorem 1 in the presented form and its proof was proposed by the referee. My formulation concerned the function $f : [a, b] \to \mathbb{R}$ and the Euclidean topology and its proof was more complicated.

2 Decreasing Sequences of Right Continuous Functions

In this part we assume that $X = [a, b)$ and τ is the topology of right continuity. This topology τ is perfectly normal and Hausdorff but is not σ-compact. So, the limit f of a decreasing sequence of right upper semicontinuous functions f_n, $n = 1, 2, \ldots$, is a right upper semicontinuous function and Remark 1 is valid for (X, τ). Thus we have the following assertion.

Remark 2. *For every right upper semicontinuous function f there is a decreasing sequence of right continuous functions f_n, $n = 1, 2, \ldots$, such that $f = \lim_{n\to\infty} f_n$.*

From the last remark by an elementary proof we obtain the next assertion.

Remark 3. *If a function $f : [a, b) \to \mathbb{R}$ is right upper semicontinuous, then there is a decreasing sequence of functions $f_n : [a, b) \to \mathbb{R}$ such that*

the functions f_n are right continuous ;

$f = \lim_{n\to\infty} f_n$;

all functions f_n, $n = 1, 2, \ldots$, are locally constant from the right, **i.e.** *for each point $x \in [a, b)$ there is a positive real $r_{x,n}$ such that*

$$I_{x,n} = [x, x + r_{x,n}] \subset [a, b) \text{ and } f \restriction I_{x,n} \text{ is constant };$$

if $\limsup_{t\to x+} f(t) < f(x)$, then for n sufficiently large $f_n(x) = f(x)$;

for every integer n the inclusion $f_n([a, b)) \subset \mathrm{cl}(f[a, b)))$, (where cl denotes the closure operation) holds.

Proof. The set A of all points x at which $\limsup_{t\to x+} f(t) < f(x)$ is countable, i.e. if $A \neq \emptyset$, then $A = \{x_1, x_2, \ldots\}$. By Remark 2 there is a decreasing sequence of right continuous functions g_n such that $f = \lim_{n\to\infty} g_n$. Fix a positive integer n and observe that there is a sequence of intervals $I_{i,n} = [u_{i,n}, v_{i,n})$, $i = 1, 2, \ldots$, such that:

$[a, b) = \bigcup_i I_{i,n}$;

$I_{i,n} \cap I_{j,n} = \emptyset$ for $i \neq j$;

$u_{i,n} = x_i$ for $i \leq n$;

$\operatorname{osc} g_n < \frac{1}{n}$ on each interval $I_{i,n}$;

$g_n(x) > f(x)$ if $x \in I_{i,n}$ and $i \leq n$.

Let

$$h_n(x) = \begin{cases} f(x_i) & \text{for } x \in I_{i,n}, \ \ i \leq n \\ \sup_{I_{i,n}} g_n & \text{for } x \in I_{i,n} \ \ i > n. \end{cases}$$

Then the functions $f_n = \min(h_1, h_2, \dots, h_n)$, for $n = 1, 2, \dots$, satisfy all required conditions. □

Theorem 2. *If $f = \mathrm{a.\,c.}\lim_{n\to\infty} f_n$, where all functions f_n, $n = 1, 2, \dots$, are right continuous, then f satisfies the following condition.*

(1) For each nonempty perfect set $A \subset [a, b)$ there is an open interval I such that $A \cap I \neq \emptyset$ and the restricted function $f \restriction (I \cap B)$, where

$$B = \{x \in A; x \textit{ is a right limit point of } A\},$$

is right continuous at each point of the intersection $B \cap I$.

PROOF. Let $A \subset [a, b)$ be a nonempty perfect set and let B denote the set of all right limit points of A. For each point $x \in [a, b)$ there is a positive integer $n(x)$ such that $f_n(x) = f(x)$ for $n \geq n(x)$. For $n = 1, 2, \dots$ put $A_n = \{x \in [a, b); n(x) = n\}$ and observe that $[a, b) = \bigcup_{n=1}^{\infty} A_n$. So there are an open interval I and a positive integer k such that $I \cap B \neq \emptyset$ and $A_k \cap I \cap B$ is dense in $B \cap I$. Thus $f(x) = f_k(x)$ for each point $x \in I \cap A$ which is a right limit point of A and consequently $f \restriction (B \cap I)$ is right continuous at each point of $I \cap B$. □

The above proof of Theorem 2 is short. However the referee related this statement to the result of Császár and Laczkovich (Theorem 13 of [2], pp. 469) which says that if X is a Baire space, the functions $f_n : X \to \mathbb{R}$, $n = 1, 2, \dots$, are continuous and $f = \mathrm{a.\,c.}\lim_{n\to\infty} f_n$, then the points of discontinuity of f constitute a nowhere dense set in X.

The connection between these two results is the following assertion.

Let $(X, \mathcal{T}, \tau)$ be a bitopological space such that τ is finer than $\mathcal{T}$ and $(X, \mathcal{T})$ is a Baire space. Assume that for every nonempty set $A \in \tau$ there is a nonempty set $B \in \mathcal{T}$ such that $B \subset A$. Then every τ-nowhere dense set is $\mathcal{T}$-nowhere dense and $(X, \mathcal{T})$ is a Baire space.

We arrive at Theorem 2 at once if we observe that the sets $X \subset [a, b)$ having no right isolated points satisfy the conditions of the previous statement. So the quoted theorem of Császár and Laczkovich can be applied.

Example 1. Let C be the Cantor ternary set and let $I_n = (a_n, b_n)$, $n = 1, 2, \ldots$, be an enumeration of all components of the set $[0,1) \setminus C$ such that $I_n \cap I_m = \emptyset$ for $n \neq m$, $n, m = 1, 2, \ldots$. Put

$$f(x) = \begin{cases} 1 & \text{for} \quad x \in B = C \setminus \{a_n; n \geq 1\} \\ 0 & \text{for} \quad x \in [0,1) \setminus B. \end{cases}$$

Observe that the function f is not of Baire class one. For $n \geq 1$ let

$$f_n(x) = \begin{cases} 0 & \text{for} \quad x \in B_n = \bigcup_{i \leq n} [a_i, b_i) \\ 1 & \text{for} \quad x \in [0,1) \setminus B_n. \end{cases}$$

Then all functions f_n, $n = 1, 2, \ldots$, are right continuous, $f_n \geq f_{n+1}$ for $n = 1, 2, \ldots$ and $\text{a.c.}\lim_{n\to\infty} f_n = f$.

Now we introduce the following condition $(1')$.

(1') A function f satisfies condition $(1')$ if for every nonempty closed set $A \subset [0,1)$ there is an open interval I such that $I \cap A \neq \emptyset$ and the restricted function $f \restriction (A \cap I)$ is right continuous. (If $x \in A$ is right isolated in A, then $f \restriction A$ is right continuous at x by default.)

Observe that the implication $(1') \Longrightarrow (1)$ is true. The function f from Example 1 satisfies condition (1) but it does not satisfy condition $(1')$. Observe also that, by Baire's theorem on Baire 1 functions, every function f satisfying condition $(1')$ is of Baire 1 class.

Theorem 3. *A function f satisfies condition $(1')$ if and only if it satisfies the following condition.*

(2) There is a sequence of nonempty closed sets $A_n \subset [a,b)$ such that all restricted functions $f \restriction A_n$, $n = 1, 2, \ldots$, are right continuous and $[a,b) = \bigcup_{n=1}^{\infty} A_n$.

Proof. $(1') \Longrightarrow (2)$. We will apply transfinite induction. Let I_0 be an open interval with rational endpoints such that the restricted function $f \restriction I_0$ is right continuous. Fix an ordinal number $\alpha > 0$ and suppose that for every ordinal number $\beta < \alpha$ there is an open interval with rational endpoints I_β such that $H_\beta = I_\beta \setminus \bigcup_{\gamma<\beta} I_\gamma \neq \emptyset$ and the restricted function $f \restriction H_\beta$ is right continuous. If $G_\alpha = [a,b) \setminus \bigcup_{\beta<\alpha} I_\beta \neq \emptyset$, then by $(1')$ there is an open interval I_α with rational endpoints such that $I_\alpha \cap G_\alpha \neq \emptyset$ and the restricted function $f \restriction (I_\alpha \cap G_\alpha)$ is right continuous. Let ξ be the first ordinal number α such that $[a,b) \setminus \bigcup_{\beta<\xi} I_\beta = \emptyset$. Since the family of all intervals with rational endpoints is countable, ξ is a countable ordinal number. Every set H_α, $\alpha < \xi$, is an F_σ

set; so there are closed sets $H_{k,\alpha}$, $k = 1, 2, \ldots$, such that $H_\alpha = \bigcup_{k=1}^\infty H_{k,\alpha}$. Evidently, all restricted functions $f \restriction H_{k,\alpha}$, $k = 1, 2, \ldots$ and $\alpha < \xi$, are right continuous. Now enumerate in a sequence (A_n) all sets

$$H_{k,\alpha}, \ \ k = 1, 2, \ldots \ \text{ and } \ \alpha < \xi,$$

and observe that this sequence satisfies all requirements.

$(2) \Longrightarrow (1')$

Fix a nonempty closed set $A \subset [a, b)$. If A contains isolated points, then condition $(1')$ is satisfied. So we assume that A is a perfect set. By (2) there is a sequence of closed sets A_n, $n = 1, 2, \ldots$, such that $[a, b) = \bigcup_n A_n$ and all restricted functions $f \restriction A_n$, $n = 1, 2, \ldots$, are right continuous. Since $A = \bigcup_{n=1}^\infty (A \cap A_n)$, there are a positive integer k and an open interval I such that $I \cap A = I \cap A_k \neq \emptyset$. But the restricted function $f \restriction (A \cap I)$ is right continuous; so condition $(1')$ is satisfied. $\square$

Theorem 4. *If f satisfies condition $(1')$ (or equivalently (2)) from the last theorem, then there is a sequence of functions f_n, $n = 1, 2, \ldots$, which are right continuous and for which* a. c. $\lim_{n\to\infty} f_n = f$.

PROOF. There is a sequence of nonempty closed sets A_n, $n = 1, 2, \ldots$, such that $[a, b) = \bigcup_{n=1}^\infty A_n$, $A_1 \subset A_2 \subset \cdots$ and all restricted functions $f \restriction A_n$, $n = 1, 2, \ldots$, are right continuous. By Tietze's theorem for $n = 1, 2, \ldots$ there is a right continuous function $f_n : [a, b) \to \mathbb{R}$ which is equal to f on the set A_n. Then a. c. $\lim_{n\to\infty} f_n = f$. $\square$

Theorem 5. *If a function f is upper semicontinuous from the right and satisfies condition $(1')$ (or equivalently (2)), then there is a decreasing sequence of right continuous functions f_n, $n = 1, 2, \ldots$, such that* a. c. $\lim_{n\to\infty} f_n = f$.

PROOF. Let a function f satisfies the hypothesis of our theorem. There is a sequence of nonempty closed sets A_n, $n = 1, 2, \ldots$, such that $[a, b) = \bigcup_{n=1}^\infty A_n$ and all restricted functions $f \restriction A_n$, $n = 1, 2, \ldots$, are right continuous. Without loss of the generality we can suppose that $a \in A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$. Fix a positive integer n and enumerate in a sequence $(I_{n,k})_k$ all components of the set $[a, b) \setminus A_n$. If $I_{n,k} = (a_{n,k}, b_{n,k})$, $a_{n,k} \in A_n$ and $\limsup_{t\to a_{n,k}+} f(t) = f(a_{n,k})$, then we find points $c_{n,k,i}$, $i = 1, 2, \ldots$, such that

$$b_{n,k} = c_{n,k,1} > \ldots > c_{n,k,i} > \ldots \searrow a_{n,k}.$$

By Theorem 2 and Remark 1 for every $i \geq 1$ there is right constant function $h_{n,k,i} : [c_{n,k,i+1}, c_{n,k,i}) \to \mathbb{R}$ such that $h_{n,k,i} \geq f/[c_{n,k,i+1}, c_{n,k,i})$ and $h_{n,k,i}(c_{n,k,i}) < f(c_{n,k,i}) + \frac{1}{ik}$. Let $g_{n,k,i}(x) = \max(h_{n,k,i}(x), f(a_{n,k}))$ for $x \in$

$[c_{n,k,i+1}, c_{n,k,i})$. Observe that $g_{n,k,i}([c_{n,k,i+1}, c_{n,k,i})) \subset \mathrm{cl}(f([a_{n,k}, b_{n,k})))$, $i = 1, 2, \ldots$ Next in every such interval $I_{n,k}$ we define the function $g_{n,k}$ by

$$g_{n,k}(x) = g_{n,k,i}(x) \ \text{ for } \ x \in [c_{n,k,i+1}, c_{n,k,i}), \quad i = 1, 2, \ldots.$$

If $I_{n,k} = [a_{n,k}, b_{n,k})$, $a_{n,k} \in A_n$, and $\limsup_{x \to a_{n,k}+} f(x) < f(a_{n,k})$, then by Remarks 2 and 3 there is a right constant function $h_{n,k} : [a_{n,k}, b_{n,k}) \to \mathbb{R}$ such that $h_{n,k} \geq f \restriction [a_{n,k}, b_{n,k})$, $h_{n,k}(a_{n,k}) = f(a_{n,k})$ and $h_{n,k}([a_{n,k}, b_{n,k})) \subset \mathrm{cl}(f([a_{n,k}, b_{n,k})))$. Let $g_{n,k}(x) = \max(h_{n,k}(x), f(x))$ for $x \in [a_{n,k}, b_{n,k})$. Put

$$g_n(x) = \begin{cases} f(x) & \text{for} \quad x \in A_n \\ g_{n,k}(x) & \text{for} \quad x \in I_{n,k} \quad i, k = 1, 2, \ldots. \end{cases}$$

Then the function g_n is right continuous, $g_n \geq f$ and $g_n \restriction A_n = f \restriction A_n$. So, by Lemma 1 we obtain our theorem. $\square$

Observe that the last theorem is not a corollary of Theorem 1, since the topology τ is not σ-compact.

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Problem. Is Theorem 5 true if we replace condition $(1')$ by (1)?

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